# Chapter 3

# MANOVA Approaches to Longitudinal Data

As discussed in the previous chapter, there are two classical approaches to the analysis of longitudinal data, the first, called variously univariate mixed-model, split-plot, or repeated-measures ANOVA, and the second based on multivariate ANOVA (MANOVA). In this chapter, we discuss the general multivariate growth curve model. The primary advantage of the MANOVA approach versus the ANOVA approach is that the MANOVA assumes a general form for the correlation of repeated measurements over time, whereas the ANOVA assumes the much more restrictive compound-symmetric form. The disadvantage of the

MANOVA model is that it requires complete data. Subjects with incomplete data must be removed from the analysis, leading to potential bias. In addition, both MANOVA and ANOVA models focus on comparison of group means and provide no information regarding subject-specific growth curves. Finally, both ANOVA and MANOVA models require that the time-points are fixed across subjects (either evenly or unevenly spaced) and are treated as a classification variable in the ANOVA or MANOVA model. This precludes analysis of unbalanced designs in which different subjects are measured on different occasions.

In the following sections, we describe the MANOVA model in detail. While it is no longer recommended for routine application (if at all), it is important in that it helps to fix ideas for the development of the more modern and advanced methods that are the primary focus of this book.

## 3.1 MANOVA for Repeated Measurements

In the MANOVA approach to analysis of repeated measurements, the $n$ repeated measures are treated as a $n \times 1$ response vector $\boldsymbol{y}_i$. Due to the multivariate nature of the analysis, subjects with any missing $y_{ij}$ (across time) are omitted from the analysis. This is the "Achilles heel" of the MANOVA model

for repeated measurements and why it is largely of only limited use in many research fields. The MANOVA model is given by,

$$\boldsymbol{y}_i = \boldsymbol{\mu} + \boldsymbol{\varepsilon}_i \ , \tag{3.1}$$

where $\boldsymbol{\mu} = n \times 1$ mean vector for timepoints, and $\boldsymbol{\varepsilon}_i = n \times 1$ vector of errors, distributed $\sim N(\mathbf{0}, \boldsymbol{\Sigma})$ in the population. Notice, that under univariate model assumptions:

- $\boldsymbol{\Sigma} = \sigma_\pi^2 \mathbf{1}_n \mathbf{1}_n' + \sigma_e^2 \boldsymbol{I}_n$
- $\boldsymbol{\mu} = \mu + \boldsymbol{\tau}$ (*i.e.*, grand mean and time effects)
- $\Rightarrow$ univariate results can be extracted from the multivariate model (and calculations).

For the one-sample case, we can characterize the timepoint vector $\boldsymbol{\mu}$ and choose contrasts depending on structure and hypotheses of interest.

### 3.1.1 Growth Curve Analysis - polynomial representation

The polynomial growth-curve model is generally described as

$$\begin{bmatrix} \mu_1 \\ \mu_2 \\ . \\ . \\ \mu_n \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix} \beta_0 + \begin{bmatrix} t_1 \\ t_2 \\ . \\ . \\ t_n \end{bmatrix} \beta_1 + \begin{bmatrix} t_1^2 \\ t_2^2 \\ . \\ . \\ t_n^2 \end{bmatrix} \beta_2 + \ldots + \begin{bmatrix} t_1^{q-1} \\ t_2^{q-1} \\ . \\ . \\ t_n^{q-1} \end{bmatrix} \beta_{q-1}$$

$$\underset{n\times 1}{\boldsymbol{\mu}} = \underset{n\times q}{\boldsymbol{T}'} \underset{q\times 1}{\boldsymbol{\beta}}$$

where $t_1, t_2, \ldots, t_n$ represent time-point values, and $q \leq n$ represents the degree of the polynomial. It is generally advantageous to orthogonalize $\boldsymbol{T}$ as $\boldsymbol{\mu} = \boldsymbol{P}'\boldsymbol{\theta}$ where $\boldsymbol{P}$ is $q \times n$ matrix of orthogonal polynomials. The first row of $\boldsymbol{P}$ is for the constant term, and the remaining rows correspond sequentially to the linear, quadratic, etc. polynominal terms.

- $\boldsymbol{P} = \boldsymbol{S}^{-1}\boldsymbol{T}$ and $\boldsymbol{S}\boldsymbol{S}' = (\boldsymbol{T}\boldsymbol{T}')$ Cholesky factorization

- $\boldsymbol{S}$ is a $q \times q$ lower triangular matrix

The orthogonal polynomial trend model for $q = n = 4$ is given by

$$\boldsymbol{P}_4 = \begin{array}{c} \left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & \vdots & \div\sqrt{4} \\ -3 & -1 & 1 & 3 & \vdots & \div\sqrt{20} \\ 1 & -1 & -1 & 1 & \vdots & \div\sqrt{4} \\ -1 & 3 & -3 & 1 & \vdots & \div\sqrt{20} \end{array}\right] \end{array} \begin{array}{c} \boldsymbol{p}_0 \\ \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \boldsymbol{p}_3 \end{array}, \tag{3.2}$$

or equivalently as

$$\begin{aligned} \boldsymbol{P}\boldsymbol{y}_i &= \boldsymbol{P}\boldsymbol{\mu} + \boldsymbol{P}\boldsymbol{\varepsilon}_i \\ &= \boldsymbol{\theta} + \boldsymbol{\varepsilon}_i^* \end{aligned}$$

where

- $\boldsymbol{\theta} = n \times 1$ vector of transformed population means with its least squares estimate given by the transformed sample mean vector, namely $\hat{\boldsymbol{\theta}} = \boldsymbol{P}\bar{\boldsymbol{y}}_{.}$, and
- $\boldsymbol{\varepsilon}_i^* \sim N(\mathbf{0}, \boldsymbol{\Sigma}^* = \boldsymbol{P}\boldsymbol{\Sigma}\boldsymbol{P}')$.

Note that the univariate ANOVA assumes that $\boldsymbol{P}\boldsymbol{\Sigma}\boldsymbol{P}'$ is diagonal with equal

values below the first element (since $\boldsymbol{P}$, as defined above, includes the zero-order term, *i.e.*, the grand mean). The corresponding MANOVA table is

| source | df | SSCP ($n \times n$) | E(SSCP) |
|---|---|---|---|
| Time | 1 | $\text{SST}^* = N\boldsymbol{P}\bar{\boldsymbol{y}}_{.}\bar{\boldsymbol{y}}_{.}'\boldsymbol{P}'$ | $\boldsymbol{P}\left[\boldsymbol{\Sigma} + N\boldsymbol{\mu}\boldsymbol{\mu}'\right]\boldsymbol{P}'$ |
| Residual | $N-1$ | $\text{SSR}^* = \boldsymbol{P}\ \text{SSR}\ \boldsymbol{P}'$ | $(N-1)\boldsymbol{P}\boldsymbol{\Sigma}\boldsymbol{P}'$ |
| | | $= \boldsymbol{P}(\boldsymbol{Y}'\boldsymbol{Y} - N\bar{\boldsymbol{y}}_{.}\bar{\boldsymbol{y}}_{.}')\boldsymbol{P}'$ | |

where $\boldsymbol{Y}$ is the $N \times n$ matrix of all data, $\bar{\boldsymbol{y}}_{.}$ is the $n \times 1$ vector of timepoint means, $\text{SST}^*$ has as its first diagonal element $Nn\bar{y}_{..}^2$ which is a function of the grand mean, and the other $n-1$ diagonal elements are the orthogonal polynomial decomposition of Time SS $= N\sum_{j=1}^{n}(\bar{y}_{.j} - \bar{y}_{..})^2$, and $\text{SSR}^*$ has its first diagonal element $n\sum_{i=1}^{N}(\bar{y}_{i.} - \bar{y}_{..})^2$, and other $n-1$ diagonal elements are the orthogonal polynomial decomposition of Error (*i.e.*, Subject by Time) SS.

The orthogonal polynomial partition of sum of squares and products in the

one-sample case is given by

$$
\begin{array}{lcl}
 & & \text{diag element} \\
\hline
\text{Time: } df = 1 & & \\
 & & \\
\mathrm{SST}^* = & \begin{bmatrix} \mathrm{sst}_0 & & & & \\ & \mathrm{sst}_1 & & & \\ & & \mathrm{sst}_2 & & \\ & & & \cdots & \\ & & & & \mathrm{sst}_{n-1} \end{bmatrix} & \begin{array}{l} \text{constant} \\ \text{linear time} \\ \text{quad time} \\ \cdots \\ (n-1)\text{th time} \end{array} \\
 & & \\
\hline
\text{Residual: } df = N - 1 & & \\
 & & \\
\mathrm{SSR}^* = & \begin{bmatrix} \mathrm{ssr}_0 & & & & \\ & \mathrm{ssr}_1 & & & \\ & & \mathrm{ssr}_2 & & \\ & & & \cdots & \\ & & & & \mathrm{ssr}_{n-1} \end{bmatrix} & \begin{array}{l} \text{subjects} \\ \text{subj} \times \text{lin} \\ \text{subj} \times \text{quad} \\ \cdots \\ \text{subj} \times (n-1) \end{array} \\
 & & \\
\hline
\end{array}
\tag{3.3}
$$

Note that these are symmetric matrices, however the diagonals contain all of

the information of interest in SS calculations.

### 3.1.2 Extracting Univariate Repeated Measures ANOVA Results

Univariate repeated measures results can be extracted directly from the MANOVA results as follows:

| Source | df | SS | MS | E(MS) |
|---|---|---|---|---|
| Subjects | $N-1$ | $\mathrm{SS}_S = n\sum_{i=1}^{N}(\bar{y}_{i.} - \bar{y}_{..})^2$ | $\frac{\mathrm{SS}_S}{N-1}$ | $\sigma_e^2 + n\sigma_\pi^2$ |
| Time | $n-1$ | $\mathrm{SS}_T = N\sum_{j=1}^{n}(\bar{y}_{.j} - \bar{y}_{..})^2$ | $\frac{\mathrm{SS}_T}{n-1}$ | $\sigma_e^2 + N\sum(\tau_j - \tau_.)^2$ |
| Residual | $(N-1)$ $\times(n-1)$ | $\mathrm{SS}_R = \sum_{i=1}^{N}\sum_{j=1}^{n} y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..})^2$ | $\frac{\mathrm{SS}_R}{(N-1)(n-1)}$ | $\sigma_e^2$ |
| total | $Nn-1$ | $\mathrm{SS}_y = \sum_{i=1}^{N}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{..})^2$ | | |

where

- $\mathrm{SS}_S = \mathrm{ssr}_0$ from $\mathrm{SSR}^*$ matrix,
- $\mathrm{SS}_T = \sum$ lower $n-1$ $\mathrm{SST}^*$ diagonal elements ,
- $\mathrm{SS}_R = \sum$ lower $n-1$ $\mathrm{SSR}^*$ diagonal elements .

Notice that,

$$\begin{aligned} \mathrm{MS}_R &= \frac{\sum \text{ lower } n-1 \text{ SSR}^* \text{ diagonal elements}}{(N-1)(n-1)} \\ &= \frac{\text{average } n-1 \text{ SSR}^* \text{ diagonal elements}}{(N-1)} \ . \end{aligned}$$

These are sometimes referred to as "averaged" tests.

### 3.1.3 Multivariate Test of the Time Effect

In order to test the null hypothesis of no effect of time, *i.e.*, $H_0 : \boldsymbol{\mu}$ elements are all equal such that $H_0 : \boldsymbol{\tau} = \mathbf{0}$, we must extract and compare the $(n-1)\times(n-1)$ submatrices of $\mathrm{SST}^*$ and $\mathrm{SSR}^*$. To do this requires solution of the two-matrix eigen problem:

$$| \ \mathrm{SST}^*_{(n-1)} \ - \ \lambda \ \mathrm{SSR}^*_{(n-1)} | \ = \ 0 \tag{3.4}$$

Note that $\lambda = 1$ if $\mathrm{SST}^*_{(n-1)} = \mathrm{SSR}^*_{(n-1)}$. To simplify the problem, we use the Cholesky factorization $\mathrm{SSR}^*_{(n-1)} = \boldsymbol{E}\boldsymbol{E}'$ to yield a 1-matrix eigen problem,

$$|\boldsymbol{E}^{-1}\ \mathrm{SST}^*_{(n-1)}(\boldsymbol{E}^{-1})' \ - \ \lambda \boldsymbol{I}_{(n-1)}| \ = \ 0\ . \tag{3.5}$$

Overall test statistics for the null hypothesis of no time-effects include Roy's largest root statistic (eigenvalue of $\lambda$), and Wilk's Lambda ($\Lambda = 1/(1+\lambda)$). Functions of these test statistics approximately follow an $F$-distribution (under the null hypothesis), though sometimes interpolation is necessary, giving rise to fractional *df* in some cases. Other multivariate test statistics include the Hotelling-Lawley trace and the Pillai-Bartlett trace.

### 3.1.4 Tests of Specific Time Elements

To test specific components of the time effect, there are two options depending on whether sphericity is reasonable or not. First, if sphericity is reasonable, then we can obtain univariate test statistics by extracting the lower $n-1$

diagonal elements of SST*, using $\mathrm{MS}_R$ as a denominator, *i.e.*,

$$
\begin{aligned}
F_1 &= \frac{\mathrm{sst}_1}{\sum_{j=1}^{n-1} \mathrm{ssr}_j/[(n-1)(N-1)]} \quad \text{linear} \\
F_2 &= \frac{\mathrm{sst}_2}{\sum_{j=1}^{n-1} \mathrm{ssr}_j/[(n-1)(N-1)]} \quad \text{quadratic} \\
&\ \ .\quad .\quad . \qquad\qquad\qquad\qquad . \\
F_{n-1} &= \frac{\mathrm{sst}_{n-1}}{\sum_{j=1}^{n-1} \mathrm{ssr}_j/[(n-1)(N-1)]} \quad (n-1)\text{th}
\end{aligned}
\tag{3.6}
$$

Notice, again that the denominator is akin to an averaged estimate of the error variance across time.

If sphericity is rejected, then a multivariate test should be used by extracting and comparing the lower $n-1$ diagonal elements of SSM* and SSR*

$$
\begin{aligned}
F_1 &= \frac{\text{sst}_1}{\text{ssr}_1/(N-1)} \qquad \text{linear} \\
F_2 &= \frac{\text{sst}_2}{\text{ssr}_2/(N-1)} \qquad \text{quadratic} \\
&\;\;\cdot \quad \cdot \quad \cdot \qquad\qquad \cdot \\
F_{n-1} &= \frac{\text{sst}_{n-1}}{\text{ssr}_{n-1}/(N-1)} \qquad (n-1)\text{th}
\end{aligned}
\tag{3.7}
$$

Here, there is no pooling across time to obtain an averaged error term. Instead, each component has its own error term. This is why, in general, the univarate tests, which use the pooled error term, are more powerful than the multivariate tests.

## 3.2 MANOVA of Repeated Measures - $s$ Sample Case

When there are multiple groups, we have

- $h = 1, \ldots, s$ groups
- $i = 1, \ldots, N_h$ subjects in group $h$
- $j = 1, \ldots, n$ timepoints
- $N = \sum N_h$ total number of subjects

The model becomes,

$$\boldsymbol{y}_{hi} = \boldsymbol{\mu} + \boldsymbol{\gamma}_h + \boldsymbol{\varepsilon}_{hi} \tag{3.8}$$

where

- $\boldsymbol{\mu}$ is the $n \times 1$ vector of time-point means,
- $\boldsymbol{\gamma}_h$ is the $n \times 1$ vector effect for the population from which the $h$th group of subjects was drawn,
- $\boldsymbol{\varepsilon}_{hi}$ is the $n \times 1$ vector of errors $\sim N(\mathbf{0}, \boldsymbol{\Sigma})$ in each of the populations

The model assumes homogeneity of variance-covariance across the $s$ groups. Again, with orthogonal transformation for time effects

$$\boldsymbol{P}\boldsymbol{y}_{hi} \quad = \quad \boldsymbol{P}\boldsymbol{\mu} + \boldsymbol{P}\boldsymbol{\gamma}_h + \boldsymbol{P}\boldsymbol{\varepsilon}_{hi}$$

$$\boldsymbol{\varepsilon}^*_{hi} \;\sim\; N(\mathbf{0}, \boldsymbol{\Sigma}^* = \boldsymbol{P}\boldsymbol{\Sigma}\boldsymbol{P}')\ ,$$

we can test $\boldsymbol{\Sigma}^*$ for sphericity and proceed using univariate "averaged" tests. The resulting MANOVA table is given by

| source | *df* | SSCP $(n \times n)$ |
|---|---|---|
| Time | 1 | $\text{SST}^* \;=\; \boldsymbol{P}\ \text{SST}\ \boldsymbol{P}' \;=\; N\boldsymbol{P}\bar{\boldsymbol{y}}_{..}\bar{\boldsymbol{y}}'_{..}\boldsymbol{P}'$ |
| Group | $s-1$ | $\text{SSG}^* \;=\; \boldsymbol{P}\ \text{SSG}\ \boldsymbol{P}' \;=\; \boldsymbol{P}(\sum_h N_h\bar{\boldsymbol{y}}_{h.}\bar{\boldsymbol{y}}'_{h.} - \text{SST})\boldsymbol{P}'$ |
| Residual | $N-s$ | $\text{SSR}^* \;=\; \boldsymbol{P}\ \text{SSR}\ \boldsymbol{P}' \;=\; \boldsymbol{P}(\text{SSY} - \text{SSG} - \text{SST})\boldsymbol{P}'$ |
| Total | $N = \sum N_h$ | $\text{SSY}^* \;=\; \boldsymbol{P}\ \text{SSY}\ \boldsymbol{P}' \;=\; \boldsymbol{P}(\sum_h \sum_i \boldsymbol{y}_{hi}\boldsymbol{y}'_{hi})\boldsymbol{P}'$ |

The results depend only on the following summary statistics:

- Cross-product matrix from the overall mean vector $\bar{\boldsymbol{y}}_{..}\bar{\boldsymbol{y}}'_{..}$
- Sum of cross-product matrices from the group mean vectors $\sum_h N_h\bar{\boldsymbol{y}}_{h.}\bar{\boldsymbol{y}}'_{h.}$
- sum of cross-product matrices from the subject data vectors $\sum_h \sum_i \boldsymbol{y}_{hi}\boldsymbol{y}'_{hi}$

For the case of orthogonal polynomial trend coefficients, we can partition the sum of squares and products, where the diagonal elements are:

$$
\begin{array}{lcl}
 & & \text{diag element} \\
\hline
\text{Time: } df = 1 & & \\
\mathrm{SST}^* = & \begin{bmatrix} \mathrm{sst}_0 & & & & \\ & \mathrm{sst}_1 & & & \\ & & \mathrm{sst}_2 & & \\ & & & \cdots & \\ & & & & \mathrm{sst}_{n-1} \end{bmatrix} & \begin{array}{l} \text{constant} \\ \text{linear time} \\ \text{quad time} \\ \cdots \\ (n-1)\text{th time} \end{array} \\
\hline
\text{Between groups: } df = s - 1 & & \\
\mathrm{SSG}^* = & \begin{bmatrix} \mathrm{ssg}_0 & & & & \\ & \mathrm{ssg}_1 & & & \\ & & \mathrm{ssg}_2 & & \\ & & & \cdots & \\ & & & & \mathrm{ssg}_{n-1} \end{bmatrix} & \begin{array}{l} \text{groups} \\ \text{grps} \times \text{lin} \\ \text{grps} \times \text{quad} \\ \cdots \\ \text{grps} \times (n-1) \end{array} \\
\hline
\text{Subjects within groups: } df = N - s & & \\
\mathrm{SSR}^* = & \begin{bmatrix} \mathrm{ssr}_0 & & & & \\ & \mathrm{ssr}_1 & & & \\ & & \mathrm{ssr}_2 & & \\ & & & \cdots & \\ & & & & \mathrm{ssr}_{n-1} \end{bmatrix} & \begin{array}{l} \text{subjects} \\ \text{s(g)} \times \text{lin} \\ \text{s(g)} \times \text{quad} \\ \cdots \\ \text{s(g)} \times (n-1) \end{array} \\
\hline
\end{array}
\tag{3.9}
$$

### 3.2.1 Extracting Univariate Results

Given the results of the previous section, it is a simple matter to extract the univariate ANOVA results, as shown in the following.

| | | | source |
|---|---|---|---|
| SS Time | = | $\text{sst}_1 + \text{sst}_2 + \ldots + \text{sst}_{n-1}$ | SST* |
| SS Group | = | $\text{ssg}_0$ | SSG* |
| SS GT | = | $\text{ssg}_1 + \text{ssg}_2 + \ldots + \text{ssg}_{n-1}$ | SSG* |
| SS Subj within grps | = | $\text{ssr}_0$ | SSR* |
| SS Residual | = | $\text{ssr}_1 + \text{ssr}_2 + \ldots + \text{ssr}_{n-1}$ | SSR* |

For the denominator we need:

- Time, G by T, & Subj: SS Residual$/(N-s)(n-1)$

  $\Rightarrow$ average of $n-1$ residual terms

- Group: $\text{ssr}_0 \,/(N-s)$ .

### 3.2.2 Multivariate Tests

For the multivariate model, the overall group test is the same as the univariate case. In contrast, each time-related term has its own denominator, *e.g.*, after the overall multivariate test of the Group by Time interaction, individual components are tested as:

$$
\begin{aligned}
F_{\mathrm{GT}_1} &= \frac{\mathrm{SSg}_1/(s-1)}{\mathrm{SSr}_1/(N-s)} \qquad \text{group by linear} \\
F_{\mathrm{GT}_2} &= \frac{\mathrm{SSg}_2/(s-1)}{\mathrm{SSr}_2/(N-s)} \qquad \text{group by quadratic} \\
&\ \ .\quad .\quad . \qquad\qquad . \\
F_{\mathrm{GT}_{n-1}} &= \frac{\mathrm{SSg}_{n-1}/(s-1)}{\mathrm{SSr}_{n-1}/(N-s)} \qquad \text{group by } (n-1)\text{th}
\end{aligned}
\tag{3.10}
$$

each on $s-1$ and $N-s$ degrees of freedom. The multivariate test of the Group by Time interaction is obtained by extracting the $(n-1) \times (n-1)$ submatrices of $\mathrm{SSG}^*$ and $\mathrm{SSR}^*$, and solving the 2-matrix eigen problem for $\min(s-1, n-1)$ roots as:

$$
|\ \mathrm{SSG}^*_{(n-1)} \ - \ \boldsymbol{\lambda}\ \mathrm{SSR}^*_{(n-1)}| \ = \ 0 \tag{3.11}
$$

Several test statistics are generally provided by standard statistical software for MANOVA: Roy's largest root statistic, Wilk's Lambda, Hotelling-Lawley Trace, and Pillai's Trace. Roy's largest root statistic is given by the first eigenvalue $\lambda_1$. Wilk's Lambda is computed as

$$\Lambda = \prod_{h=1}^{s-1} 1 \,/\, (1 + \lambda_h) \,. \tag{3.12}$$

These same tests can be constructed for $n - q - 1$ terms if only a $q < n$ degree trend is considered (*e.g.*, only test for linear and quadratic trends even if $n > 3$). If the test is nonsignificant, we may want to pool the Group $\times$ Time interaction into the Residual for (multivariate) testing of Time effects. Unfortunately, despite the increase in statistical power, pooling is not easily accomplished with most MANOVA statistical software.

### 3.2.3 General Framework

To consider a general framework for the multivariate model, let A and B be two between-subjects factors (*e.g.*, sex and group), and T be the within-subjects

factor (*e.g.*, time). The general setup is:

| within-subjects | between-subjects | | | |
|---|---|---|---|---|
| | I (1) | A $(a-1)$ | B $(b-1)$ | AB $(a-1)(b-1)$ |
| I (1) | grand mean | sex | group | sex $\times$ group |
| T $(n-1)$ | time | sex $\times$ time | group $\times$ time | sex $\times$ group $\times$ time |

Note that we may want to test time-related terms with less than $(n-1)$ df, especially with orthogonal polynomials, which are difficult to interpret beyond the cubic term. In other cases, we may have a design on the repeated mea-

surements. For example, consider the following two-period crossover study:

| period 1 | | period 2 | |
|---|---|---|---|
| drug | placebo | drug | placebo |
| drug | placebo | placebo | drug |
| placebo | drug | drug | placebo |
| placebo | drug | placebo | drug |

The MANOVA table has the following form.

| within-subjects | between-subjects I(1) | between-subjects Order $(c-1)$ |
|---|---|---|
| I (1) | grand mean | order |
| Group $(a-1)$ | group | order $\times$ group |
| Period $(b-1)$ | period | order $\times$ period |
| Group $\times$ Period $(a-1)(b-1)$ | grp $\times$ period | order $\times$ group $\times$ period |

with $n-1 = (a-1)+(b-1)+(a-1)(b-1)$, where $n$ is the total number

of time-points.

## 3.3 Illustration

To illustrate the multivariate approach, we return to the vocabulary growth data previously presented in Chapter 2 and in Bock (1975). Summary statistics (vocabulary score means, standard deviations, and correlations between the grades) are presented in Table 2.1, and are displayed graphically in Figure 2.1. Pre-multiplying the vector mean

$$\bar{\boldsymbol{y}} = \begin{bmatrix} 1.14 \\ 2.54 \\ 2.99 \\ 3.47 \end{bmatrix},$$

by the orthogonal polynomial matrix

$$\boldsymbol{P} = \begin{bmatrix} .5 & .5 & .5 & .5 \\ -.67082 & -.22361 & .22361 & .67082 \\ .5 & -.5 & -.5 & .5 \\ -.22361 & .67082 & -.67082 & .22361 \end{bmatrix},$$

yields following orthogonal polynomial coefficients.

| | |
|---|---|
| Constant | 5.0694 |
| Linear | 1.6658 |
| Quadratic | -0.4606 |
| Cubic | 0.2224 |

The polynomial SST* matrix, presented in Table 3.1, is obtained by multiplying the squares and cross products of these coefficients by the number of subjects ($N = 64$). Similarly, the residual SSR* matrix in Table 3.1 is obtained by the matrix calculation:

$$\text{SSR}^* = \boldsymbol{P}(\boldsymbol{Y}'\boldsymbol{Y} - N\bar{\boldsymbol{y}}_{.}\bar{\boldsymbol{y}}_{.}')\boldsymbol{P}' \ .$$

These are referred to as sums of squares and cross product (SSCP) matrices. In the present case, the SSR* matrix conforms to mixed-model assumptions in that the off-diagonal elements (of the lower three by three portion of this matrix) are small relative to their corresponding diagonal elements, and the last three diagonal elements are of the same magnitude. A test of this assumption (not shown) does not reject sphericity. However, for illustration, we will interpret the multivariate results for these data and compare them to the

univariate results presented in Chapter 2.

To obtain the multivariate test for the grade effect, the eigen problem

$$|\boldsymbol{E}^{-1}\ \mathrm{SST}^*_{(n-1)}(\boldsymbol{E}^{-1})'\ -\ \lambda \boldsymbol{I}_{(n-1)}|\ =\ 0\ , \tag{3.13}$$

where we use the Cholesky factorization $\mathrm{SSR}^*_{(n-1)}\ =\ \boldsymbol{E}\boldsymbol{E}'$, yields 4.7432 as Roy's largest root statistic (eigenvalue of $\lambda$). Similarly, Wilks' Lambda equals $1/(1+4.7432) = .17412$. It can be shown that these translate to a $F$-value of 96.45 (see Finn, 1974) with df = 3, 61, which is highly significant $p < .0001$. Thus, there is clearly a grade effect on average vocabulary scores.

In terms of the individual trend components, results of the analysis are quite similar to the univariate results presented in Chapter 2, though the cubic term is even less significant here. As in the univariate analysis, both the linear and quadratic trend components are highly significant. Notice that the multivariate analysis uses separate denominators for forming these $F$-values for the trend components (*e.g.*, Linear $F$ = 177.59 / (50.42/63) = 221.88), whereas the univariate analysis uses the pooled MSE (*e.g.*, Linear $F$ = 177.59 / (((50.42 + 43.95 + 60.57)/3)/63) = 216.57).

The univariate sums of squares in Tables 2.2 and 2.3, may be obtained from Table 3.1 as follows: (a) the constant term is the first diagonal element of $\mathrm{SST}^*$, and the linear, quadratic, and cubic polynomial grade sums of squares

are the remaining diagonal elements of SST*; (b) the subject sums of squares is the first diagonal element of SSR*, and (c) the residual sums of squares is the sum of the remaining three diagonal elements of SSR*.

Table 3.1

Multivariate Analysis of Variance of the

Orthogonal Polynomial Transformed Vocabulary-Growth Data

| Source | df | SSCP | | | | F | p |
|---|---|---|---|---|---|---|---|
| Time SST* | 1 | | | | | | |
| Constant | | 1644.71 | | | | | |
| Linear | | 540.45 | 177.59 | | | 221.88 | .0001 |
| Quadratic | | -149.45 | -49.11 | 13.58 | | 19.46 | .0001 |
| Cubic | | 72.16 | 23.71 | -6.56 | 3.17 | 3.29 | .075 |
| Residual SSR* | 63 | | | | | | |
| Between subjects | | 873.60 | | | | | |
| Linear error | | 3.84 | 50.42 | | | | |
| Quadratic error | | -49.82 | 12.05 | 43.95 | | | |
| Cubic error | | -23.76 | -3.36 | -4.27 | 60.57 | | |

## 3.4 Summary

In summary, ANOVA and MANOVA approaches for analysis of longitudinal data represent well understood and well developed statistical methodologies. In addition, there is considerable available computer software for their computation. The results are based on relatively simple and non-iterative calculations. Both models, unfortunately, have features which limit their usage in longitudinal data analysis. The ANOVA model for repeated measurements assumes sphericity, which is unrealistic for many applications where variances tend to increase with time and correlation decreases with increasing intervals in time. Alternatively, while the MANOVA model allows for a general variance-covariance structure for the repeated measures, it has the disadvantage of requiring complete data for all subjects and identical measurement occasions. Unfortunately, this overly stringent requirement is violated in most cases. Furthermore, software implementations of the multivariate model often provide only limited ways of handling covariates. In the following chapters, we consider more general models that overcome the limitations of these traditional approaches.