Conditional Symmetry Models for Three-Way Contingency Tables

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ABSTRACT

We generalize the conditional symmetry model for an $I \times I \times I$ contingency table with commensurable ordinal classification variables. The new family of models is constructed in such a way so that possesses the usual desirable properties that appear in the similar generalizations of symmetry and marginal homogeneity models, that is connections between the models above obey a coherent structure.

Key Words: Marginal homogeneity; ordinal variables; quasi-symmetry; symmetry.

1 Introduction

For square tables with commensurable classification variables the models of symmetry (S), quasi-symmetry (QS), marginal homogeneity (MH) and conditional (or triangular) symmetry (T) are classical and well known; see, for example, Agresti (2002), Goodman (1985), Bishop et al. (1975). The widespread applicability and physical interpretation of the S, MH and T models is straightforward and has been exhibited in all basic contingency tables textbooks. The QS model emanated mainly from mathematical arguments and its interpretation is harder, see the key reference McCullagh (1982).

While models S, QS and MH have been generalized to higher order contingency tables in Bhapkar (1979a, 1979b) and Bhapkar and Darroch (1990), such a generalization does not exist for the T-model. The only attempts towards this direction have been given by Read (1978) and Sobel (1988), where the conditional symmetry within levels model for $I \times I \times J$ tables was proposed.

This generalization is in fact the simple T model for two-way tables considered for each level of the third classification variable.

In this paper we propose a generalization to T models for three-way tables. This construction is achieved by retaining all desirable properties between models that connect the T models to the existing generalizations for the S, MH and QS models. This is rigorously shown through a series of mathematical derivations. Apart from mathematical completeness, these models have potential applications in a range of statistical applications including repeated measures and ordered categorical responses. This is demonstrated by analyzing a $4 \times 4 \times 4$ contingency table with real data.

The rest of the paper proceeds as follows. A brief review of the symmetry models for two-way tables, with particular emphasis given to the T model, and the existing generalizations of the symmetry models to higher order contingency tables is provided in Section 2. In Section 3 we introduce a new family of T models for three-way tables whereas in Section 4 the corresponding properties are extended and proved. Estimation and fit of the new models is discussed in Section 5 whereas a real data illustrative example is presented in Section 6. We conclude with a brief discussion in Section 7.

2 Preliminaries

Let π_{ij} be the cell probabilities of an $I \times I$ table with commensurable classification variables. Then, under the T model, these probabilities satisfy the relation

$$\pi_{ij} = \frac{\tau_1}{\tau_2}\pi_{ji}, \quad i < j, \quad \tau_1 + \tau_2 = 2, \; i,j = 1,\ldots,I \tag{2.1}$$

which render the physical interpretation of the model expressed as a constant proportionality between the symmetric cell probabilities. In terms of symmetry departure, the T model may equivalently be defined as

$$\pi_{ij} = \begin{cases} \tau_1 \pi_{ij}^S, & i < j \\ \tau_2 \pi_{ij}^S, & i > j \end{cases}, \quad i,j = 1,\ldots,I, \tag{2.2}$$

where π_{ij}^S denote the probabilities under the complete symmetry model S

$$\pi_{ij} = \pi_{ji}, \quad i \neq j, \quad i,j = 1,\ldots,I. \tag{2.3}$$

A basic property of the T model is that when it holds simultaneously with the marginal homogeneity model MH

$$\pi_{i\cdot} = \pi_{\cdot i}, \quad i = 1, \ldots, I, \tag{2.4}$$

then model S also holds. On the other hand, S implies T and MH, see Agresti (2002). In the sequel, we shall be denoting such properties as $S = T \wedge MH$.

Extensions of the S, QS and MH models to contingency tables of orders $c > 2$ can be best described by following the notation of Bhapkar and Darroch (1990). The complete symmetry model S_c is defined as the model under which the probabilities of all cells corresponding to permutated indexes are equal. Let P_c denote the set of all permutations $\rho(i_{(1)}, i_{(2)}, \ldots, i_{(c)})$ of the indexes $i_{(1)} \leq i_{(2)} \leq \ldots \leq i_{(c)}$ with $i_{(1)} < i_{(c)}$ and $i_{(j)} = 1, \ldots, I$ for $j = 1, \ldots, c$. Then model S_c is defined as

$$\pi_{\rho(i_{(1)}, i_{(2)}, \ldots, i_{(c)})} = \pi_{i_{(1)}, i_{(2)}, \ldots, i_{(c)}}, \quad \forall \rho(i_{(1)}, i_{(2)}, \ldots, i_{(c)}) \in \mathrm{P}_c. \tag{2.5}$$

Marginal homogeneity is extended to marginal homogeneity of order 1 up to $c-1$ according to the order of the marginals on which homogeneity applies. Thus, under $MH_c(1)$

$$\pi_1(i) = \pi_2(i) = \ldots = \pi_c(i) \ , \quad i = 1, \ldots, I, \tag{2.6}$$

where $\pi_r(i)$, $r = 1, \ldots, c$, are the one-dimensional marginal probability functions of the table, summing over all classification variables except the r-th. If $\pi_{rq}(i,j)$, $r, q = 1, \ldots, c$, are the two-dimensional marginal probability functions, by summing over all classification variables except the r-th and q-th, $MH_c(2)$ is defined by

$$\begin{aligned} \pi_{12}(i,j) &= \pi_{13}(i,j) = \ldots = \pi_{c-1.c}(i,j) \ , \\ \pi_{12}(i,j) &= \pi_{12}(j,i) \ . \end{aligned}$$

Marginal homogeneity models of higher order are defined accordingly.

3 Extension of Conditional Symmetry Model to Three-Dimensional Tables

Let π_{ijk} be the cell probabilities for $i, j, k = 1, \ldots, I$. The models defined in Section 2 apply in this case for $c = 3$. To simplify notation, we do not use the ordered indexes $i_{(1)}, i_{(2)}, i_{(3)}$ but, instead, the indexes $i, j, k = 1, \ldots, I$, with their ordering carefully specified. Following corresponding extensions of the MH and QS models we define models of first and second order.

We base our definitions on the cells with $i \leq j \leq k$ but not $i = j = k$. For these reference cells the subscripts i, j and k stand for the first, second and third subscripts respectively. The T-asymmetry model of second order, denoted as $T_3(2)$, is defined as

$$\pi_{\rho(ijk)} = \tau^r_{m(1)}\tau^r_{m(2)}\pi_{ijk}, \ i \leq j \leq k \text{ but not } i = j = k, \tag{3.1}$$

with $\rho(ijk)$ denoting any permutation of the indexes i, j and k for $i, j, k = 1, ..., I$, and

$$r = \begin{cases} 0, & i \neq j \neq k \\ 1, & i = j \text{ or } j = k \text{ or } i = k \end{cases},$$

$$m(1) = \begin{cases} 0, & \text{there occurs no violence of the other } i \leq j \leq k \text{ for adjacent subscripts} \\ 1, & \text{violence of the order of the } 1^{st} \text{ pair of adjacent subscripts } (i, j) \\ 2, & \text{violence of the order of the } 2^{d} \text{ pair of adjacent subscripts } (j, k) \end{cases},$$

$$m(2) = \begin{cases} 0, & \text{no violence of the order of } \textit{not} \text{ adjacent subscripts} \\ 12, & \text{violence of of the order of } \textit{not} \text{ adjacent subscripts } (i, k) \end{cases}.$$

The $T_3(2)$ asymmetry parameters are $\tau^0_1, \tau^0_2, \tau^0_{12}, \tau^1_1, \tau^1_2, \tau^1_{12}$ while $\tau^0_0 = \tau^1_0 = 1$. Analytically, considering all possible permutations, and taking care the first of the most distant violence of the subscripts order, model (3.1) expands to

$$\left.\begin{array}{l} \pi_{jik} = \tau^0_1\pi_{ijk}, \ i < j < k \\ \pi_{ikj} = \tau^0_2\pi_{ijk}, \ i < j < k \\ \pi_{kji} = \tau^0_{12}\pi_{ijk}, \ i < j < k \\ \pi_{kij} = \tau^0_{12}\pi_{jik} = \tau^0_1\tau^0_{12}\pi_{ijk}, \ i < j < k \\ \pi_{jki} = \tau^0_{12}\pi_{ikj} = \tau^0_2\tau^0_{12}\pi_{ijk}, \ i < j < k \\ \pi_{jii} = \tau^1_{12}\pi_{iij}, \ i < j \\ \pi_{iji} = \tau^1_2\pi_{iij}, \ i < j \\ \pi_{jji} = \tau^1_{12}\pi_{ijj}, \ i < j \\ \pi_{jij} = \tau^1_1\pi_{ijj}, \ i < j \end{array}\right\}. \tag{3.2}$$

Definition (3.2) is analogous to (2.1). An equivalent expression for model $T_3(2)$ which corresponds to definition (2.2) and visualizes the nature of the model (departure from the complete symmetry model) is given by

$$\pi_{ijk} = \begin{cases} \frac{6\pi^S_{ijk}}{(\tau_1^0+\tau_2^0+1)(\tau_{12}^0+1)} \begin{cases} 1,\ i<j<k \\ \tau_1^0,\ j<i<k \\ \tau_2^0,\ i<k<j \\ \tau_{12}^0,\ k<j<i \\ \tau_1^0\tau_{12}^0,\ j<k<i \\ \tau_2^0\tau_{12}^0,\ k<i<j \end{cases} \\ \frac{3\pi^S_{ijk}}{\tau_2^1+\tau_{12}^1+1} \begin{cases} 1,\ i=j<k \\ \tau_2^1,\ i=k<j \\ \tau_{12}^1,\ j=k<i \end{cases} \\ \frac{3\pi^S_{ijk}}{\tau_1^1+\tau_{12}^1+1} \begin{cases} 1,\ i<j=k \\ \tau_1^1,\ j<i=k \\ \tau_{12}^1,\ k<i=j \end{cases} \end{cases}, \tag{3.3}$$

where, as it is evident, subscripts i, j, k correspond to the first, second and third classification variable irrespectively of their magnitude, and π^S_{ijk} is the probability of cell (i, j, k) under the model of complete symmetry. In the following we shall be using definition (3.3) and not (3.1) or (3.2).

If we impose the additional restriction

$$\tau_{12}^r = \tau_1^r\tau_2^r,\ r = 0, 1 \tag{3.4}$$

in any of the expressions of the $T_3(2)$ model we obtain the T-asymmetry model of first order, denoted as $T_3(1)$.

Two more models that we introduce at this stage are the $T_3(1)$ and the $T_3(2)$ homogeneous models denoted by $T_3(1) - h$ and $T_3(2) - h$ respectively. The $T_3(2) - h$ is the $T_3(2)$ model with the additional restriction

$$\tau^1_{m(i)} = \tau^0_{m(i)},\ i = 1, 2 \tag{3.5}$$

while if we impose on $T_3(1)$ the restriction

$$\tau^1_{m(1)} = \tau^0_{m(1)} \tag{3.6}$$

it reduces to the $T_3(1) - h$ model.

The practical implications of the models we propose become evident if we consider a repeated measurement scenario in which individuals produce 3 responses across time for a factor with I levels. If the τ-parameters are taken as $(\tau_1^0, \tau_2^0, \tau_{12}^0)$ the individual responses belong to different categories in every time point; if they are taken as $(\tau_1^1, \tau_2^1, \tau_{12}^1)$ the individual responses stay once, or return once in the same category. To distinguish whether one stays at first $(i_1 = i_2)$ or second place $(i_2 = i_3)$, or returns in the initial category $(i_1 = i_3)$, we can further generalize the model by introducing different τ-parameters for each of the three cases above.

In the context of $I \times I \times I$ tables with commensurable ordinal classification variables, the T symmetry within levels model proposed in Read (1978) is specified by

$$\pi_{ijk} = \frac{\tau_1^k}{\tau_2^k}\pi_{jik},\ i<j,\ \tau_1^k + \tau_2^k = 2,\ i,j,k = 1,\ldots,I \tag{3.7}$$

and it can be simplified by considering the proportionality to be homogeneous across the levels of the third variable, that is $\tau_i^k = \tau_i$ for $i = 1, 2$ and $k = 1, \ldots, I$. Essentially, the T model is defined three times corresponding to all possible combination of variable pairs, denoted by T_{12}, T_{23} and T_{13}. The precise definitions of these models are

$$T_{12}: \quad \pi_{ijk} = t_{12}\pi_{jik},\ i<j,\ i,j = 1,\ldots,I \tag{3.8}$$

$$T_{23}: \quad \pi_{ijk} = t_{23}\pi_{ikj},\ j<k,\ j,k = 1,\ldots,I \tag{3.9}$$

$$T_{13}: \quad \pi_{ijk} = t_{13}\pi_{kji},\ i<k,\ i,k = 1,\ldots,I. \tag{3.10}$$

Generalization of the T models to c-way tables with $c > 3$ is possible though complicated. There will be sets of τ^r - parameters, $r = 0, 1, ..., c-2$ and the subscripts will take the form $\tau_1^r, ..., \tau_{c-1}^r$, τ_{ij}^r, $i, j = 1, ..., c-1$, $\tau_{ijl}^r, i, j, l = 1, ..., c-1$, and so on up to τ parameters with $c-1$ subscripts.

4 Relations to Other Models

In correspondence to the properties of the simple T model of two-way tables and in analogy to the properties of the QS_3(k) and MH_3(k), $k = 1, 2$ models of three-way tables, we state the following properties for the newly introduced T models.

***Proposition*4.1 :** $T_3(2) \Rightarrow T_3(1)$

Proof: The fact that $T_3(2)$ implies $T_3(1)$ and $T_3(2) - h$ implies $T(_31) - h$ is obvious from the corresponding definitions.□

Proposition4.2 : $MH_3(k) \wedge T_3(k) = S_3$, $k = 1, 2$

Proof:

- $k = 2 : MH_3(2) \wedge T_3(2) = S_3$:

  The direction $S_3 \Rightarrow MH_3(2)$ and $T_3(2)$ is obvious since

$$S_3 \Rightarrow \begin{cases} T_3(2) & \text{with } \tau_1^r = \tau_2^r = \tau_{12}^r = 1,\ r = 0, 1 \\ MH_3(2) & \text{(cf. Bhapkar and Darroch, 1990)} \end{cases}$$

  Thus, we need to prove that $MH_3(2) \wedge T_3(2) \Rightarrow S_3$. The $MH_3(2)$ model is defined by

$$\pi_{ij.} = \pi_{ji.},\ i, j = 1, ..., I \tag{4.1}$$

$$\pi_{ij.} = \pi_{i.j} = \pi_{.ij},\ i, j = 1, ..., I \tag{4.2}$$

  Considering (4.2) for the special case $i = j = 1$ and $i = j = I$ and expressing the corresponding probabilities through (3.3), since $T_3(2)$ also holds, we obtain

$$\left.\begin{array}{l}
\left.\begin{array}{c} \pi_{11.} = \pi_{1.1} \Rightarrow \sum\limits_{k=2}^{I} \frac{3\pi^S_{11k}}{1+\tau_2^1+\tau_{12}^1} = \sum\limits_{k=2}^{I} \frac{3\tau_2^1\pi^S_{1k1}}{1+\tau_2^1+\tau_{12}^1} \\ \text{complete symmetry: } \pi^S_{11k} = \pi^S_{1k1},\ k = 2, \ldots, I \end{array}\right\} \Rightarrow \tau_2^1 = 1 \\[2ex]
\left.\begin{array}{c} \pi_{11.} = \pi_{.11} \Rightarrow \sum\limits_{k=2}^{I} \frac{3\pi^S_{11k}}{1+\tau_2^1+\tau_{12}^1} = \sum\limits_{k=2}^{I} \frac{3\tau_{12}^1\pi^S_{k11}}{1+\tau_2^1+\tau_{12}^1} \\ \text{complete symmetry: } \pi^S_{11k} = \pi^S_{k11},\ k = 2, \ldots, I \end{array}\right\} \Rightarrow \tau_{12}^1 = 1 \\[2ex]
\left.\begin{array}{c} \pi_{II.} = \pi_{I.I} \Rightarrow \sum\limits_{k=1}^{I-1} \frac{3\tau_{12}^1\pi^S_{IIk}}{1+\tau_1^1+\tau_{12}^1} = \sum\limits_{k=1}^{I-1} \frac{3\tau_1^1\pi^S_{IkI}}{1+\tau_1^1+\tau_{12}^1} \\ \text{complete symmetry: } \pi^S_{IIk} = \pi^S_{IkI},\ k = 1, ..., I-1 \end{array}\right\} \Rightarrow \tau_1^1 = \tau_{12}^1
\end{array}\right\} \Rightarrow \tau_1^1 = \tau_2^1 = \tau_{12}^1 = 1$$

  Further on, since $\tau_1^1 = \tau_2^1 = \tau_{12}^1 = 1$,

$$\begin{aligned}
\pi_{12.} &= \pi_{21.} \Rightarrow \pi^S_{121} + \pi^S_{122} + \sum_{k \geq 3} \pi_{12k} = \pi^S_{211} + \pi^S_{212} + \sum_{k \geq 3} \pi_{21k} \\
&\Rightarrow \frac{6}{(\tau_1^0 + \tau_2^0 + 1)(\tau_{12}^0 + 1)} \sum_{k \geq 3} \pi^S_{12k} = \frac{6\tau_1^0}{(\tau_1^0 + \tau_2^0 + 1)(\tau_{12}^0 + 1)} \sum_{k \geq 3} \pi^S_{21k} \\
&\Rightarrow \tau_1^0 = 1
\end{aligned}$$

$$\pi_{12.} = \pi_{1.2} \Rightarrow \frac{6}{(\tau_1^0+\tau_2^0+1)(\tau_{12}^0+1)}\sum_{k\geq 3}\pi_{12k}^S = \frac{6\tau_2^0}{(\tau_1^0+\tau_2^0+1)(\tau_{12}^0+1)}\sum_{k\geq 3}\pi_{1k2}^S$$
$$\Rightarrow \tau_2^0 = 1$$

$$\pi_{12.} = \pi_{.21} \Rightarrow \frac{6}{(\tau_1^0+\tau_2^0+1)(\tau_{12}^0+1)}\sum_{k\geq 3}\pi_{12k}^S = \frac{6\tau_{12}^0}{(\tau_1^0+\tau_2^0+1)(\tau_{12}^0+1)}\sum_{k\geq 3}\pi_{1k2}^S$$
$$\Rightarrow \tau_{12}^0 = 1$$

Hence, since $\tau_1^r = \tau_2^r = \tau_{12}^r = 1$, $r = 0, 1$, model $T_3(2)$ reduces to the complete symmetry model

- $k = 1$: $MH_3(1) \wedge T_3(1) = S_3$

The direction $S_3 \Rightarrow MH_3(1)$ and $T_3(1)$ is obvious since

$$S_3 \Rightarrow \begin{cases} T_3(1) & \text{with } \tau_1^r = \tau_2^r = 1,\ r = 0,1 \\ MH_3(1) & \text{-(cf.Bhapkar and Darroch, 1990)} \end{cases}$$

We will prove that $MH_3(1) \wedge T_3(1) \Rightarrow S_3$. The $MH_3(1)$ model is defined by

$$\pi_{i..} = \pi_{.i.} = \pi_{..i},\ i = 1, ..., I \tag{4.3}$$

Considering (4.3) for $i = 1$, we have

$$\begin{aligned}
\pi_{1..} &= \pi_{.1.} \Rightarrow \sum_{j=2}^{I}\pi_{1jj} + \sum_{j=2}^{I}\pi_{1j1} + \sum_{j=2}^{I}\pi_{11j} + \sum_{1<j<k}\pi_{1jk} + \sum_{1<k<j}\pi_{1jk} + \pi_{111} \\
&= \sum_{j=2}^{I}\pi_{j1j} + \sum_{j=2}^{I}\pi_{j11} + \sum_{j=2}^{I}\pi_{11j} + \sum_{1<j<k}\pi_{j1k} + \sum_{1<k<j}\pi_{j1k} + \pi_{111} \Rightarrow \text{due to (3.3)} \\
& \frac{3}{\tau_1^1+\tau_1^1\tau_2^1+1}\sum_{j=2}^{I}\pi_{1jj}^S + \frac{3\tau_2^1}{\tau_2^1+\tau_1^1\tau_2^1+1}\sum_{j=2}^{I}\pi_{1j1}^S + \frac{6\sum_{j\neq k\neq 1}\pi_{1jk}^S(1+\tau_2^0)}{(\tau_1^0+\tau_2^0+1)(\tau_1^0\tau_2^0+1)} \\
&= \frac{3\tau_1^1}{\tau_1^1+\tau_1^1\tau_2^1+1}\sum_{j=2}^{I}\pi_{1jj}^S + \frac{3\tau_1^1\tau_2^1}{\tau_2^1+\tau_1^1\tau_2^1+1}\sum_{j=2}^{I}\pi_{1j1}^S + \frac{6\sum_{j\neq k\neq 1}\pi_{1jk}^S(\tau_1^0+\tau_1^0(\tau_1^0\tau_2^0))}{(\tau_1^0+\tau_2^0+1)(\tau_1^0\tau_2^0+1)} \\
&\Rightarrow (1-\tau_1^1)\left\{\frac{\sum_{j=2}^{I}\pi_{1jj}^S}{\tau_1^1+\tau_1^1\tau_2^1+1} + \frac{\tau_2^1\sum_{j=2}^{I}\pi_{1j1}^S}{\tau_2^1+\tau_1^1\tau_2^1+1}\right\}
\end{aligned}$$

$$+\frac{2[1+\tau_2^0-\tau_1^0(1+\tau_1^0\tau_2^0)]}{(\tau_1^0+\tau_2^0+1)(\tau_1^0\tau_2^0+1)}\sum_{j\neq k\neq 1}\pi_{1jk}^S=0 \quad (4.4)$$

$$\begin{aligned}
\pi_{.1.} &= \pi_{..1} \Rightarrow \sum_{j=2}^{I}\pi_{j1j}+\sum_{j=2}^{I}\pi_{j11}+\sum_{j=2}^{I}\pi_{11j}+\sum_{1<i<k}\pi_{i1k}+\sum_{1<k<i}\pi_{i1k}+\pi_{111} \\
&= \sum_{j=2}^{I}\pi_{jj1}+\sum_{j=2}^{I}\pi_{j11}+\sum_{j=2}^{I}\pi_{1j1}+\sum_{1<i<j}\pi_{ij1}+\sum_{1<j<i}\pi_{ij1}+\pi_{111} \Rightarrow \\
&\frac{3\tau_1^1}{\tau_1^1+\tau_1^1\tau_2^1+1}\sum_{j=2}^{I}\pi_{1jj}^S+\frac{3\tau_2^1}{\tau_2^1+\tau_1^1\tau_2^1+1}\sum_{j=2}^{I}\pi_{11j}^S+\frac{6\sum_{j\neq i\neq 1}\pi_{1jk}^S(\tau_1^0+\tau_1^0(\tau_1^0\tau_2^0))}{(\tau_1^0+\tau_2^0+1)(\tau_1^0\tau_2^0+1)} \\
&= \frac{3\tau_1^1\tau_2^1}{\tau_1^1+\tau_1^1\tau_2^1+1}\sum_{j=2}^{I}\pi_{1jj}^S+\frac{3}{\tau_2^1+\tau_1^1\tau_2^1+1}\sum_{j=2}^{I}\pi_{1j1}^S+\frac{6\sum_{j\neq k\neq 1}\pi_{1jk}^S(\tau_1^0\tau_2^0(1+\tau_2^0))}{(\tau_1^0+\tau_2^0+1)(\tau_1^0\tau_2^0+1)} \\
&\Rightarrow (1-\tau_2^1)\left\{\frac{\tau_1^1\sum_{j=2}^{I}\pi_{1jj}^S}{\tau_1^1+\tau_1^1\tau_2^1+1}+\frac{\sum_{j=2}^{I}\pi_{1j1}^S}{\tau_2^1+\tau_1^1\tau_2^1+1}\right\} \\
&\quad +\frac{2[\tau_1^0+\tau_1^0(\tau_1^0\tau_2^0)-\tau_1^0\tau_2^0(1+\tau_2^0)]}{(\tau_1^0+\tau_2^0+1)(\tau_1^0\tau_2^0+1)}\sum_{j\neq k\neq 1}\pi_{1jk}^S=0 \quad (4.5)
\end{aligned}$$

Expression (4.4) is equivalent to

$$\begin{aligned}
&(1-\tau_1^1)\left\{\frac{\sum_{j=2}^{I}\pi_{1jj}^S}{\tau_1^1+\tau_1^1\tau_2^1+1}+\frac{\tau_2^1\sum_{j=2}^{I}\pi_{1j1}^S}{\tau_2^1+\tau_1^1\tau_2^1+1}\right\} \\
&+(1-\tau_1^0)\frac{2[1+\tau_2^0(1+\tau_1^0)]\sum_{j\neq k\neq 1}\pi_{1jk}^S}{(\tau_1^0+\tau_2^0+1)(\tau_1^0\tau_2^0+1)}=0 \quad (4.6)
\end{aligned}$$

This expression is of the form

$$(1-\tau_1^1)\{+\}+(1-\tau_1^0)\{+\}=0 \quad (4.7)$$

and holds for any $(I\times I\times I)$ table when both the $MH_3(1)$ and $T_3(1)$ models do hold. From the definition of the $T_3(1)$ model [through (3.3) and (3.4)] it is clear that there is no relation between the group of cells with $i\neq j\neq k$ and the one with $i=j$ or $j=k$ or $i=k$. Note for example that under $T_3(1)$ [as well as under $T_3(2)$]:

$$\begin{aligned}
\sum_{\rho}\pi_{\rho(ijk)} &= 6\pi_{ijk}^S,\ i\neq j\neq k \\
\sum_{\rho}\pi_{\rho(ijk)} &= 3\pi_{ijk}^S,\ i=j<k \text{ or } i=k<j \text{ or } j=k<i \\
\sum_{\rho}\pi_{\rho(ijk)} &= 3\pi_{ijk}^S,\ i=j>k \text{ or } i=k>j \text{ or } j=k>i
\end{aligned}$$

where ρ represents all possible permutations under the restrictions imposed on the indexes i, j and k.

Hence, since the fact that $\tau_1^1 < 1$ does not imply in general that $\tau_1^0 > 1$ we have to accept (due to (4.7)) that

$$\tau_1^0 = \tau_1^1 = 1.$$

Similarly,

$$\begin{aligned}
\pi_{.I.} &= \pi_{..I} \Rightarrow \sum_{i<k\neq I} \pi_{iIk} + \sum_{i>k\neq I} \pi_{iIk} + \sum_{i=1}^{I-1} \pi_{iIi} + \sum_{i=1}^{I-1} \pi_{iII} + \sum_{i=1}^{I-1} \pi_{IIi} + \pi_{III} = \\
& \sum_{i<k\neq I} \pi_{ikI} + \sum_{i>k\neq I} \pi_{ikI} + \sum_{i=1}^{I-1} \pi_{iiI} + \sum_{i=1}^{I-1} \pi_{iII} + \sum_{i=1}^{I-1} \pi_{IiI} + \pi_{III} \\
&\Rightarrow \frac{6\tau_2^0(1+\tau_1^0\tau_2^0)\sum_{i<k} \pi_{ikI}^S}{(\tau_1^0+\tau_2^0+1)(\tau_1^0\tau_2^0+1)} + \frac{3\tau_2^1}{\tau_2^1+\tau_1^1\tau_2^1+1}\sum \pi_{iIi}^S + \frac{3\tau_1^1\tau_2^1}{\tau_1^1+\tau_1^1\tau_2^1+1}\sum \pi_{IIi}^S = \\
& \frac{6(1+\tau_1^0)\sum_{i<k} \pi_{ikI}^S}{(\tau_1^0+\tau_2^0+1)(\tau_1^0\tau_2^0+1)} + \frac{3}{\tau_2^1+\tau_1^1\tau_2^1+1}\sum \pi_{iiI}^S + \frac{3\tau_1^1}{\tau_1^1+\tau_1^1\tau_2^1+1}\sum \pi_{IIi}^S \\
&\Rightarrow (\tau_2^0-1)\frac{2[1+\tau_1^0(\tau_2^0+1)]\sum_{i<k} \pi_{ikI}^S}{(\tau_1^0+\tau_2^0+1)(\tau_1^0\tau_2^0+1)} \\
& \qquad + (\tau_2^1-1)\left\{\frac{\sum \pi_{iIi}^S}{\tau_2^1+\tau_1^1\tau_2^1+1} + \frac{\tau_1^1\sum \pi_{IIi}^S}{\tau_1^1+\tau_1^1\tau_2^1+1}\right\} = 0. \qquad (4.8)
\end{aligned}$$

With the same reasoning as in (4.6) we have that

$$(4.8) \Rightarrow \tau_2^0 = \tau_2^1 = 1.$$

Hence, since $\tau_2^r = \tau_2^r = 1$, $r = 0, 1$, the $T_3(1)$ model reduces to that of complete symmetry S_3. $\square$

***Proposition*4.3 :** $\quad MH_3(k) \wedge T_3(k) - h = S_3,\ k = 1, 2$

Proof:

- $k = 2$: $MH_3(2) \wedge T_3(2) - h = S_3$

  Since

$$S_3 \Rightarrow \begin{cases} T_3(2) - h & \text{with } \tau_1 = \tau_2 = 1 \\ MH_3(2) \end{cases}$$

we have to prove only the opposite. Assume that $MH_3(2)$ and $T_3(2)-h$ do hold. Then they will be true also for the special cases $\pi_{11.}=\pi_{1.1}$,$\pi_{11.}=\pi_{.11}$, and $\pi_{II.}=\pi_{I.I}$, which after some algebra result to $\tau_2=1$, $\tau_{12}=1$ and $\tau_1=1$ respectively. Hence, since $\tau_1=\tau_2=\tau_{12}=1$, the S_3 model holds.

- $k=1$: $MH_3(1)\wedge T_3(1)-h=S_3$

  Expression (4.6) is simplified to

$$(1-\tau_1)\left\{\frac{\sum\pi^S_{1jj}}{\tau_1+\tau_1\tau_2+1}+\frac{\tau_2\sum\pi^S_{1j1}}{\tau_2+\tau_1\tau_2+1}+\frac{2[1+\tau_2(1+\tau_1)]\sum\limits_{j\neq k}\pi^S_{1jk}}{(\tau_1+\tau_2+1)(\tau_1\tau_2+1)}\right\}=0$$

  which implies $\tau_1=1$ since the argument in the brackets is always positive. Analogously (4.8) is reduced to

$$(\tau_2-1)\left\{\frac{\sum\pi^S_{iIi}}{\tau_2+\tau_1\tau_2+1}+\frac{\tau_1\sum\pi^S_{IIi}}{\tau_1+\tau_1\tau_2+1}+\frac{2[1+\tau_1(1+\tau_2)]\sum\limits_{i\neq k}\pi^S_{ikI}}{(\tau_1+\tau_2+1)(\tau_1\tau_2+1)}\right\}=0$$

  which implies $\tau_2=1$ with the same reasoning as above.

  Hence $T_3(1)-h\wedge MH_3(1)\Rightarrow S_3$. The opposite is obvious and the proof is completed. □

***Proposition*4.4 :** $T_{12}\wedge T_{23}\wedge T_{13}=S_3$

Proof: If T_{12} and T_{23} hold simultaneously then after some algebraic manipulations it can be proved that $t_{12}=t_{23}$ and analogously $T_{12}\cap T_{13}$ implies $t_{13}=t_{12}^{-1}$ and $T_{23}\cap T_{13}$ implies $t_{13}=t_{23}^{-1}$, i.e. $t_{12}=t_{23}=t_{13}^{-1}$. Further on, similar calculations lead to $t_{12}=t_{23}=t_{13}^{-1}=1$ which implies S_3.

The opposite is immediate since S_3 implies T_{12},T_{23} and T_{13} with $t_{12}=t_{23}=t_{13}^{-1}=1$. □

***Proposition*4.5 :** $MH_3(2)\wedge T_{ij}=S_3$, $i=1,2,\ j=2,3,\ i\neq j$.

Proof: Since S_3 implies $MH_3(2)$ and T_{ij}, $i=1,2$, $j=2,3$, $i\neq j$, we only need to prove the opposite. Let $MH_3(2)$ and T_{12} hold. By defining T_{12} in terms of departure from symmetry

$$\pi_{ijk}=\begin{cases}c_{12}\pi^S_{ijk}, & i<j\\ (2-c_{12})\pi^S_{ijk}, & i>j\end{cases},\quad i,j,k=1,\ldots,I,$$

it can be proved that $c_{12}=1$ which implies $t_{12}=1$ since $t_{12}=c_{12}/(2-c_{12})$. Therefore S_3 holds. The proof for T_{23} and T_{13} is similar. □

5 Maximum Likelihood Estimation and Model Fit

Our objective is to determine the maximum likelihood estimates (MLEs) of the parameters of the conditional symmetry models $T_3(1)$ and $T_3(2)$ based on an $I \times I \times I$ contingency table of sampling proportions p_{ijk}. The corresponding log-likelihood kernel is

$$L = -\sum_{i,j,k} p_{ijk} log(\pi_{ijk}) \tag{5.1}$$

where the probabilities π_{ijk} are expressed by (3.3) for the $T_3(2)$ model and by (3.3) with the additional constraint (3.4) for $T_3(1)$. The MLEs of the probabilities under complete symmetry π^S_{ijk} are

$$\hat{\pi}^S_{ijk} = \frac{1}{6}\sum_{\rho} \{\mathcal{I}(i \neq j \neq k) + 2 \cdot \mathcal{I}(i = j \vee j = k \vee i = k) \cdot [1 - \mathcal{I}(i = j = k)]\} \cdot p_{\rho(i,j,k)} \ , \tag{5.2}$$

where $\mathcal{I}(\cdot)$ denotes the indicator function and the sum is over all possible permutations. By substituting (5.2) in (5.1), taking derivatives with respect to the τ parameters of the model under consideration and by equating them to zero, we readily obtain the corresponding likelihood equations. For $T_3(2)$ they yield closed form expressions for the MLEs while for $T_3(1)$, $T_3(1) - h$ and $T_3(2) - h$ they have to be solved numerically.

The goodness of fit of the $T_3(1)$ and $T_3(2)$ models can be tested by a large sample chi-square test. The degrees of freedom for $T_3(1)$ and $T_3(2)$ are $df_{T1} = df_S - 4$ and $df_{T2} = df_S - 6$, respectively, where $df_S = \frac{I(I-1)(5I+2)}{6}$ are the degrees of freedom of the complete symmetry model S_3.

An alternative way to fit the symmetry models is via the generalized linear model approach by defining each time the appropriate design matrix. Let

$$d_q = \begin{cases} 1, & i = j = k \\ 0, & \text{else} \end{cases} \quad , \quad q = 1, , \dots, I,$$

be the vectors indicating the I main diagonal entries of the table and define the vectors indicating the entries of the cube contingency table corresponding to permutations of a choice of indexes as

$$c_{i_{(1)}i_{(2)}i_{(3)}} = \begin{cases} 1, & (i,j,k) \in P_3 \\ 0, & \text{else} \end{cases} \quad , \ i_{(1)}i_{(2)}i_{(3)} \in C_S \ ,$$

with C_S denoting the set of indexes satisfying $i_{(1)} \leq i_{(2)} \leq i_{(3)}$ in $\{1, \dots, I\}$, $i_{(1)} < i_{(3)}$, and P_3 the set of all 6 permutations of the indexes $i_{(1)}$, $i_{(2)}$, $i_{(3)}$. Then the complete symmetry model is defined as

$$S_3 = \sum_{q=1}^{I} d_q + \sum_{C_S} c_{i_{(1)}i_{(2)}i_{(3)}} \ .$$

Similarly, the $T_3(2)$ model is defined by the terms of the S_3 model, with some additional parameters corresponding to τ. Thus, for

$$t_{0.w} = \begin{cases} 1, & (i > j)\mathcal{I}(w=1) + (j > k)\mathcal{I}(w=2) + (i > k)\mathcal{I}(w=12), \quad i \neq j \neq k \\ 0, & \text{else} \end{cases} , \quad w = 1, 2, 12,$$

$$t_{1.w} = \begin{cases} 1, & j(i > j)\mathcal{I}(w=1) + (j > k)\mathcal{I}(w=2), \quad i = k \\ 0, & \text{else} \end{cases} , \quad w = 1, 2,$$

$$t_{1.12} = \begin{cases} 1, & i > k, \quad i = j \text{ or } j = k \\ 0, & \text{else} \end{cases} ,$$

the $T_3(2)$ model is defined as

$$T_3(2) = S_3 + \sum_{q=0}^{1} \left[\sum_{w=1}^{2} t_{q.w} + t_{q.12} \right]$$

with 6 parameters more that of S_3. Setting

$$t^c_{q.12} = t_{q.1} + t_{q.2}, \quad q = 0, 1$$

the $T_3(1)$ model is defined as

$$T_3(1) = S_3 + \sum_{q=0}^{1} \left[\sum_{w=1}^{2} t_{q.w} + t^c_{q.12} \right] ,$$

and has 4 parameters more than S_3, since the $t^c_{q.12}$ are redundant. Furthermore, through this procedure, the homogeneous models are easily obtained by just summing the sets of vectors that correspond to the homogeneous parameters.

Furthermore, estimation and goodness-of-fit testing of the models introduced here can be based on the ϕ-divergence, extending the work of Menendez et al. (2005) for two-way tables. For three-way tables goodness-of fit tests based on *phi*-divergence have been developed so far only for the complete symmetry model (Menendez et al., 2004). In two-dimensional contingency tables, the test of marginal homogeneity conditional on quasi-symmetry is well-known. Menendez et al. (2007) developed this conditional test based on the ϕ-divergence test statistic. For three-way tables, marginal homogeneity of 1st or 2nd order can be tested conditional on the $T_3(1)$ or $T_3(2)$ models, respectively, based on Proposition 4.2, by formulating the corresponding ϕ-divergence test statistics.

6 Illustration

We apply the conditional symmetry models on a classical $4 \times 4 \times 4$ contigency table data that cross-classifies the diagnoses of carcinoma in situ of the uterine cervix by three pathologists on a 4-items scale; see Table 1. This data set was first introduced by Landis and Koch (1977) and analyzed in the context of symmetry models by Lovison (2000). Symmetry models were applied on this data set and their fit is summarized in Table 2. All models were fitted by glm() function in R through the approach described in Section 5.

Table 1: Diagnoses of carcinoma in situ of the uterine cervix by three pathologists (N = Negative, ASH = Atypical Squamous Hyperplasia, CiS = Carcinoma in Situ, S+I = Squamous Carcinoma with early stromal invasion or Invasive Carcinoma). *Source: Landis & Koch, 1977*

		Pathologist E			
Path. A	**Path. C**	N	ASH	CiS	S+I
N	N	11	8	0	0
	ASH	2	5	0	0
	CiS	0	0	0	0
	S+I	0	0	0	0
ASH	N	1	6	2	0
	ASH	2	6	7	2
	CiS	0	0	0	0
	S+I	0	0	0	0
CiS	N	0	0	0	0
	ASH	0	4	13	1
	CiS	0	0	17	3
	S+I	0	0	0	0
S+I	N	0	1	1	1
	ASH	0	0	0	0
	CiS	0	0	11	6
	S+I	0	0	3	5

Lovison(2000) proposed the $QS_3(1)$ model and concluded that the three pathologists have significantly different perceptions of the signals sent out by the slides. However, he did not describe the way of differentiation. We propose a more parsimonious model that gives insight into the pathologists' differentiation. In particular, we suggest the $T_3^r(2) - h$ model which is the

Table 2: Goodness-of-fit of the models applied on the data of Table 1.

Model	G^2	df	p-value
S_3	97.055	44	0.0000
$QS_3(1)$	16.850	38	0.9988
$T_3(1)$	79.719	40	0.0002
$T_3(2)$	45.050	38	0.2008
$T_3(1)-h$	80.805	42	0.0003
$T_3(2)-h$	47.281	41	0.2315
$T_3^r(2)-h$	48.108	42	0.2393

$T_3(2)-h$ with the additional restriction $\tau_2^0 = \tau_2^1 = 1$. From (3.2) it is implied that, for all $i < j$, $P_C(A = i, E = j) = P_C(A = j, E = i)$, where $P_C(A = i, E = j)$ is the probability pathologists A and E evaluated a slide as i and j respectively, conditional on the evaluation of pathologist C. This model implies that conditional on the evaluation of pathologist C, A and E exhibit symmetric evaluations.

The maximum likelihood estimates of the parameters of the $T_3^r(2)-h$ model are $\hat{\tau}_1^0 = \hat{\tau}_1^1 = 0.168$ and $\hat{\tau}_{12}^0 = \hat{\tau}_{12}^1 = 0.141$. For $i < j$, (3.2) leads to $P_A(E = i, C = j) = 0.168 \cdot P_A(E = j, C = i)$, with the interpretation that conditional on the evaluation of pathologist A, E evaluates constantly more severe than C. Similarly, $\frac{P_E(A=i,C=j)}{P_E(A=j,C=i)} = 0.141 \cdot I(E \geq i) + 0.168 \cdot 0.141 \cdot I(E < i)$. According to this last relation, conditional on the evaluation of E, A evaluates constantly more severely than C.

7 Discussion and Extensions

We have generalized the T symmetry model for $I \times I \times I$ contingency tables. The definitions for the first and second order T models were constructed in such a way that the usual desirable properties which connect T with S and MH models hold.

The key element of our construction is certainly the fact that it allows the existence of desired properties expressed in Propositions 4.1-4.4. Since these properties impose a coherency inter-model structure, any other generalization would not be aesthetically or mathematically acceptable.

In some application areas of the three-way symmetry models, such as raters disagreement modeling, the observed contingency tables are often highly sparse. Thus, symmetry models are particularly important in sparse contingency tables, in which inference with asymptotic arguments becomes harder. Therefor, we currently focus in developing inference for three-way symmetry models through algebraic statistics, based on Krampe et al. (2011).

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