1.4.3 Modelling Approaches

New modeling aspects expressed in terms of four models of the problem of outstanding claims are presented in this section. For comparison purposes, we present a Bayesian analysis of two models used in the past, the log-normal (model 1.1) and the state space model (model 1.3). We enhance these models by simultaneously modelling claim amounts and counts and using the total claim counts to specify appropriate parameter constraints. These modifications result in Models 1.2 and 1.4.

1.4.3.1 Model 1: Log-Normal Model

The simplest model for the data in Table 1.1 is a log-normal (anova-type) model. This model was investigated by Renshaw and Verrall (1994), Renshaw (1989) and Verrall (1991, 1993, 1996). Verrall (1990) produced Bayes estimates for the parameters of this model. The model is given by the formulation

$$\Upsilon_{\imath\jmath} = log\frac{Y_{\imath\jmath}}{inf_{\imath\jmath}}, \quad \Upsilon_{\imath\jmath} \sim N(\mu_{\imath\jmath}, \sigma^2), \quad \mu_{\imath\jmath} = b_0 + a_\imath + b_\jmath, \quad \imath, \jmath = 1, \ldots, r \qquad (1.1)$$

where $\Upsilon_{\imath\jmath}$ are called log-adjusted claim amounts and $N(\mu_{\imath\jmath}, \sigma^2)$ denotes the normal distribution with mean $\mu_{\imath\jmath}$ and variance σ^2. In Taylor and Ashe (1983) and Verrall (1991, 1993, 1996) the alternative parametrization $\Upsilon_{\imath\jmath} = log(Y_{\imath\jmath}) - log(inf_{\imath\jmath} \times E_\imath)$ with $\Upsilon_{\imath\jmath} \sim N(\mu_{\imath\jmath}, \sigma^2)$ is used, where $E_\imath$ is a measure of exposure (for example size of portfolio for year $\imath$). This reparametrization can be easily adopted for all following models. Finally, (1.1) requires appropriate constraints to achieve identifiability of the parameters, so here we adopt the usual sum-to-zero parametrization, that is, $\sum_\imath a_\imath = \sum_\jmath b_\jmath = 0$. Consequently, expression (1.1) assumes that the expected log-adjusted claim amount $\mu_{\imath\jmath}$ originated at year $\imath$ and paid with delay of $\jmath - 1$ years is modelled via a linear predictor which consists of the average log-adjusted claim amount b_0, a factor which reflects expected changes due to origin year $a_\imath$, and a factor depending on the delay pattern $b_\jmath$.

To complete the Bayesian formulation we use the priors

$$b_0 \sim N(0, \sigma^2_{b_0}), \quad a_\imath \sim N(0, \sigma^2_{a_\imath}), \quad b_\jmath \sim N(0, \sigma^2_{b_\jmath}), \; \imath, \jmath = 2, ..., r, \quad \tau = \sigma^{-2} \sim G(a_\tau, b_\tau)$$

with $G(a, b)$ denoting gamma distribution with mean a/b. For the kind of problems we are interested in, vague diffuse proper priors (Kass and Wasserman, 1996) are produced by using

$$\sigma^2_{b_0} = 1000, \quad \sigma^2_{a_\imath} = 100, \; \imath = 2, ..., r, \quad \sigma^2_{b_\jmath} = 100, \; \jmath = 2, ..., r, \quad a_\tau = b_\tau = 0.001.$$

A disadvantage of the above model is that it does not use any information from the observed counts. That is, any prediction of the missing claim amounts will be based only on the observed claim amounts. As a result, a source of information for a year (or cell) such as a sudden increase of accidents will not affect the prediction of the claim amounts.

1.4.3.2 Model 2: Log-Normal & Multinomial Model

We suggest here a two stage hierarchical model which uses both data sets in Tables 1.1 and 1.2 and the can be written, assuming $n_{\imath\jmath} > 0$ for all $\imath, \jmath$, in the two stage formulation

$$\Upsilon_{\imath\jmath} = log\frac{Y_{\imath\jmath}}{inf_{\imath\jmath}}, \quad \Upsilon_{\imath\jmath} \sim N(\mu_{\imath\jmath}, \sigma^2), \quad \mu_{\imath\jmath} = b_0 + a_\imath + b_\jmath + log(n_{\imath\jmath}),$$

$$(n_{\imath 1}, n_{\imath 2}, \ldots, n_{\imath r})^T \sim Multinomial(p'_1, p'_2, \ldots, p'_r; T_\imath), \qquad log(p'_\jmath/p'_1) = b^*_\jmath \tag{1.2}$$

where $(n_{\imath 1}, n_{\imath 2}, \ldots, n_{\imath r})^T$ are the number of claims originated at year $\imath$ and $p'_\jmath$ is the probability for a claim to be settled with a delay of $\jmath - 1$ years. For the first stage of the model we use as in Model 1 sum-to-zero constraints. Compared to (1.1), the linear predictor in this stage has been enhanced with the term $log(n_{\imath\jmath})$. As a result, b_0 represents the average log-adjusted amount per claim finalized and $a_\imath$, $b_\jmath$ reflect expected differences from b_0 due to origin and delay years respectively. For the second stage we use corner constraints ($b^*_1 = 0$) to facilitate its straightforward interpretation: $b^*_\jmath$ represents the log-odds of an accident to be paid with a delay of $\jmath - 1$ years versus an accident paid without delay.

The second (multinomial) stage of Model 1.2 is equivalent, to the log-linear model

$$n_{\imath\jmath} \sim Poisson(\lambda_{\imath\jmath}), \qquad log(\lambda_{\imath\jmath}) = b^*_0 + a^*_i + b^*_\jmath$$

under the constraints $\sum_{\jmath=1}^{r} n_{\imath\jmath} = n_{\imath.} = T_\imath$, $\sum_{\jmath=1}^{r} \lambda_{\imath\jmath} = \lambda_{\imath.} = T_\imath$, where b^*_0 and a^*_i are nuisance parameters; for more details see Agresti (1990). Under the assumption that $n_{\imath\jmath} > 0$ for all $\imath, \jmath$ it is precise to assume that $n_{\imath\jmath}$ follows a 'truncated at zero' $Poisson(\lambda_{\imath\jmath})$. However, for the size of the data we are interested in, the above distribution is practically identical to $Poisson(\lambda_{\imath\jmath})$. Had we assumed that $T_\imath$ is unknown, we would have used the above log-linear model without constraints on $\lambda_{\imath\jmath}$ and $n_{\imath\jmath}$. This could be useful, for example, if some kind of exposure measure is available, say the size of portfolio. Then, Model 1.2 without the constraints on λ_{ij} and n_{ij} is appropriate for predicting 'incurred but not reported claims'.

We suggest similar prior distributions as Model 1.1

$$b_0 \sim N(0, \sigma^2_{b_0}), \quad a_i \sim N(0, \sigma^2_{a_i}),\ i = 2, ..., r, \quad b_j \sim N(0, \sigma^2_{b_j}),\ j = 2, ..., r,$$

$$\tau = \sigma^{-2} \sim G(a_\tau, b_\tau), \quad b^*_j \sim N(0, \sigma^2_{b^*_j}),\ j = 2, ..., r.$$

The same values for $\sigma^2_{b_0}$, $\sigma^2_{a_i}$, $\sigma^2_{b_j}$ as in Model 1.1 can be used. For the additional parameters b^*_j we suggest $\sigma^2_{b^*_j} = 100$, for $j = 2, ..., r$.

1.4.3.3 Model 3: State Space Modelling of Claim Amounts

An alternative modelling perspective for this kind of problems is the state space (or dynamic linear) models where the parameters depend on each other in a time recursive way. A general description of MCMC in dynamic models is given by Gamerman (1998). Carter and Kohn (1994) describe how to use Gibbs sampler for general state space models and Carlin (1992) applies Gibbs sampler for state space models for actuarial time series. For application of state space models in claim amounts problem see De Jong and Zehnwirth (1983) and Verrall (1989, 1994). The state space model can be written as

$$\Upsilon_{ij} = log\frac{Y_{ij}}{inf_{ij}},\ \Upsilon_{ij} \sim N(\mu_{ij}, \sigma^2),\ \mu_{ij} = b_0 + a_i + b_{ij} \tag{1.3}$$

with the recursive associations

$$b_{ij} = b_{i-1,j} + \varepsilon_i, \quad \varepsilon_i \sim N(0, \sigma^2_\varepsilon), \quad i = 2, ..., r,$$

$$a_i = a_{i-1} + \zeta_i, \quad \zeta_i \sim N(0, \sigma^2_\zeta), \quad i = 2, ..., r$$

and corner constraints $a_1 = b_{i1} = 0, \quad i = 1, 2, \ldots, r.$

Comparing Model 3 with Model 1 we first note that b_j has been replaced by b_{ij}. Thus, the delay effect on the log-adjusted claim amounts changes with the origin year. Second, the introduced recursive associations express the belief that the parameters a_i and b_j evolve through time via known stochastic mechanisms. In fact, these mechanisms are determined by the disturbance terms ε_i and ζ_i; as σ^2_ε approaches zero (1.3) degenerates to Model 1, whereas when σ^2_ζ approaches zero the parameters a_i tend to zero. The corner constraints imply that b_0 is the expected log-adjusted claim amount for the first origin year paid without delay and $a_\imath$ and $b_{\imath\jmath}$ are interpreted accordingly.

In (1.3) we only need to define prior distributions for the first state space parameters; for more details see Carlin *et al*. (1992), Carlin (1992) and Gamerman (1998). We propose priors $b_{1\jmath} \sim N(0, \sigma^2_{b_{1\jmath}})$ and $b_0 \sim N(0, \sigma^2_{b_0})$ with $\sigma^2_{b_{1\jmath}} = 100$ and $\sigma^2_{b_0} = 1000$. The prior for the precision $\tau = \sigma^{-2}$ is a $G(a_\tau, b_\tau)$ density as in Model 1.1. We additionally use non-informative gamma priors for the parameters $\sigma_\varepsilon^{-2} \sim G(a_\varepsilon, b_\varepsilon)$ and $\sigma_\zeta^{-2} \sim G(a_\zeta, b_\zeta)$ with proposed values $a_\varepsilon = b_\varepsilon = a_\zeta = b_\zeta = 10^{-10}$. Finally, as in Model 1.1, we note that this model does not use any information from claim counts.

1.4.3.4 Model 4: State Space Modelling of Average Claim Amount per Accident

Here we generalise the Model 1.3 by incorporating information from data in Table 1.2. Assuming that $n_{\imath\jmath} > 0$ for all $\imath, \jmath$, we suggest

$$\Upsilon_{\imath\jmath} = log\frac{Y_{\imath\jmath}}{inf_{\imath\jmath}}, \quad \Upsilon_{\imath\jmath} \sim N(\mu_{\imath\jmath}, \sigma^2), \quad \mu_{\imath\jmath} = b_0 + a_\imath + b_{\imath\jmath} + log(n_{\imath\jmath}),$$

$$(n_{\imath 1}, n_{\imath 2}, \ldots, n_{\imath r})^T \sim Multinomial(p'_1, p'_2, \ldots, p'_r; T_\imath), \quad log(p'_\jmath / p'_1) = b^*_\jmath, \quad b^*_1 = 0 \qquad (1.4)$$

with the recursive associations

$$b_{\imath\jmath} = b_{\imath-1,\jmath} + \varepsilon_\imath, \quad \varepsilon_\imath \sim N(0, \sigma^2_\varepsilon), \quad \imath = 2, ..., r,$$

$$a_\imath = a_{\imath-1} + \zeta_\imath, \quad \zeta_\imath \sim N(0, \sigma^2_\zeta), \quad \imath = 2, ..., r.$$

In analogy with Model 2, we have added the term $log(n_{\imath\jmath})$ in the linear predictor. Thus, b_0 represents the log-adjusted amount per claim finalized for the first origin year paid without delay, and $a_\imath$, $b_{\imath\jmath}$ are interpreted accordingly. The multinomial second stage formulation is interpreted exactly as in Model 2. The priors can be defined similarly as in Models 1.2 and 1.3.

1.4.4 A Real Data Example

The following data came from a major Greek insurance company. Tables 1.3-1.5 give the claim amounts, the claim counts, the total counts for car accidents and the inflation factor. Due to their nature, the main source of delay is due to claims that are notified but settled after the accident year. Liabilities that have arisen but reported later are assumed to be minimal. Moreover, the assumption of no partial payments is plausible since only a small proportion of car accident claims are paid in more than one installments.

		B						
	Year	1	2	3	4	5	6	7
	1989	527003	220645	130250	84085	72182	21656	49868
	1990	715247	341364	166001	99845	108648	91958	
	1991	1166119	428365	166410	155376	191644		
A	**1992**	1686294	647331	335290	427069			
	1993	2780948	961010	444610				
	1994	3619446	1328151					
	1995	4002087						

Table 1.3: Outstanding Claim Amounts from a Greek Insurance Company(thousands drachmas).

The analysis of the data above was initiated by deinflating the data in Table 1.3 using the inflation factors in Table 1.5. Therefore, the resulting predictive amounts presented in this section should be multiplied by the corresponding inflation factor to represent amounts for a specific year (for example multiply by 257/100=2.57 to get the inflated amount for year 1996).

Posterior summaries of Models 1-4 are given in Tables 1.7 and 1.8. Note the striking difference of our proposed models 1.2 and 1.4 when compared with the existing approaches expressed by Models 1.1 and 1.3 for outstanding claim amounts for 1991 and 1992. This deviation is easily explainable if we examine carefully the data in Table 1.4. The remaining outstanding claims for 1991 are only 132 and account for the 1.05% of the total claim counts (12,601). This percentage is comparably much smaller than the corresponding outstanding claim counts of 1989 and 1990 which were 3.03% and 4.48% respectively. This decrease is being taken into account by our models and the produced estimates for 1991 are appropriately adjusted.

Table 1.9 gives the posterior summaries for variance components for all models. For the

		B							
	Year	1	2	3	4	5	6	7	Total
	1989	6622	1943	489	138	61	223	66	9542
	1990	6943	2133	632	154	162	390		10496
	1991	8610	2216	736	651	256			12601
A	**1992**	9791	3167	1570	624				15565
	1993	11722	3192	1773					17735
	1994	13684	3664						19746
	1995	13068							18600

Table 1.4: Outstanding Claim Counts from a Greek Insurance Company.

Year	1989	1990	1991	1992	1993	1994	1995	1996
Inflation (%)	100.0	120.4	143.9	166.6	190.6	214.2	235.6	257.0

Table 1.5: Inflation Factor for Greece.

	Year					
Model	1990	1991	1992	1993	1994	1995
1	1107(17)	1374(22)	1904(69)	2505(118)	3026(238)	3112(556)
2	1105(19)	1322(6)	1787(48)	2400(121)	2892(271)	2950(698)
3	1103(13)	1379(27)	1896(68)	2533(167)	3013(282)	3091(475)
4	1101(3)	1330(2)	1789(8)	2433(31)	2752(59)	2767(168)

Table 1.6: Posterior Mean (Standard Deviation) for Total Claim Amounts Paid for Each Accident Year (million drachmas; adjusted for inflation).

	Year					
Model	1990	1991	1992	1993	1994	1995
1	34(17)	65(22)	215(69)	409(118)	773(238)	1413(555)
2	32(19)	13(6)	97(48)	304(121)	639(271)	1251(698)
3	30(13)	70(27)	206(68)	436(167)	760(282)	1393(475)
4	28(3)	21(2)	99(8)	336(31)	498(59)	1068(168)

Table 1.7: Posterior Mean (Standard Deviation) for Total Outstanding Claim Amounts of Each Accident Year (million drachmas; adjusted for inflation).

	Year						
Model	1996	1997	1998	1999	2000	2001	Total
1	1222(338)	679(177)	470(140)	299(110)	152(59)	88(54)	2909(670)
2	1085(450)	582(215)	375(171)	191(109)	66(40)	37(29)	2336(806)
3	1166(289)	677(225)	496(227)	310(179)	161(154)	85(99)	2895(834)
4	937(136)	456(70)	353(78)	179(43)	87(24)	41(20)	2052(226)

Table 1.8: Posterior Mean (Standard Deviation) of Total Claim Amounts to be Paid in Each Future Year (million drachmas; adjusted for inflation).

data we analysed, we noticed that the state space model for claim amounts (Model 1.3) did not differ very much from the simple log-normal model. This is due to the small posterior values of σ_ε^2 and may imply that no dynamic term is needed when modelling the total claim amounts. On the other hand, incorporation the claim counts (Model 4) resulted in a posterior density of σ_ε^2 which gives evidence for a non-constant dynamic term. Therefore, Model 3 implies that the total payments have a similar delay pattern across years while Model 4 implies that 'payments per claim finalized' for origin year $\imath$ and delay year $\jmath$ change from year to year.

Posterior	Model Parameters							
	σ^2				σ_ε^2		σ_ζ^2	
Value	Model 1	Model 2	Model 3	Model 4	Model 3	Model 4	Model 3	Model 4
mean	0.0893	0.1366	0.0623	0.00008	0.0379	0.1249	0.1091	0.0150
median	0.0816	0.1231	0.0603	0.00008	0.0002	0.1197	0.0777	0.0112
st.dev.	0.0409	0.0596	0.0399	0.00002	0.0777	0.0324	0.1227	0.0145

Table 1.9: Posterior Summaries for Model Parameters σ^2, σ_ε^2 and σ_ζ^2.

1.4.5 Discussion

In this case study we developed new models in order to analyse the well known problem of outstanding claims of insurance companies using Bayesian theory and MCMC methodology.

The models fitted can be divided in two categories. The first category contains models that use only the information from claim amounts (Table 1.1) while the second exploits both claim amounts and counts (Tables 1.1 and 1.2). Thus the enriched family attempts to model the average payment per claim finalized or paid; this is the approach we advocate, and we believe that it improves the predictive behaviour of the model.

The models dealt with in this illustrated example can be generalised by adding other factors in the first (log-normal) stage. For example, we may assume that the variance of $\Upsilon_{\imath\jmath}$ depends on the claim counts of the corresponding cell. Since our suggested models are already multiplicative in the error, this adjustment will improve, at least in our data, only slightly the fit.

Finally, we would like to mention that the Bayesian paradigm used in this case study did not utilize the advantage of using informative prior densities. By illustrating our results with non-informative priors, we only provide a yardstick for comparison with other approaches. However, any prior knowledges can be incorporated in our models using usual quantification arguments.

1.4.6 Full Conditional Posterior Densities of Case Study

Conditional posterior distributions needed for the MCMC implementation of the four models presented in Section 1.4.3 are given here in detail. Iterative samples from these conditional densities provide, after some burn-in period and by using an appropriate sample lag, the required samples from the posterior density.

1.4.6.1 Computations for Model 1

The model described in Section 1.4.3.1 includes parameters $b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}$. The claim amounts and counts are divided in known/observed (data) for $i + j \leq r + 1$ and unknown missing (parameters) for $i + j > r + 1$. Denote by $\boldsymbol{\Upsilon}^U$ the observed (inflation adjusted) log-amounts by $\boldsymbol{\Upsilon}^L$ the missing (inflation adjusted) log-amounts and by $\boldsymbol{\Upsilon}$ the matrix containing both observed and missing claim (inflation adjusted) log-amounts. Assuming that the missing data $\boldsymbol{\Upsilon}^L$ are a further set of parameters, the parameter vector is given by $(b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}, \boldsymbol{\Upsilon}^L)$ and the data vector is given by $(\boldsymbol{\Upsilon}^U)$. Using Bayes theorem and denoting by f the prior, conditional and marginal densities, the posterior distribution is given by

$$\begin{aligned} f(b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}, \boldsymbol{\Upsilon}^L|\boldsymbol{\Upsilon}^U) &\propto \\ &\propto f(\boldsymbol{\Upsilon}^U|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}, \boldsymbol{\Upsilon}^L) f(b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}, \boldsymbol{\Upsilon}^L) \\ &\propto f(\boldsymbol{\Upsilon}^U|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}) f(\boldsymbol{\Upsilon}^L|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}) f(b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}) \\ &\propto f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}) f(b_0) f(\boldsymbol{a}) f(\boldsymbol{b}) f(\sigma^{-2}) \end{aligned}$$

where $\boldsymbol{b} = (b_2, \ldots, b_r)$ and $\boldsymbol{a} = (a_2, \ldots, a_r)$.

The full conditional distributions are therefore given by

1. $f(b_0|.) \propto f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}) f(b_0)$

2. $f(\boldsymbol{a}|.) \propto f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}) f(\boldsymbol{a})$

3. $f(\boldsymbol{b}|.) \propto f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}) f(\boldsymbol{b})$

4. $f(\sigma^{-2}|.) \propto f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}) f(\sigma^{-2})$

5. $f(\boldsymbol{\Upsilon}^L|.) \propto f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2})$

In the above posterior the conditional $f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2})$ is the full likelihood assuming that there are no missing data in the claim amount table; therefore,

$$f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}) = (2\pi\sigma^2)^{-r^2/2} exp\left(-\frac{1}{2\sigma^2} \sum_{\imath=1}^{r} \sum_{\jmath=1}^{r} [\Upsilon_{\imath\jmath} - b_0 - a_\imath - b_\jmath)]^2 \right)$$

Thus, the resulting conditional distributions are

1.
$$f(b_0|.) = N\left(\frac{\Upsilon_{..}}{r^2 + \sigma^2/\sigma^2_{b_0}}, \frac{\sigma^2}{r^2 + \sigma^2/\sigma^2_{b_0}} \right), \tag{1.5}$$
where $\Upsilon_{..} = \sum_{\imath=1}^{r} \sum_{\jmath=1}^{r} \Upsilon_{\imath\jmath}$.

2. **[a]**
$$f(a_\imath|.) = N\left(\frac{\Upsilon_{\imath.} - \Upsilon_{1.} - r \sum_{k\neq 1,i} a_k}{2r + \sigma^2/\sigma^2_{a_\imath}}, \frac{\sigma^2}{2r + \sigma^2/\sigma^2_{a_\imath}} \right), \ \imath = 2, \ldots, r, \tag{1.6}$$
where $\Upsilon_{\imath.} = \sum_{\jmath=1}^{r} \Upsilon_{\imath\jmath}$.

[b] Set $a_1 = -\sum_{\imath=2}^{r} a_\imath$.

3. **[a]**
$$f(b_\jmath|.) = N\left(\frac{\Upsilon_{.\jmath} - \Upsilon_{.1} - r \sum_{k\neq 1,j} b_k}{2r + \sigma^2/\sigma^2_{b_\jmath}}, \frac{\sigma^2}{2r + \sigma^2/\sigma^2_{b_\jmath}} \right), \ \jmath = 2, \ldots, r, \tag{1.7}$$
where $ln_{.j} = \sum_{\imath=1}^{r} log(n_{\imath\jmath})$ and $\Upsilon_{.j} = \sum_{\imath=1}^{r} \Upsilon_{\imath\jmath}$

[b] Set $b_1 = -\sum_{\jmath=2}^{r} b_\jmath$.

4.
$$f(\tau = \sigma^{-2}|.) = G(a_\tau + r^2/2, b_\tau + SS/2), \tag{1.8}$$
with $SS = \sum_{\imath=1}^{r} \sum_{\jmath=1}^{r} (\Upsilon_{\imath\jmath} - \mu_{\imath\jmath})^2$ and $\mu_{\imath\jmath} = b_0 + a_\imath + b_\jmath)$.

5.
$$f(\Upsilon_{ij}|.) = N(\mu_{ij}, \sigma^2), \quad i = 2, \dots, r, \quad j = r - i + 2, \dots, r, \tag{1.9}$$
with $\mu_{ij} = b_0 + a_i + b_j$.

1.4.6.2 Computations for Model 2

The model introduced in 1.4.3.2 is more complicated and includes parameters $b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}$ from stage one, and $\boldsymbol{b}^*$ from stage two. Similar to above, the claim (inflation-adjusted) log-amounts and counts are divided in known/observed (data) for $i + j \leq r + 1$ and unknown missing (parameters) for $i+j > r+1$. Denote by $\boldsymbol{N}^U$ and $\boldsymbol{\Upsilon}^U$ the observed claim counts and amounts, respectively, by $\boldsymbol{N}^L$ and $\boldsymbol{\Upsilon}^L$ the missing claim counts and amounts, respectively, and by $\boldsymbol{N}$ and $\boldsymbol{\Upsilon}$ the matrices containing both observed and missing claim counts and amounts, respectively. Assuming that the missing data $\boldsymbol{N}^L$ and $\boldsymbol{\Upsilon}^L$ are a further set of parameters, the parameter vector is given by $(b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}, \boldsymbol{b}^*, \boldsymbol{N}^L, \boldsymbol{\Upsilon}^L)$ and the data vector is given by $(\boldsymbol{N}^U, \boldsymbol{\Upsilon}^U)$. Using Bayes theorem and denoting by f the prior, conditional and marginal densities, the posterior distribution is given by

$$
\begin{aligned}
f(b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}, \boldsymbol{b}^*, \boldsymbol{N}^L, \boldsymbol{\Upsilon}^L|\boldsymbol{N}^U, \boldsymbol{\Upsilon}^U) \quad &\propto \\
&\propto \; f(\boldsymbol{N}^U, \boldsymbol{\Upsilon}^U|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}, \boldsymbol{b}^*, \boldsymbol{N}^L, \boldsymbol{\Upsilon}^L) f(b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}, \boldsymbol{b}^*, \boldsymbol{N}^L, \boldsymbol{\Upsilon}^L) \\
&\propto \; f(\boldsymbol{N}^U, \boldsymbol{\Upsilon}^U|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}, \boldsymbol{b}^*) f(\boldsymbol{N}^L, \boldsymbol{\Upsilon}^L|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}, \boldsymbol{b}^*) f(b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}, \boldsymbol{b}^*) \\
&\propto \; f(\boldsymbol{N}, \boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}, \boldsymbol{b}^*) f(b_0) f(\boldsymbol{a}) f(\boldsymbol{b}) f(\sigma^{-2}) f(\boldsymbol{b}^*) \\
&\propto \; f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}, \boldsymbol{N}) f(\boldsymbol{N}|\boldsymbol{b}^*) f(b_0) f(\boldsymbol{a}) f(\boldsymbol{b}) f(\sigma^{-2}) f(\boldsymbol{b}^*)
\end{aligned}
$$

where $\boldsymbol{b} = (b_2, \dots, b_r)$, $\boldsymbol{a} = (a_2, \dots, a_r)$ and $\boldsymbol{b^*} = (b_2^*, \dots, b_r^*)$.

The full conditional distributions are therefore given by

1. $f(b_0|.) \propto f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}, \boldsymbol{N}) f(b_0)$

2. $f(\boldsymbol{a}|.) \propto f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}, \boldsymbol{N}) f(\boldsymbol{a})$

3. $f(\boldsymbol{b}|.) \propto f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}, \boldsymbol{N}) f(\boldsymbol{b})$

4. $f(\sigma^{-2}|.) \propto f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}, \boldsymbol{N}) f(\sigma^{-2})$

5. $f(\boldsymbol{\Upsilon}^L|.) \propto f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}, \boldsymbol{N})$

6. $f(\boldsymbol{b}^*|.) \propto f(\boldsymbol{N}|\boldsymbol{b}^*)f(\boldsymbol{b}^*)$

7. $f(\boldsymbol{N}^L|.) \propto f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}, \boldsymbol{N})f(\boldsymbol{N}|\boldsymbol{b}^*)$

In the above posterior the conditional $f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}, \boldsymbol{N})$ is the full likelihood for the first stage assuming that there are no missing data in the claim amount table; therefore,

$$f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}, \boldsymbol{N}) = (2\pi\sigma^2)^{-r^2/2} exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{r} \sum_{j=1}^{r} [\Upsilon_{ij} - b_0 - a_i - b_j - log(n_{ij})]^2 \right)$$

The full likelihood $f(\boldsymbol{N}|\boldsymbol{b}^*)$ of the second stage, assuming no missing claim counts, can be written as

$$f(\boldsymbol{N}|\boldsymbol{b}^*) = exp\left(\sum_{i=1}^{r} log(T_i!) - \sum_{i=1}^{r} \sum_{j=1}^{r} log(n_{ij}!) + \sum_{j=2}^{r} n_{.j} b_j^* - n_{..} log(\sum_{k=1}^{r} e^{b_k^*}) \right),$$

where $n_{..} = \sum_{i=1}^{r} \sum_{j=1}^{r} n_{ij} = \sum_{i=1}^{r} T_i$ and $n_{.j} = \sum_{i=1}^{r} n_{ij}$. Thus, the resulting conditional distributions are

1. $$f(b_0|.) = N\left(\frac{\Upsilon_{..} - ln_{..}}{r^2 + \sigma^2/\sigma^2_{b_0}}, \frac{\sigma^2}{r^2 + \sigma^2/\sigma^2_{b_0}} \right), \quad (1.10)$$

   where $ln_{..} = \sum_{i=1}^{r} \sum_{j=1}^{r} log(n_{ij})$ and $\Upsilon_{..} = \sum_{i=1}^{r} \sum_{j=1}^{r} \Upsilon_{ij}$.

2. **[a]**

   $$f(a_i|.) = N\left(\frac{\Upsilon_{i.} - \Upsilon_{1.} - (ln_{i.} - ln_{1.}) - r \sum_{k \neq 1,i} a_k}{2r + \sigma^2/\sigma^2_{a_i}}, \frac{\sigma^2}{2r + \sigma^2/\sigma^2_{a_i}} \right), \; i = 2, \ldots, r, \quad (1.11)$$

   where $ln_{i.} = \sum_{j=1}^{r} log(n_{ij})$ and $\Upsilon_{i.} = \sum_{j=1}^{r} \Upsilon_{ij}$.

   [b] Set $a_1 = -\sum_{i=2}^{r} a_i$.

3. **[a]**

   $$f(b_j|.) = N\left(\frac{\Upsilon_{.j} - \Upsilon_{.1} - (ln_{.j} - ln_{.1}) - r \sum_{k \neq 1,j} b_k}{2r + \sigma^2/\sigma^2_{b_j}}, \frac{\sigma^2}{2r + \sigma^2/\sigma^2_{b_j}} \right), \; j = 2, \ldots, r, \quad (1.12)$$

   where $ln_{.j} = \sum_{i=1}^{r} log(n_{ij})$ and $\Upsilon_{.j} = \sum_{i=1}^{r} \Upsilon_{ij}$

   [b] Set $b_1 = -\sum_{j=2}^{r} b_j$.

4. $f(\tau = \sigma^{-2}|.)$ is given by (1.8) with $\mu_{ij} = b_0 + a_i + b_j + log(n_{ij})$.

5. $f(\Upsilon_{ij}|.)$ is given by (1.9) with $\mu_{ij} = b_0 + a_i + b_j + log(n_{ij})$.

6. [**a**]

$$f(b_j^*|.) \propto exp\left(b_j^* n_{.j} - n_{..}log(\sum_{k=1}^{r} e^{b_k^*}) - 0.5 b_j^{*2}/\sigma^2_{b_j^*}\right),\ j = 2, \ldots, r, \quad (1.13)$$

where $n_{.j} = \sum_{i=1}^{r} n_{ij}$.

[**b**] Set $b_1^* = 0$.

To obtain a sample from (1.13) we may use either Metropolis-Hastings algorithm or Gilks and Wild (1992) adaptive rejection sampling for log-concave distributions. Both methods provide similar convergence rates.

7. The full conditional posterior of the missing counts n_{ij} for $j = r - i + 2, \ldots, r - 1$, $i = 3, \ldots, r$ is complicated since

$$f(n_{ij}|.) \propto f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{N}) f(\boldsymbol{N}|\boldsymbol{b}^*)$$

The constraint $T_i = \sum_{j=1}^{r} n_{ij}$ reduces the above posterior to

$$f(n_{ij}|.) \propto f(\Upsilon_{ij}|b_0, a_i, b_j, n_{ij}, \Upsilon_{ir}) f(\Upsilon_{ir}|b_0, a_i, b_r, n_{ir}) f(n_{ij}|b_j^*, n_{,r}) f(n_{ir}|b_r^*)$$

where $n_{ir} = T_i - \sum_{j=1}^{r-1} n_{ij}$. Therefore,

$$f(n_{ij}|.) \propto \frac{[p_j']^{n_{ij}}}{n_{ij}!} \frac{[p_r']^{\Omega_{ij} - n_{ij}}}{(\Omega_{ij} - n_{ij})!} \exp\left(\Psi_{ij}(n_{ij}) + \Psi_{ir}(\Omega_{ij} - n_{ij})\right) \quad (1.14)$$

where $\Omega_{ij} = T_i - \sum_{k \neq j, r} n_{ik}$ and $\Psi_{ij}(n_{ij}) = -\frac{1}{2\sigma^2}(\Upsilon_{ij} - b_0 - a_i - b_j - log(n_{ij}))^2$.

We sample from $f(n_{ij}|.)$ by using the following Metropolis-Hastings step. Propose missing n'_{ij} and $n'_{ir} = \Omega_{ij} - n'_{ij}$, with $i = 3, \ldots, r$ and $j = r - i + 2, \ldots, r - 1$ from

$$Binomial(p_j'', \Omega_{ij}),\ p_j'' = \frac{p_j'}{p_j' + p_r'} = (1 + exp(b_r^* - b_j^*))^{-1}.$$

Accept the proposed move with probability

$$a = min\{1, exp(\Psi_{ij}(n'_{ij}) + \Psi_{ir}(\Omega_{ij} - n'_{ij}) - \Psi_{ij}(n_{ij}) - \Psi_{ir}(\Omega_{ij} - n_{ij}))\}. \quad (1.15)$$

1.4.6.3 Computations for Model 3

The dynamic model described in Section 1.4.3.3 is an extension of model of Section 1.4.3.1 includes parameters $b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}, \sigma_\varepsilon^{-2}, \sigma_\zeta^{-2}$, where $\boldsymbol{b} = (b_{12}, \ldots, b_{1r}, b_{22}, \ldots, b_{2r}, \ldots, b_{rr})$. Using Bayes theorem the posterior distribution is given by

$$f(b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}, \sigma_\epsilon^{-2}, \sigma_\zeta^{-2}, \boldsymbol{\Upsilon}^L|\boldsymbol{\Upsilon}^U) \propto$$

$$\propto f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}) f(b_0) f(\boldsymbol{a}|\sigma_\zeta^{-2}) f(\boldsymbol{b}|\sigma_\varepsilon^{-2}) f(\sigma^{-2}) f(\sigma_\varepsilon^{-2}) f(\sigma_\zeta^{-2}).$$

The full conditional distributions are therefore given by

1. $f(b_0|.) \propto f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}) f(b_0)$
2. $f(\boldsymbol{a}|.) \propto f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}) f(\boldsymbol{a}|\sigma_\zeta^2)$
3. $f(\boldsymbol{b}|.) \propto f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}) f(\boldsymbol{b}|\sigma_\varepsilon^2)$
4. $f(\sigma^{-2}|.) \propto f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}) f(\sigma^{-2})$
5. $f(\sigma_\varepsilon^{-2}|.) \propto f(\boldsymbol{b}|\sigma_\varepsilon^{-2}) f(\sigma_\varepsilon^{-2})$
6. $f(\sigma_\zeta^{-2}|.) \propto f(\boldsymbol{a}|\sigma_\zeta^{-2}) f(\sigma_\zeta^{-2})$
7. $f(\boldsymbol{\Upsilon}^L|.) \propto f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2})$

Similar to Model 1, the conditional $f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2})$ is the full likelihood assuming that there are no missing data in the claim amount table; therefore,

$$f(\boldsymbol{\Upsilon}|b_0, \boldsymbol{a}, \boldsymbol{b}, \sigma^{-2}) = (2\pi\sigma^2)^{-r^2/2} exp\left(-\frac{1}{2\sigma^2} \sum_{\imath=1}^{r} \sum_{\jmath=1}^{r} [\Upsilon_{\imath\jmath} - b_0 - a_\imath - b_{\imath\jmath})]^2 \right)$$

Thus, the resulting conditional distributions are

1. $$f(b_0|.) = N\left(\frac{\Upsilon_{..} - r^2\bar{a} - r^2\bar{b}}{r^2 + \sigma^2/\sigma_{b_0}^2}, \frac{\sigma^2}{r^2 + \sigma^2/\sigma_{b_0}^2} \right), \quad (1.16)$$

   where $\bar{a} = r^{-1}\sum_\imath a_\imath$ and $\bar{b} = r^{-2}\sum_{\imath\jmath} b_{\imath\jmath}$.

2. **[a]** Set $a_1 = 0$.

   [b]

$$f(a_\imath|.) = N\left(\frac{\Upsilon_{\imath.} - rb_0 - b_{\imath.} + (a_{\imath+1} + a_{\imath-1})\sigma^2/\sigma_\zeta^2}{r + 2\sigma^2/\sigma_\zeta^2}, \frac{\sigma^2}{r + 2\sigma^2/\sigma_\zeta^2}\right), \ \imath = 2, \ldots, r-1 \tag{1.17}$$

   where $b_{\imath.} = \sum_\jmath b_{\imath\jmath}$.

   [c]

$$f(a_r|.) = N\left(\frac{\Upsilon_{r.} - rb_0 - b_{r.} + a_{r-1}\sigma^2/\sigma_\zeta^2}{r + \sigma^2/\sigma_\zeta^2}, \frac{\sigma^2}{r + \sigma^2/\sigma_\zeta^2}\right). \tag{1.18}$$

3. **[a]** Set $b_{\imath 1} = 0$ for $\imath = 1, \ldots, r$.

   [b]

$$f(b_{1\jmath}|.) = N\left(\frac{\Upsilon_{1\jmath} - b_0 + b_{2\jmath}\sigma^2/\sigma_\varepsilon^2}{1 + \sigma^2/\sigma_\varepsilon^2 + \sigma^2/\sigma_{b_{1\jmath}}^2}, \frac{\sigma^2}{1 + \sigma^2/\sigma_\varepsilon^2 + \sigma^2/\sigma_{b_{1\jmath}}^2}\right), \ \jmath = 2, \ldots, r. \tag{1.19}$$

   [c]

$$f(b_{\imath\jmath}|.) = N\left(\frac{\Upsilon_{\imath\jmath} - b_0 - a_\imath + (b_{\imath-1,\jmath} + b_{\imath+1,\jmath})\sigma^2/\sigma_\varepsilon^2}{1 + 2\sigma^2/\sigma_\varepsilon^2}, \frac{\sigma^2}{1 + 2\sigma^2/\sigma_\varepsilon^2}\right), \tag{1.20}$$

   for $\imath = 2, \ldots, r-1, \ \jmath = 2, \ldots, r$.

   [d]

$$f(b_{r\jmath}|.) = N\left(\frac{\Upsilon_{r\jmath} - b_0 - a_r + b_{r-1,\jmath}\sigma^2/\sigma_\varepsilon^2}{1 + \sigma^2/\sigma_\varepsilon^2}, \frac{\sigma^2}{1 + \sigma^2/\sigma_\varepsilon^2}\right), \ \jmath = 2, \ldots, r. \tag{1.21}$$

4. $f(\tau|.)$ is given by equation (1.8), using $\mu_{\imath\jmath} = b_0 + a_\imath + b_{\imath\jmath}$.

5. $f(\Upsilon_{\imath\jmath}|.)$, for $i + j > r + 1$, is given by equation (1.9) using $\mu_{\imath\jmath} = b_0 + a_\imath + b_{\imath\jmath}$.

6.

$$f(\sigma_\varepsilon^{-2}|.) = G\left(a_\varepsilon + (r-1)^2/2, \ b_\varepsilon + \sum_{\imath=2}^{r}\sum_{\jmath=2}^{r}(b_{\imath\jmath} - b_{\imath-1,\jmath})^2/2\right). \tag{1.22}$$

7.

$$f(\sigma_\zeta^{-2}|.) = G\left(a_\zeta + (r-1)/2, \ b_\zeta + \sum_{\imath=2}^{r}(a_\imath - a_{\imath-1})^2/2\right). \tag{1.23}$$

1.4.6.4 Computations for Model 4

The first stage of Model 4 is similar to model 3 but we substitute Υ_{ij} by $\Upsilon^*_{ij} = log[Y_{ij}]$ - $log[n_{ij} inf_{ij}]$ in all conditional distributions (1.16 - 1.21). The stage two is equivalent to the second stage of Model 2. In more detail we have

1. $f(b_0|.)$ is given by (1.16) if we substitute Υ_{ij} by Υ^*_{ij}.

2. **[a]** Set $a_1 = 0$.
   [b] $f(a_i|.)$ for $i = 2, \ldots, r-1$ is given by (1.17) if we substitute Υ_{ij} by Υ^*_{ij}.
   [c] $f(a_r|.)$ is given by (1.18) if we substitute Υ_{ij} by Υ^*_{ij}.

3. **[a]** Set $b_{i1} = 0$ for $i = 1, \ldots, r$.
   [b] $f(b_{1j}|.)$ for $j = 2, \ldots, r$ is given by (1.19) if we substitute Υ_{ij} by Υ^*_{ij}.
   [c] $f(b_{ij}|.)$ for $i = 2, \ldots, r-1,\ j = 2, \ldots, r$ is given by (1.20) if we substitute Υ_{ij} by Υ^*_{ij}.
   [d] $f(b_{rj}|.)$ for $j = 2, \ldots, r$ is given by (1.21) if we substitute Υ_{ij} by Υ^*_{ij}.

4. $f(\tau|.)$ is given by equation (1.8), using $\mu_{ij} = b_0 + a_i + b_{ij} - log(n_{ij})$.

5. $f(\Upsilon_{ij}|.)$, for $i + j > r + 1$, is given by equation (1.9) using $\mu_{ij} = b_0 + a_i + b_{ij} - log(n_{ij})$.

6. $f(\sigma_\varepsilon^{-2}|.)$ is given by (1.22).

7. $f(\sigma_\zeta^{-2}|.)$ is given by (1.23).

8. $f(b^*_j|.)$ is given by (1.13).

9. The full conditional posterior of the missing counts $f(n_{ij}|.)$ for $j = r - i + 2, \ldots, r-1$, $i = 3, \ldots, r$ is given by (1.14) with $\Psi_{ij}(n_{ij}) = -0.5\sigma^{-2}[\Upsilon_{ij} - b_0 - a_i - b_{ij} - log(n_{ij})]^2$.

   In order to achieve to achieve an optimal acceptance rate we propose a simultaneous updating scheme of n_{ij}, b_{ij} and b_{ir} when $i = 3, \ldots, r$ and $j = r - i + 2, \ldots, r$. The corresponding joint full conditional posterior of these parameters is given by given by an equation of type (1.14) substituting Ψ_{ij} with $\Psi^*_{ij}(n_{ij}, b_{ij}) = -0.5\sigma^{-2}[\Upsilon_{ij} - b_0 - a_i - b_{ij} - log(n_{ij})]^2 - 0.5\sigma_\varepsilon^{-2}[(b_{ij} - b_{i-1,j})^2 + (b_{i+1,j} - b_{i,j})^2]$.

We used the following metropolis step. We propose candidate $n'_{ij}, b'_{ij}, n'_{ir}$ from the proposal densities

$$q(n'_{ij}, b'_{ij}, b'_{ir}|n_{ij}, b_{ij}, b_{ir}) = q(n'_{ij}|n_{ij}, b_{ij}, b_{ir})q(b'_{ij}|n'_{ij}, n_{ij}, b_{ij}, b_{ir})q(b'_{ir}|n'_{ij}, n_{ij}, b_{ij}, b_{ir})$$

with

$$\begin{aligned} q(n'_{ij}|n_{ij}, b_{ij}, b_{ir}) &= Binomial([1 + exp(b^*_r - b^*_j)]^{-1}, \Omega_{ij}) \\ q(b'_{ij}|n'_{ij}, n_{ij}, b_{ij}, b_{ir}) &= N(b_{ij} + log(n_{ij}) - log(n'_{ij}), \bar{\sigma}^2_{ij}), \\ q(b'_{ir}|n'_{ij}, n_{ij}, b_{ij}, b_{ir}) &= N(b_{ir} + log(\Omega_{ij} - n_{ij}) - log(\Omega_{ij} - n'_{ij}), \bar{\sigma}^2_{ir}) \end{aligned}$$

where $\bar{\sigma}^2_{ij}$ and $\bar{\sigma}^2_{ir}$ are metropolis parameters that should be calibrated appropriately to achieve a desired acceptance rate.

Accept the proposed move with probability

$$\alpha = min\left\{1, \frac{exp[\Psi^*_{ij}(n'_{ij}, b'_{ij}) + \Psi^*_{ir}(\Omega_{ij} - n'_{ij}, b'_{ir})]q(b_{ij}, b_{ir}|n_{ij}, n'_{ij}, b'_{ij}, b'_{ir})}{exp[\Psi^*_{ij}(n_{ij}, b_{ij}) + \Psi^*_{ir}(\Omega_{ij} - n_{ij}, b_{ir})]q(b'_{ij}, b'_{ir}|n'_{ij}, n_{ij}, b_{ij}, b_{ir})}\right\}. \qquad (1.24)$$