Predictability and Model Selection in the Context of ARCH Models

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Abstract

Most of the methods used in the ARCH literature for selecting the appropriate model are based on evaluating the ability of the models to describe the data. An alternative model selection approach is examined based on the evaluation of the predictability of the models on the basis of standardized prediction errors.

Keywords and Phrases: ARCH models, Model selection, Predictability, Correlated Gamma Ratio distribution, Prediction Error Criterion

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1. Introduction

ARCH models have widely been used in financial time series analysis and particularly in analyzing the risk of holding an asset, evaluating the price of an option, forecasting time varying confidence intervals and obtaining more efficient estimators under the existence of heteroscedasticity.

In the recent literature, numerous parametric specifications of ARCH models have been considered for the description of the characteristics of financial markets. In the linear ARCH(q) model originally introduced by Engle (1982), the conditional variance is postulated to be a linear function of the past q squared innovations. Bollerslev (1986) proposed the generalized ARCH, or GARCH(p,q), model, where the conditional variance is postulated to be a linear function of both the past q squared innovations and the past p conditional variances. Nelson (1991) proposed the exponential GARCH, or EGARCH, model. The EGARCH model belongs to the family of asymmetric GARCH models, which capture the phenomenon that negative returns predict higher volatility than positive returns of the same magnitude. Other popular asymmetric models are the GJR model of Glosten et al. (1993), the threshold GARCH, or TARCH, model, introduced by Zakoian (1990) and the quadratic ARCH, or QGARCH, model, introduced by Sentana (1995). ARCH models go by such exotic names as AARCH, NARCH, PARCH, PNP-ARCH and STARCH among others.

The richness of the family of parametric ARCH models certainly complicates the search for the true model, and leaves quite a bit of arbitrariness in the model selection stage. The problem of selecting the model that describes best the movement of the series under study is therefore of practical importance.

The aim of this paper is to develop a model selection method based on the evaluation of the predictability of the ARCH models. In section 2 of the paper, the ARCH process is presented. Section 3 provides a brief description of the methods used in the literature for selecting the appropriate model based on evaluating the ability of the models to describe the data. In section 4, Panaretos et al.'s (1997) model selection method based on a standardized prediction error criterion is examined in the context of ARCH models. In section 5 the suggested model selection method is applied using return data for the Athens Stock Exchange (ASE) index over the period August 30th, 1993 to November 4th, 1996, while in section 6 a selection method based on the ability of the models describing the data is investigated. Finally, in section 7 a brief discussion of the results is provided.

2. The ARCH Process

Let $\{y_t(\theta)\}$ refer to the univariate discrete time real-valued stochastic process to be predicted (e.g. the rate of return of a particular stock or market portfolio from time $t-1$ to t) where θ is a vector of unknown parameters and $E(y_t(\theta) \mid I_{t-1}) \equiv E_{t-1}(y_t(\theta)) \equiv \mu_t(\theta)$ denotes the conditional mean given the information set available at time $t-1$, I_{t-1}. The innovation process for the conditional mean, $\{\varepsilon_t(\theta)\}$, is then given by $\varepsilon_t(\theta) = y_t(\theta) - \mu_t(\theta)$ with corresponding unconditional variance $V(\varepsilon_t(\theta)) = E(\varepsilon_t^2(\theta)) \equiv \sigma^2(\theta)$, zero unconditional mean and $E(\varepsilon_t(\theta)\varepsilon_s(\theta)) = 0$, $\forall t \neq s$. The conditional variance of the process given I_{t-1} is defined by $V(y_t(\theta) \mid I_{t-1}) \equiv V_{t-1}(y_t(\theta)) \equiv E_{t-1}(\varepsilon_t^2(\theta)) \equiv \sigma_t^2(\theta)$. Since investors would know the information set I_{t-1} when they make their investment decisions at time $t-1$, the relevant expected return to the investors and volatility are $\mu_t(\theta)$ and $\sigma_t^2(\theta)$, respectively.

An ARCH process, $\{\varepsilon_t(\theta)\}$, can be presented as:

$$\begin{gathered}
\varepsilon_t(\theta) = z_t \sigma_t(\theta) \\
z_t \overset{i.i.d.}{\sim} f[E(z_t) = 0, V(z_t) = 1] \qquad (2.1) \\
\sigma_t^2(\theta) = g(\sigma_{t-1}(\theta), \sigma_{t-2}(\theta), \ldots; \varepsilon_{t-1}(\theta), \varepsilon_{t-2}(\theta), \ldots; \upsilon_{t-1}, \upsilon_{t-2}, \ldots),
\end{gathered}$$

where $E(z_t) = 0$, $V(z_t) = 1$, $f(.)$ is the density function of z_t, $\sigma_t(\theta)$ is a time-varying, positive and measurable function of the information set at time $t-1$, υ_t is a vector of predetermined variables included in I_t, and $g(.)$ is a linear or nonlinear functional form. By definition, $\varepsilon_t(\theta)$ is serially uncorrelated with mean zero, but with a time varying conditional variance equal to $\sigma_t^2(\theta)$. The conditional variance is a linear or nonlinear function of lagged values of σ_t and ε_t, and predetermined variables included in I_{t-1}, $(\upsilon_{t-1}, \upsilon_{t-2}, \ldots)$. The standard ARCH models assume that $f(.)$ is the density function of the normal distribution. Bollerslev (1987) proposed using the student t distribution with an estimated kurtosis regulated by the degrees of freedom parameter. Nelson (1991) proposed the use of the generalized error distribution (Harvey (1981), Box and Tiao (1973)), which is also referred to as the exponential power distribution. Other distributions, that have been employed, include the generalized t distribution (Bollerslev et al. (1994)), the normal Poisson mixture distribution (Jorion (1988)), the normal lognormal mixture (Hsieh (1989)), and a serially dependent mixture of normally distributed variables (Cai (1994)) or student t distributed variables (Hamilton and Susmel (1994)). In the sequel, for notational convenience, no explicit indication of the dependence on the vector of parameters, θ, is given when obvious from the context.

Since very few financial time series have a constant conditional mean of zero, an ARCH model can be presented in a regression form by letting ε_t be the innovation process in a linear regression:

$$\begin{gathered} y_t = x_t'\beta + \varepsilon_t \\ \varepsilon_t \mid I_{t-1} \sim f\left(0, \sigma_t^2\right) \\ \sigma_t^2 = g\left(\sigma_{t-1}(\theta), \sigma_{t-2}(\theta), \ldots; \varepsilon_{t-1}(\theta), \varepsilon_{t-2}(\theta), \ldots; \upsilon_{t-1}, \upsilon_{t-2}, \ldots\right), \end{gathered} \tag{2.2}$$

where x_t is a $k \times 1$ vector of endogenous and exogenous explanatory variables included in the information set I_{t-1} and β is a $k \times 1$ vector of unknown parameters.

Let us assume that the conditional mean, $\mu_t = E\left(y_t \mid I_{t-1}\right)$, can be adequately described by a κ^{th} order autoregressive $\left[AR(\kappa)\right]$ model:

$$y_t = c_0 + \sum_{i=1}^{\kappa}\left(c_i y_{t-i}\right) + \varepsilon_t . \tag{2.3}$$

Usually, the conditional mean is either the overall mean or a first order autoregressive process. Theoretically, the $AR(1)$ process allows for the autocorrelation induced by discontinuous (or non-synchronous) trading in the stocks making up an index (Scholes and Williams (1977), Lo and MacKinlay (1988)). According to Campbell et al. (1997), “the non-synchronous trading arises when time series, usually asset prices, are taken to be recorded at time intervals of a fixed length when in fact they are recorded at time intervals of other, possible irregular lengths.” The Scholes and Williams model suggests the 1^{st} order moving average process for index returns, while the Lo and MacKinlay model suggests an $AR(1)$ form. Higher orders of the autoregressive process are considered in order to investigate if they are adequate to produce more accurate predictions.

Engle (1982) introduced the original form of $\sigma_t^2 = g(.)$ as a linear function of the past q squared innovations:

$$\sigma_t^2 = a_0 + \sum_{i=1}^{q} \left(a_i \varepsilon_{t-i}^2 \right). \tag{2.4}$$

For the conditional variance to be positive, the parameters must satisfy $\alpha_0 > 0$, $a_i \geq 0$, for $i = 1,...,q$. In empirical applications of ARCH(q) models, a long lag length and a large number of parameters are often called for. To circumvent this problem Bollerslev (1986) proposed the generalized ARCH, or GARCH(p,q), model:

$$\sigma_t^2 = a_0 + \sum_{i=1}^{q} \left(a_i \varepsilon_{t-i}^2 \right) + \sum_{j=1}^{p} \left(b_j \sigma_{t-j}^2 \right), \tag{2.5}$$

where $\alpha_0 > 0$, $a_i \geq 0$, for $i = 1,...,q$, and $b_j \geq 0$, for $j = 1,...,p$. If $\sum_{i=1}^{q} a_i + \sum_{j=1}^{p} b_j < 1$, then $\{\varepsilon_t\}$ is covariance stationary and its unconditional variance is equal to $\sigma^2 = a_0 \left(1 - \sum_{i=1}^{q} a_i - \sum_{j=1}^{p} b_j \right)^{-1}$. Note that even though the innovation process for the conditional mean is serially uncorrelated, it is not independent through time. The innovations for the variance are denoted as:

$$E_t\left(\varepsilon_t^2\right) - E_{t-1}\left(\varepsilon_t^2\right) = \varepsilon_t^2 - \sigma_t^2 \equiv v_t. \tag{2.6}$$

The innovation process $\{v_t\}$ is a martingale difference sequence in the sense that it cannot be predicted from its past. However, its range may depend upon the past, making it neither serially independent nor identically distributed.

The unconditional distribution of ε_t has fatter tails than the time invariant distribution of z_t. For example, in the case of the ARCH process in (2.1) with the density function $f(.)$ being the normal distribution and the functional form of σ_t^2 denoted as in the ARCH(1) model, the kurtosis of ε_t is $E\left(\varepsilon_t^4\right) / E\left(\varepsilon_t^2\right)^2 = 3\left(1 - \alpha_1^2\right) / \left(1 - 3\alpha_1^2\right)$ always greater than 3, the kurtosis value of the normal distribution.

The GARCH(p,q) model successfully captures several characteristics of financial time series, such as thick tailed returns and volatility clustering first noted by Mandelbrot (1963): "… large changes tend to be followed by large changes of either sign, and small changes tend to be followed by small changes…". On the other hand, the GARCH structure imposes important limitations. The variance only depends on the magnitude and not on the sign of ε_t, which is somewhat at odds with the empirical behavior of stock market prices where a *leverage effect* may be present. The term *leverage effect*, first noted by Black (1976), refers to the tendency for changes in stock returns to be negatively correlated with changes in returns volatility, i.e. volatility tends to rise in response to *bad news*, $(\varepsilon_t < 0)$, and to fall in response to *good news*, $(\varepsilon_t > 0)$.

In order to capture the asymmetry exhibited by the data, a new class of models was introduced, termed the *asymmetric ARCH models*. The most popular model proposed to capture the asymmetric effects is Nelson's (1991) exponential GARCH, or EGARCH(p,q), model:

$$\ln\left(\sigma_t^2\right) = a_0 + \sum_{i=1}^{q}\left(a_i\left(\left|\frac{\varepsilon_{t-i}}{\sigma_{t-i}}\right| - E\left|\frac{\varepsilon_{t-i}}{\sigma_{t-i}}\right|\right) + \gamma_i\left(\frac{\varepsilon_{t-i}}{\sigma_{t-i}}\right)\right) + \sum_{j=1}^{p}\left(b_j \ln\left(\sigma_{t-j}^2\right)\right). \tag{2.7}$$

The parameter γ_1 allows for the asymmetric effect. If $\gamma_1 = 0$ then a *positive surprise*, $(\varepsilon_t > 0)$, has the same effect on volatility as a *negative surprise*, $(\varepsilon_t < 0)$. Here, the term surprise at time t refers to the unexpected return, which is the rate of return from time $t-1$ to t minus the relevant expected return to the investors, e.g., $\varepsilon_t = y_t - \mu_t$. If $-1 < \gamma_1 < 0$, a positive surprise increases volatility less than a negative surprise. If $\gamma_1 < -1$, a positive surprise actually reduces volatility while a negative surprise increases volatility. For $\gamma_1 < 0$, the leverage effect exists. Because of the logarithmic transformation, the forecasts of the variance are guaranteed to be non-negative. Thus, in contrast to the GARCH model, no restrictions need to be imposed on the model estimation.

The number of possible conditional volatility formulations is vast. The threshold GARCH, or TARCH(p,q), model is one of the widely used models:

$$\sigma_t^{\delta} = a_0 + \sum_{i=1}^{q}\left(a_i d\left(\varepsilon_{t-i} > 0\right)\left|\varepsilon_{t-i}\right|^{\delta} + \gamma_i d\left(\varepsilon_{t-i} \le 0\right)\left|\varepsilon_{t-i}\right|^{\delta}\right) + \sum_{j=1}^{p}\left(b_j \sigma_{t-j}^{\delta}\right), \tag{2.8}$$

where $d(.)$ denotes the indicator function (i.e. $d\left(\varepsilon_{t-i} > 0\right) = 1$ if $\varepsilon_{t-i} > 0$, and $d\left(\varepsilon_{t-i} > 0\right) = 0$ otherwise). Zakoian's (1990) model is a special case of the TARCH model with $\delta = 1$, while Glosten et al. (1993) consider a version of the TARCH model with $\delta = 2$. The TARCH model allows a response of volatility to news with different coefficients for good and bad news.

A wide range of ARCH models proposed in the literature has been reviewed by Bollerslev et al. (1992), Bollerslev et al. (1994), Bera and Higgins (1993), Hamilton (1994) and Gourieroux (1997). Hentschel (1995) considers a complete parametric family of ARCH models. This family nests the most popular symmetric and asymmetric ARCH models, thereby highlighting the relation between the models and their treatment of asymmetry.

3. Model Selection Methods

Most of the methods used in the literature for selecting the appropriate model are based on evaluating the ability of the models to describe the data. Standard model selection criteria such as the Akaike Information Criterion (AIC) (Akaike (1973)) and the Schwarz Bayesian Criterion (SBC) (Schwarz (1978)) have widely been used in the ARCH literature, despite the fact that their statistical properties in the ARCH context are unknown. These are defined in terms of $l_T\left(\hat{\theta}\right)$, the maximized value of the log-likelihood function of a model, where $\hat{\theta}$ is the maximum likelihood estimator of θ based on a sample of size n and $\breve{\theta}$ denotes the dimension of θ, thus:

$$AIC = l_n\left(\hat{\theta}\right) - \breve{\theta} \tag{3.1}$$

$$SBC = l_n\left(\hat{\theta}\right) - 2^{-1}\breve{\theta}\ln(n). \tag{3.2}$$

In addition, the evaluation of loss functions for alternative models is mainly used in model selection. When we focus on estimation of means, the loss function of choice is typically the mean squared error (MSE):

$$MSE = n^{-1}\sum_{t=1}^{n}\varepsilon_t^2 \ . \tag{3.3}$$

When the same strategy is applied to variance estimation, the choice of the mean squared error is much less clear. Because of high non-linearity in volatility models, a number of researchers constructed heteroscedasticity-adjusted loss functions. Bollerslev et al. (1994) present four types of loss functions:

$$L_1 = \sum_{t=1}^{n}\left(\varepsilon_t^2 - \sigma_t^2\right)^2, \tag{3.4}$$

$$L_2 = \sum_{t=1}^{n}\ln\left(\frac{\varepsilon_t^2}{\sigma_t^2}\right)^2, \tag{3.5}$$

$$L_3 = \sum_{t=1}^{n} \frac{\left(\varepsilon_t^2 - \sigma_t^2\right)^2}{\sigma_t^4}, \tag{3.6}$$

$$L_4 = \sum_{t=1}^{n} \left(\frac{\varepsilon_t^2}{\sigma_t^2} + \ln\left(\sigma_t^2\right) \right). \tag{3.7}$$

Pagan and Schwert (1990) used the first two of the loss functions to compare alternative estimators with in-sample and out-of-sample data sets. Andersen et al. (1999) and Heynen and Kat (1994) are some examples from the literature that applied loss functions to compare the forecast performance of various volatility models.

Moreover, loss functions have been constructed, based upon the goals of the particular application. West et al. (1993) developed such a criterion based on the portfolio decisions of a risk averse investor. Engle et al. (1993) assumed that the objective was to price options and developed a loss function from the profitability of a particular trading strategy.

4. Model Selection Based on a Prediction Error Criterion (PEC)

Let us assume that a researcher is interested in evaluating the ability of the ARCH models to forecast the conditional variance. Consider the simple case of a regression model: $y_t = x_t'\beta + \varepsilon_t$ where β is a vector of k unknown parameters to be estimated, x_t is a vector of variables included in the information set at time $t-1$ and $\varepsilon_t \overset{i.i.d.}{\sim} N\left(0, \sigma^2\right)$. At time $t-1$, the expected value μ_t of y_t is estimated on the basis of the information available at time $t-1$, i.e. $\hat{y}_{t|t-1} = \hat{\mu}_t = x_t'\hat{\beta}_{t-1}$, where $\hat{\beta}_{t-1} = \left(\mathbf{X}_{t-1}'\mathbf{X}_{t-1}\right)^{-1}\left(\mathbf{X}_{t-1}'\mathbf{Y}_{t-1}\right)$ is the least square estimator of β at time $t-1$, $\mathbf{Y}_t$ is the $\left(l_t \times 1\right)$ vector of l_t observations on the dependent variable y_t, and $\mathbf{X}_t$ is the $\left(l_t \times k\right)$ matrix of the k variables included in the information set. In a manner of speaking, $\hat{y}_{t|t}$ and $\hat{y}_{t|t-1}$ can be considered as in-sample and out-of-sample forecasts, respectively. In other words, $\hat{y}_{t|t}$ is measured on the basis of I_t, the information set available at time t, while $\hat{y}_{t|t-1}$ is measured on the basis of I_{t-1}, the information set available at time $t-1$.

In the sequel, the density function $f(.)$, in equation (2.1), is assumed to be that of the normal distribution. For an ARCH process being presented as

$$
\begin{gathered}
y_t = x_t'\beta + \varepsilon_t \\
\varepsilon_t \mid I_{t-1} \sim N\left(0, \sigma_t^2\right) \\
\sigma_t^2(\theta) = g\left(\sigma_{t-1}(\theta), \sigma_{t-2}(\theta), \ldots; \varepsilon_{t-1}(\theta), \varepsilon_{t-2}(\theta), \ldots; \upsilon_{t-1}, \upsilon_{t-2}, \ldots\right),
\end{gathered}
$$

and θ being the vector of unknown parameters, let $\hat{z}_{t|t-1} \equiv \hat{\varepsilon}_{t|t-1}\hat{\sigma}_{t|t-1}^{-1}$ denote the standardized one step ahead prediction errors[1]. The vector θ denotes the set of parameters to be estimated for both the conditional mean and the conditional variance. The most commonly used way to model the conditional variance is the GARCH(p,q) process:

$$
\sigma_t^2 = a_{0,t} + \sum_{i=1}^{q}\left(a_{i,t}\varepsilon_{t-i}^2\right) + \sum_{j=1}^{p}\left(b_{j,t}\sigma_{t-j}^2\right),
$$

The parameters, $\left(a_{0,t}, a_{1,t}, \ldots, a_{q,t}, b_{1,t}, \ldots, b_{p,t}\right)$, are indexed by the subscript t to indicate that they may vary with time. The GARCH(p,q) process may be rewritten as:

$$
\sigma_t^2 = \left(u_t', \eta_t', w_t'\right)\left(v_t, \zeta_t, \omega_t\right),
$$

where $u_t' = \left(1, \varepsilon_{t-1}^2, \ldots, \varepsilon_{t-q}^2\right)$, $\eta_t' = 0$, $w_t' = \left(\sigma_{t-1}^2, \ldots, \sigma_{t-p}^2\right)$, $v_t' = \left(a_{0,t}, a_{1,t}, \ldots, a_{q,t}\right)$, $\zeta_t' = 0$, $\omega_t' = \left(b_{1,t}, \ldots, b_{p,t}\right)$.

The vector $\theta_t = \left(\beta_t', v_t', \zeta_t', \omega_t'\right)$ denotes the set of parameters to be estimated for both the conditional mean and the conditional variance at time t.

The residual $\hat{\varepsilon}_{t|t-1} \equiv y_t - \hat{y}_{t|t-1}$ reflects the difference between the forecast and the observed value of the stochastic process. Panaretos et al. (1997) suggested measuring the predictive behaviour of linear regression models on the basis of the standardized distance between the predicted and the observed value of the dependent random variable. The estimate of the standardized distance was defined by:

$$
r_t = \frac{y_t - \hat{y}_{t|t-1}}{\sqrt{V\left(\hat{y}_{t|t-1}\right)}},
$$

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[1] Consider the case of the AR(1)GARCH(1,1) model as defined by equations (2.3) and (2.5), for $\kappa = 1$ and $p = q = 1$, respectively. The estimators of the one step ahead prediction error and its variance conditional on the information set available at time $t-1$ are given by $\hat{\varepsilon}_{t|t-1} = y_t - \hat{c}_{0,t-1} - \hat{c}_{1,t-1}y_{t-1}$ and $\hat{\sigma}_{t|t-1}^2 = \hat{a}_{0,t-1} + \hat{a}_{1,t-1}\hat{\varepsilon}_{t-1|t-1}^2 + \hat{b}_{1,t-1}\hat{\sigma}_{t-1|t-1}^2$, respectively

where $V\left(\hat{y}_{t|t-1}\right)=\left(\mathbf{Y}_{t-1}-\mathbf{X}_{t-1}\hat{\beta}_{t-1}\right)'\left(\mathbf{Y}_{t-1}-\mathbf{X}_{t-1}\hat{\beta}_{t-1}\right)\left(1+x_t\left(\mathbf{X}'_{t-1}\mathbf{X}_{t-1}\right)^{-1}x'_t\right)\left(l_{t-1}-k\right)^{-1}$. A scoring rule to rate the performance of the model at time t for a series of T points in time, $(t=1,...,T)$, was defined by

$$R_T = T^{-1}\sum_{t=1}^{T} r_t^2 ,$$

the average of the squared standardized residuals. In the sequel, this approach is adopted using the average of the squared standardized one step ahead prediction errors as a scoring rule in order to rate the performance of an ARCH model to forecast the conditional variance, in particular,

$$R_T = \frac{\sum_{t=1}^{T} \hat{z}_{t|t-1}^2}{T} . \tag{4.1}$$

$\hat{z}_{t|t-1} \equiv \hat{\varepsilon}_{t|t-1}\hat{\sigma}_{t|t-1}^{-1}$ is the estimated standardized distance between the predicted and the observed value of the dependent random variable, when the conditional standard deviation of the dependent variable given I_{t-1} is defined by an ARCH model, $V(y_t \mid I_{t-1}) \equiv \sigma_t^2$.

<u>Theorem 1:</u> Let (θ_t) denote the vector of unknown parameters to be estimated at time t. Under the assumption of constancy of parameters over time, $(\theta_1)=(\theta_2)=...=(\theta_T)=(\theta)$, the estimated standardized one step ahead prediction errors $\hat{z}_{t|t-1}, \hat{z}_{t+1|t}, ..., \hat{z}_{T|T-1}$ are asymptotically independently standard normally distributed, i.e.,

$$\hat{z}_{t|t-1} \equiv \left(y_t - \hat{y}_{t|t-1}\right)\hat{\sigma}_{t|t-1}^{-1} \overset{i.i.d.}{\sim} N(0,1). \tag{4.2}$$

<u>Proof:</u> To prove the theorem, we need the following lemmas.

<u>Lemma 1</u>: (Slutsky's theorem) (see, e.g. Greene (1997, p.118)): For a continuous function $g(x_n)$ that is not a function of n, $p\lim g(x_n) = g(p\lim x_n)$.

(Here $p\lim$ denotes the limit in probability as $n \to \infty$.)

The following two Lemmas are implications of Slutsky's theorem.

Lemma 2: (see, e.g. Hamilton, 1994, p. 182): Let $\{X_n\}$ denote a sequence of $(T \times 1)$ random vectors with $p\lim X_n = c$, i.e., $X_n \overset{p}{\rightarrow} c$. Let $g(.)$ be a vector-valued function, $g : R^T \rightarrow R^m$, which is continuous at c and does not depend on n. Then $g(X_n) \overset{p}{\rightarrow} g(c)$.

Lemma 3: (see, e.g. Hamilton (1994, p. 182)): Let $\{X_{1n}\}$ denote a sequence of $(T \times T)$ random matrices with $X_{1n} \overset{p}{\rightarrow} C_1$, where C_1 is a non-singular matrix. Let X_{2n} denote a sequence of $(T \times 1)$ random vectors with $X_{2n} \overset{p}{\rightarrow} c_2$, where c_2 is a constant. Then, $(X_{1n})^{-1} X_{2n} \overset{p}{\rightarrow} (C_1)^{-1} c_2$, or $p\lim(X_{1n})^{-1} X_{2n} = (C_1)^{-1} c_2$.

We now prove the following lemma.

Lemma 4: Let $\{X_{in}\}$, for $i = 1,...,T$, denote a sequence of random vectors with $p\lim X_{in} = W_i$, where W_i, $i = 1,...,T$ are independently and identically distributed with some distribution function $F(.)$. Then $p\lim(X_{1n}, X_{2n},..., X_{Tn}) = (W_1, W_2,..., W_T)$, and $X_{1n}, X_{2n},..., X_{Tn}$ are asymptotically independently and identically distributed with distribution function $F(.)$.

Proof of Lemma 4: Let $\tilde{g}(.)$ be a vector-valued real function, $\tilde{g}(.): R^T \rightarrow R^T$:

$$(x_1, x_2,..., x_T) \rightarrow \tilde{g}(x_1, x_2,..., x_T) \equiv (g_1(x_1, x_2,..., x_T), g_2(x_1, x_2,..., x_T),..., g_T(x_1, x_2,..., x_T)).$$

Assume that $\tilde{g}(.)$ is continuous at z_i, $\forall i = 1,...,T$, and does not depend on n.

According to Slutsky's theorem (Lemma 1), for a continuous function $g(x_n)$ that is not a function of n, $p\lim g(x_n) = g(p\lim x_n)$. Thus,

$$p\lim \tilde{g}(X_{1n}, X_{2n},..., X_{Tn}) = (g_1(X_1, X_2,..., X_T), g_2(X_1, X_2,..., X_T),..., g_T(X_1, X_2,..., X_T)).$$

By setting $\tilde{g}(x_1, x_2,..., x_T) = (x_1, x_2,..., x_T)$, (i.e. $g_i(x_1, x_2,..., x_T) = x_i, \forall i = 1,...,T$), and applying Slutsky's theorem we obtain

$$p\lim \tilde{g}(X_{1n}, X_{2n},..., X_{Tn}) \equiv p\lim(X_{1n}, X_{2n},..., X_{Tn}) = \tilde{g}(W_1, W_2,..., W_T) \equiv (W_1, W_2,..., W_T)$$

Let $F_{(X_{1n},X_{2n},...,X_{Tn})}(x_1,x_2,...,x_T)$ denote the joint density distribution of the random variables $X_{1n},X_{2n},...,X_{Tn}$. As convergence in probability implies convergence in distribution, we have

$$\lim_{n\to\infty} F_{(X_{1n},X_{2n},...,X_{Tn})}(x_1,x_2,...,x_T) = F_{(W_1,W_2,...,W_T)}(x_1,x_2,...,x_T) =$$

$$= F_{W_1}(x_1)\cdot F_{W_2}(x_2)\cdot ...\cdot F_{W_T}(x_T) = \lim_{n\to\infty} F_{X_{1n}}(x_1)\cdot \lim_{n\to\infty} F_{X_{2n}}(x_2)\cdot ...\cdot \lim_{n\to\infty} F_{X_{Tn}}(x_T)$$

As the joint density is asymptotically the product of the marginal densities, $X_{1n},X_{2n},...,X_{Tn}$ are asymptotically independently distributed, each with distribution function $F(.)$.

Let us now return to the proof of Theorem 1: At time $t-1$, the expected value of y_t is estimated on the basis of the information available at time $t-1$, i.e. $\hat{y}_{t|t-1} = x_t'\hat{\beta}_{t-1}$ and the expected value of the conditional variance is estimated on the basis of the information available at time $t-1$, i.e. $\hat{\sigma}^2_{t|t-1} = (u_t',\eta_t',w_t')(\hat{v}_{t-1},\hat{\zeta}_{t-1},\hat{\omega}_{t-1})$. Note that the elements of the vector (u_t',η_t',w_t') belong to the I_{t-1}, so are considered as known values. The $\hat{z}_{t|t-1}$ can be written as:

$$\hat{z}_{t|t-1} = \frac{(y_t - \hat{y}_{t|t-1})}{\sqrt{\hat{\sigma}^2_{t|t-1}}} =$$

$$= \frac{(x_t'\beta_t + \varepsilon_t - x_t'\hat{\beta}_{t-1})}{\sqrt{\hat{\sigma}^2_{t|t-1}}} =$$

$$= \frac{\varepsilon_t}{\sqrt{\hat{\sigma}^2_{t|t-1}}} + \frac{(x_t'(\beta_t - \hat{\beta}_{t-1}))}{\sqrt{\hat{\sigma}^2_{t|t-1}}} =$$

$$= \frac{z_t\sqrt{\sigma_t^2}}{\sqrt{\hat{\sigma}^2_{t|t-1}}} + \frac{(x_t'(\beta_t - \hat{\beta}_{t-1}))}{\sqrt{\hat{\sigma}^2_{t|t-1}}} =$$

$$= \frac{z_t((u_t',\eta_t',w_t')(v_t,\zeta_t,\omega_t))^{1/2}}{((u_t',\eta_t',w_t')(\hat{v}_{t-1},\hat{\zeta}_{t-1},\hat{\omega}_{t-1}))^{1/2}} + \frac{(x_t'(\beta_t - \hat{\beta}_{t-1}))}{((u_t',\eta_t',w_t')(\hat{v}_{t-1},\hat{\zeta}_{t-1},\hat{\omega}_{t-1}))^{1/2}}$$

We assume that a sample of n observations has been used to estimate the vector of unknown parameters. According to Bollerslev (1986), the maximum likelihood estimate $\hat{\theta}_t$ is strongly consistent for θ_t and asymptotically normal with mean θ_t. In other words,

$p\lim\left(\hat{\theta}_t\right)=\theta_t \Leftrightarrow p\lim\left(\hat{\beta}_t',\hat{v}_t',\hat{\zeta}_t',\hat{\omega}_t'\right)=\left(\beta_t',v_t',\zeta_t',\omega_t'\right)$, where $p\lim$ denotes limit in probability as the size of the sample, n, goes to infinity. According to Lemma 2:

$$p\lim\left(\hat{z}_{t|t-1}\right)=$$

$$=p\lim\left(\frac{z_t\left(\left(u_t',\eta_t',w_t'\right)\left(v_t,\zeta_t,\omega_t\right)\right)^{1/2}}{\left(\left(u_t',\eta_t',w_t'\right)\left(\hat{v}_{t-1},\hat{\zeta}_{t-1},\hat{\omega}_{t-1}\right)\right)^{1/2}}\right)+p\lim\left(\frac{\left(x_t'\left(\beta_t-\hat{\beta}_{t-1}\right)\right)}{\left(\left(u_t',\eta_t',w_t'\right)\left(\hat{v}_{t-1},\hat{\zeta}_{t-1},\hat{\omega}_{t-1}\right)\right)^{1/2}}\right)=$$

Then, based on Lemma 3:

$$=\frac{z_t\left(\left(u_t',\eta_t',w_t'\right)\left(v_t,\zeta_t,\omega_t\right)\right)^{1/2}}{\left(u_t',\eta_t',w_t'\right)\left(p\lim\left(\hat{v}_{t-1},\hat{\zeta}_{t-1},\hat{\omega}_{t-1}\right)\right)^{1/2}}+\frac{\left(x_t'p\lim\left(\beta_t-\hat{\beta}_{t-1}\right)\right)}{\left(p\lim\left(\left(u_t',\eta_t',w_t'\right)\left(\hat{v}_{t-1},\hat{\zeta}_{t-1},\hat{\omega}_{t-1}\right)\right)\right)^{1/2}}=$$

$$=\frac{z_t\left(\left(u_t',\eta_t',w_t'\right)\left(v_t,\zeta_t,\omega_t\right)\right)^{1/2}}{\left(u_t',\eta_t',w_t'\right)\left(\left(v_t,\zeta_t,\omega_t\right)\right)^{1/2}}+\frac{\left(x_t'p\lim\left(\beta_t-\hat{\beta}_{t-1}\right)\right)}{\left(\left(u_t',\eta_t',w_t'\right)p\lim\left(\hat{v}_{t-1},\hat{\zeta}_{t-1},\hat{\omega}_{t-1}\right)\right)^{1/2}}=$$

$$=z_t+\frac{\left(x_t'\right)(0)}{\left(\left(u_t',\eta_t',w_t'\right)\left(v_t,\zeta_t,\omega_t\right)\right)^{1/2}}=$$

$$=z_t$$

As convergence in probability implies convergence in distribution, the $\hat{z}_{t|t-1},\hat{z}_{t+1|t},\ldots,\hat{z}_{T|T-1}$ are asymptotically standard normally distributed:

$$\hat{z}_{t|t-1}\xrightarrow{p}z_t \Rightarrow \hat{z}_{t|t-1}\xrightarrow{d}z_t\sim N(0,1)$$

This result, combined with Lemma 4, implies that the $\hat{z}_{t|t-1},\hat{z}_{t+1|t},\ldots,\hat{z}_{T|T-1}$ are asymptotically independently standard normally distributed, i.e.,

$$\hat{z}_{t|t-1}\xrightarrow{d}z_t\overset{i.i.d.}{\sim}N(0,1).$$

Hence, the theorem has been established.

The result of the theorem is valid for all the conditional variance functions with consistent estimators of the parameters.

Remark: As concerns the EARCH and the TARCH models, the maximum likelihood estimator $\hat{\theta}_t=\left(\hat{\beta}_t',\hat{v}_t',\hat{\zeta}_t',\hat{\omega}_t'\right)$ is consistent and asymptotically normal.

Consider the EGARCH(p,q) model in the following form

$$\ln\left(\sigma_t^2\right) = a_{0,t} + \sum_{i=1}^{q}\left(a_{i,t}\left|\frac{\varepsilon_{t-i}}{\sigma_{t-i}}\right| + \gamma_{i,t}\left(\frac{\varepsilon_{t-i}}{\sigma_{t-i}}\right)\right) + \sum_{i=1}^{p}\left(b_{i,t}\ln\left(\sigma_{t-i}^2\right)\right)$$

which can be written as:

$$\ln\sigma_t^2 = \left(u_t', \eta_t', w_t'\right)\left(v_t, \zeta_t, \omega_t\right)$$

where $u_t' = \left(1, \left|\varepsilon_{t-1}/\sigma_{t-1}\right|, \ldots, \left|\varepsilon_{t-q}/\sigma_{t-q}\right|\right)$, $\eta_t' = \left(\left[\varepsilon_{t-1}/\sigma_{t-1}\right], \ldots, \left\lfloor\varepsilon_{t-q}/\sigma_{t-q}\right\rfloor\right)$,

$w_t' = \left(\ln\sigma_{t-1}^2, \ldots, \ln\sigma_{t-p}^2\right)$, $v' = \left(a_{0,t}, a_{1,t}, \ldots, a_{q,t}\right)$, $\zeta' = \left(\gamma_{1,t}, \ldots, \gamma_{q,t}\right)$, $\omega' = \left(b_{1,t}, \ldots, b_{p,t}\right)$

The parameters, $\left(a_{0,t}, a_{1,t}, \ldots, a_{q,t}, \gamma_{1,t}, \ldots, \gamma_{q,t}, b_{1,t}, \ldots, b_{p,t}\right)$, are indexed by the subscript t to indicate that they may vary with time. According to Nelson (1991), under sufficient regularity conditions, the maximum likelihood estimator $\hat{\theta}_t = \left(\hat{\beta}_t', \hat{v}_t', \hat{\zeta}_t', \hat{\omega}_t'\right)$ is consistent and asymptotically normal. Also, for the TARCH(p,q) process, the conditional variance can take the form:

$$\sigma_t^2 = a_{0,t} + \sum_{i=1}^{q}\left(a_{i,t}\varepsilon_{t-i}^2\right) + \gamma_t\varepsilon_{t-1}^2 d_{t-1} + \sum_{i=1}^{p}\left(b_{i,t}\sigma_{t-i}^2\right),$$

which can be written as:

$\sigma_t^2 = \left(u_t', \eta_t', w_t'\right)\left(v_t, \zeta_t, \omega_t\right)$

where $u_t' = \left(1, \varepsilon_{t-1}^2, \ldots, \varepsilon_{t-q}^2\right)$, $\eta_t' = \left(d_{t-1}\varepsilon_{t-1}^2\right)$, $w_t' = \left(\sigma_{t-1}^2, \ldots, \sigma_{t-p}^2\right)$, $v' = \left(a_{0,t}, a_{1,t}, \ldots, a_{q,t}\right)$,

$\zeta' = \left(\gamma_t\right)$, $\omega' = \left(b_{1,t}, \ldots, b_{p,t}\right)$, $d_t = 1$ if $\varepsilon_t < 0$, and $d_t = 0$ otherwise.

As pointed out by Glosten et al. (1993), as long as the conditional mean and variance are correctly specified, the maximum likelihood estimates will be consistent and asymptotically normal.

According to Lemma 1, if $p\lim\hat{z}_{t|t-1} = z_t \sim N(0,1)$ and $g\left(\hat{z}_{t|t-1}\right) = \sum_{t=1}^{T}\left(\hat{z}_{t|t-1}^2\right)$, which is a continuous function, then $p\lim\sum_{t=1}^{T}\left(\hat{z}_{t|t-1}^2\right) = \sum_{t=1}^{T}\left(z_t^2\right)$. As convergence in probability implies convergence in distribution, $\sum_{t=1}^{T}\left(\hat{z}_{t|t-1}^2\right) \xrightarrow{d} \sum_{t=1}^{T}\left(z_t^2\right) \sim \chi_T^2$. Hence, as $\hat{z}_{t|t-1}$ are asymptotically standard normal variables, the variable TR_T is asymptotically χ^2 distributed with T degrees of freedom, i.e.,

$$TR_T \overset{d}{\to} \chi_T^2 . \tag{4.3}$$

Also, for two processes A and B with T_1 and T_2 observations, respectively, the ratio of the scoring rules $R_{T_1}^{(A)} \equiv T_1^{-1}\sum_{t=1}^{T}\hat{z}_{t|t-1}^{(A)2}$ and $R_{T_2}^{(B)} \equiv T_2^{-1}\sum_{t=1}^{T}\hat{z}_{t|t-1}^{(B)2}$ is F distributed with T_1 and T_2 degrees of freedom, i.e.,

$$R_{T_1T_2} \equiv \frac{R_{T_1}^{(A)}}{R_{T_2}^{(B)}} \sim F_{T_1,T_2} , \tag{4.4}$$

if $R_{T_1}^{(A)}$ and $R_{T_2}^{(B)}$ are independently distributed.

According to Kibble (1941), if, for $t = 1,2,...,T$, $\hat{z}_{t|t-1}^{(A)}$ and $\hat{z}_{t|t-1}^{(B)}$ are standard normally distributed variables, following jointly the bivariate standard normal distribution, then the joint distribution of $\left(R_T^{(A)}, R_T^{(B)}\right)$ has a bivariate gamma distribution with probability density function (p.d.f) given by:

$$f\left(R_T^{(A)}, R_T^{(B)}\right) = \frac{\exp\left(-\frac{R_T^{(A)} + R_T^{(B)}}{1-\rho^2}\right)}{\Gamma(T/2)\left(1-\rho^2\right)^{T/2}} \sum_{i=0}^{\infty}\left(\frac{\left(\rho/\left(1-\rho^2\right)\right)^{2i}}{\Gamma(i+1)\Gamma\left(i+(T/2)\right)}\left(R_T^{(A)}R_T^{(B)}\right)^{(T/2)-1-i}\right), \tag{4.5}$$

where $\Gamma(.)$ is the gamma function and ρ is the correlation coefficient between $\hat{z}_{t|t-1}^{(A)}$ and $\hat{z}_{t|t-1}^{(B)}$, $\rho \equiv Cor\left(\hat{z}_{t|t-1}^{(A)}, \hat{z}_{t|t-1}^{(B)}\right)$. Panaretos et al. (1997) showed that, when the joint distribution of $\left(R_T^{(A)}, R_T^{(B)}\right)$ is Kibble's bivariate gamma, the distribution of the ratio $Z_T^{(A,B)} \equiv R_T^{(A)}\big/R_T^{(B)}$ is defined by the following p.d.f.:

$$f_{Z_T^{(A,B)}}\left(Z_T^{(A,B)}\right) = \frac{\left(1-\rho^2\right)^{T/2}}{B\left(T/2, T/2\right)} Z_T^{(A,B)T/2-1}\left(1+Z_T^{(A,B)}\right)^{-T}\left[1-\left(\frac{2\rho}{Z_T^{(A,B)}+1}\right)^2 Z_T^{(A,B)}\right]^{-\frac{T+1}{2}} , \tag{4.6}$$

where $B\left(\frac{T}{2},\frac{T}{2}\right) = \Gamma\left(\frac{T}{2}\right)^2 \Big/ \Gamma(T)$.

$$Z_T^{(A,B)} \equiv \sum_{t=1}^{T}\hat{z}_{t|t-1}^{2(B)} \Big/ \sum_{t=1}^{T}\hat{z}_{t|t-1}^{2(A)} \sim CGR(k,\rho), \tag{4.7}$$

where $k = T/2$. Panaretos et al. (1997) referred to the distribution in (4.6) as the Correlated gamma ratio (CGR) distribution. (A sample of tables of its percentage points and of graphs depicting its probability density function is given in the Appendix).

As pointed out by Panaretos et al. (1997), $R_T^{(A)}$ and $R_T^{(B)}$ could represent the sum of the squared standardized prediction errors from two regression models (not necessarily nested) but with a common dependent variable. Thus, two regression models can be compared through testing a null hypothesis of equivalence of the models in their predictability against the alternative that model (A) produces "better" predictions. Here, the notion of the equivalence of two models with respect to their predictive ability is considered in Panaretos et al.'s (1997) sense to be defined implicitly through their mean squared prediction errors. Following Panaretos et al.'s (1997) rationale, the closest description of the hypothesis to be tested is

H_0: Models A and B have equal mean squared prediction errors

Versus

H_1: Model A has lower mean squared prediction error than model B

using $Z_T^{(A,B)}$ as a test statistic, i.e., using the ratio of the sum of the squared standardized one step ahead prediction errors $\hat{z}_{t|t-1}$ of the two competing models.

The null hypothesis is rejected if $Z_T^{(A,B)} > CGR(k,\rho,a)$, where $CGR(k,\rho,a)$ is the $100(1-a)$ percentile of the CGR distribution.

In the case of independence between $R_T^{(A)}$ and $R_T^{(B)}$, the CGR density function reduces to the form:

$$f_{Z_T^{(A,B)}}\left(Z_T^{(A,B)}\right)=\frac{1}{B\left(T/2,T/2\right)}Z_T^{(A,B)T/2-1}\left(1+Z_T^{(A,B)}\right)^{-T}, \tag{4.8}$$

which is the p.d.f. of the F distribution with T and T degrees of freedom.

Since very few financial time series have a constant conditional mean of zero, in order to estimate the conditional variance, the conditional mean should have been defined. Thus, both the conditional mean and variance are estimated simultaneously. According to the PEC model selection algorithm, the models that are considered as having a "better" ability to predict future values of the dependent variable, are those with the lowest sum of squared standardized one-step-ahead prediction errors. It becomes evident, therefore, that these models can potentially be regarded as the most appropriate to use for volatility forecasts too.

5. Empirical Results

The suggested model selection procedure is illustrated on data referring to the daily returns of the Athens Stock Exchange (ASE) index. Let $y_t = \ln\left(P_t / P_{t-1}\right)$ denote the continuously compound rate of return from time $t-1$ to t, where P_t is the ASE closing price at time t. The data set covers the period from August 30th, 1993 to November 4th, 1996, a total of 800 trading days. Table 1 presents the descriptive statistics. For an estimated kurtosis equal to 7.25 and an estimated skewness equal to 0.08, the distribution of returns is flat (platykurtic) and has a long right tail relative to the normal distribution. The Jarque Bera (JB) statistic (Jarque and Bera (1980)) is used to test whether the series is normally distributed. The test statistic measures the difference of the skewness and kurtosis of the series from those of the normal distribution. The JB statistic is computed as:

$$JB = n\left(S^2 + \left((K-3)^2 / 4\right)\right) / 6, \tag{5.1}$$

where n is the number of observations, S is the skewness and K is the kurtosis. Under the null hypothesis of a normal distribution, the JB statistic is χ^2 distributed with 2 degrees of freedom.

Table (1). Descriptive Statistics of the daily returns of the ASE index

(30th August 1993 to 4th November 1996 (800 observations))

Observations	800
Mean	5.72E-05
Median	-0.00018
Standard Deviation	0.012
Skewness	0.08
Kurtosis	7.25
Jarque Bera (JB)	602.38
probability	<0.000001
Augmented Dickey Fuller (ADF)	-12.67
1% critical value	-3.44
Phillips Perron (PP)	-24.57
1% critical value	-3.44

The skewness of a symmetric distribution, as the normal distribution, is zero. Positive skewness implies that the distribution has a long right tail. Negative skewness implies a long left tail distribution.

The kurtosis of the normal distribution is 3. If the kurtosis exceeds 3, the distribution is peaked (leptokurtic) relative to the normal. If the kurtosis is less than 3, the distribution is flat (platykurtic) relative to the normal.

Under the null hypothesis of a normal distribution, the JB statistic is χ^2 distributed with 2 degrees of freedom. The reported probability is the probability that the JB statistic exceeds, in absolute value, the observed value under the null hypothesis.

ADF: The null hypothesis of non-stationarity is rejected if the ADF value is less than the critical value. (4 lagged differences).

PP: The null hypothesis of non-stationarity is rejected if the PP value is less than the critical value. (4 truncation lags).

From Table 1, the value of the JB statistic obtained is 602.38 with a very low p-value (practically zero). So, the null hypothesis of normality is rejected. In order to determine whether $\{y_t\}$ is a stationary process, the Augmented Dickey Fuller test (ADF) (Dickey and Fuller (1979)) and the nonparametric Phillips Perron (PP) test (Phillips (1987), Phillips and Perron (1988)) are conducted.

The ADF test examines the null hypothesis, $H_0 : \gamma = 0$, versus the alternative, $H_1 : \gamma < 0$, in the following regression:

$$\Delta y_t = c + \gamma y_{t-1} + \sum_{i=1}^{\kappa} \varphi_i \Delta y_{t-i} + \varepsilon_t, \tag{5.2}$$

where Δ denotes the difference operator. According to the ADF test, the null hypothesis of non-stationarity is rejected at the 1% level of significance for any lag order up to $\kappa = 12$. The test regression for the PP test is the AR(1) process:

$$\Delta y_t = c + \gamma y_{t-1} + \varepsilon_t. \tag{5.3}$$

While the ADF test corrects for higher order serial correlation by adding lagged differenced terms on the right hand side, the PP test makes a correction to the t statistic of the γ coefficient from the AR(1) regression to account for the serial correlation in ε_t. The correction is nonparametric since an estimate of the spectrum of ε_t at frequency zero, that is robust to heteroscedasticity and autocorrelation of unknown form, is used. According to the PP test, the null hypothesis is also rejected at the 1% level of significance.

Table (2). Lagrange multiplier (LM) test. Test the null hypothesis of no ARCH effects in the residuals up to order q.

$$\varepsilon_t^2 = \beta_0 + \sum_{i=1}^{q} \beta_i \varepsilon_{t-i}^2 + u_t$$

$$\varepsilon_t = y_t - c$$

Q	LM statistic	p-value
1	108.203	0.00
2	113.315	0.00
3	127.947	0.00
4	128.577	0.00
5	130.691	0.00
6	133.467	0.00
7	131.573	0.00
8	129.496	0.00

The LM statistic is computed as the number of observations times the R^2 from the auxiliary test regression. It converges in distribution to a χ^2_q.

The most commonly used test, for examining the null hypothesis of homoscedasticity against the alternative hypothesis of heteroscedasticity, is Engle's (1982) Lagrange multiplier (LM) test. The ARCH LM test statistic is computed from an auxiliary test regression. To test the null hypothesis of no ARCH effects up to order q in the residuals, the regression model

$$\varepsilon_t^2 = \beta_0 + \sum_{i=1}^{q} \beta_i \varepsilon_{t-i}^2 + u_t , \tag{5.4}$$

with $\varepsilon_t = y_t - c$ is run. Engle's test statistic is computed as the product of the number of observations times the value of the coefficient of variation R^2 of the auxiliary test regression. From Table 2, the values of the LM test statistic for $q = 1,\ldots,8$ are highly significant at any reasonable level.

As, according to the results of the above tests, the assumptions of stationarity and ARCH effects seem to be plausible for the process $\{y_t\}$ of daily returns, several ARCH models are considered in the sequel. It is assumed, specifically, that the conditional mean is considered as a κ^{th} order autoregressive process:

$$\begin{gathered} y_t = \mu_t + z_t \sigma_t \\ \mu_t = c_0 + \sum_{i=1}^{\kappa} \left(c_i y_{t-i} \right) \\ z_t \overset{i.i.d.}{\sim} N(0,1), \end{gathered} \tag{5.5}$$

and the conditional variance σ_t^2 is assumed to be related to lagged values of ε_t and σ_t according to a GARCH(p,q) model, an EGARCH(p,q) model or a TARCH(p,q) model. In particular, σ_t^2 is assumed to be determined by one of the following models:

The GARCH(p,q) model

$$\sigma_t^2 = a_0 + \sum_{i=1}^{q} \left(a_i \varepsilon_{t-i}^2 \right) + \sum_{j=1}^{p} \left(b_j \sigma_{t-j}^2 \right) \tag{5.6}$$

The EGARCH(p,q) model

$$\ln\left(\sigma_t^2\right) = a_0 + \sum_{i=1}^{q} \left(a_i \left| \frac{\varepsilon_{t-i}}{\sigma_{t-i}} \right| + \gamma_i \left(\frac{\varepsilon_{t-i}}{\sigma_{t-i}} \right) \right) + \sum_{j=1}^{p} \left(b_j \ln\left(\sigma_{t-j}^2\right) \right) \tag{5.7}$$

The TARCH(p,q) model

$$\sigma_t^2 = a_0 + \sum_{i=1}^{q}\left(a_i \varepsilon_{t-i}^2\right) + \gamma\varepsilon_{t-1}^2 d_{t-1} + \sum_{j=1}^{p}\left(b_j \sigma_{t-j}^2\right), \tag{5.8}$$

where $d_t = 1$ if $\varepsilon_t < 0$, and $d_t = 0$ otherwise. Thus, the AR(κ)GARCH(p, q), AR(κ)EGARCH(p, q) and AR(κ)TARCH(p, q) models are applied, for $\kappa = 0,\ldots,4$, $p = 0,1,2$ and $q = 1,2$, yielding a total of 90 cases.

Since, in estimating non-linear models, no closed form expressions are obtainable for the parameter estimators, an iterative method has to be employed. The value of the parameter vector θ that maximizes $l_t(\theta)$, the log likelihood contribution for each observation t, is to be found. Iterative optimization algorithms work by starting with an initial set of values for the parameter vector θ, say $\theta^{(0)}$, and obtaining a set of parameter values $\theta^{(1)}$ which corresponds to a higher value of $l_t(\theta)$. This process is repeated until the objective function $l_t(\theta)$ no longer improves between iterations. In the sequel, the Marquardt algorithm (Marquardt (1963)) is used. This algorithm modifies the Berndt, Hall, Hall and Hausman, or BHHH, algorithm (Berndt et al. (1974)) by adding a correction matrix to the Hessian approximation (i.e., to the sum of the outer product of the gradient vectors for each observation's contribution to the objective function). The Marquardt updating algorithm is computed as:

$$\theta^{(i+1)} = \theta^{(i)} + \left(\sum_{t=1}^{n}\frac{\partial l_t^{(i)}}{\partial\theta}\frac{\partial l_t^{(i)}}{\partial\theta'} - aI\right)^{-1}\sum_{t=1}^{n}\frac{\partial l_t^{(i)}}{\partial\theta}, \tag{5.9}$$

where I is the identity matrix and a is a positive number chosen by the algorithm. The effect of this modification is to push the parameter estimates in the direction of the gradient vector. The idea is that when we are far from the maximum, the local quadratic approximation to the function may be a poor guide to its overall shape, so it may be better off to simply follow the gradient. The correction may provide a better performance at locations far from the optimum, and allows for computation of the direction vector in cases where the Hessian is near singular.

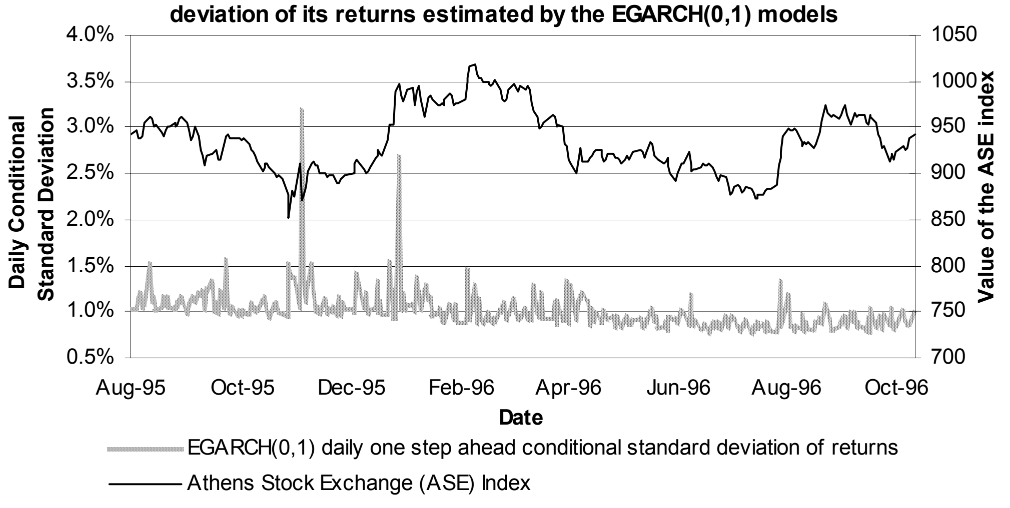
The quasi-maximum likelihood estimator (QMLE) is used, as according to Bollerslev and Wooldridge (1992), it is generally consistent, has a limiting normal distribution and provides asymptotic standard errors that are valid under non-normality.

In order to compute the sum of squared standardized one step ahead prediction errors, a rolling sample of constant size equal to 500 is used, or $s = 500$, so 300 one step ahead daily forecasts are estimated. The out-of-sample data set is split into 5 subperiods and the PEC model selection algorithm is applied in each subperiod separately. Thus, the model selection is revised every 60 trading days and the information set includes daily continuously compound returns of the two most recently years, or 500 trading days. The choice of a 60 day length for each subperiod is arbitrary. The sum of the squared one step ahead prediction errors, $\sum_{t=s+1}^{T+s}\left(\hat{z}^{2}_{t|t-1}\right)$, is estimated for each model and presented in Table 3, in the Appendix. The models selected for each subperiod and their sums of the squared standardized one step ahead prediction errors are:

	Subperiod	Model Selected	$\min\left(\sum_{t=s+1}^{T+s}\left(\hat{z}^{2}_{t\|t-1}\right)\right)$
1.	25 August 1995 - 16 November 1995	AR(2) EGARCH(0,1)	21.961
2.	17 November 1995 - 13 February 1996	AR(0) EGARCH(0,1)	76.315
3.	14 February 1996 - 14 May 1996	AR(0) EGARCH(0,1)	42.176
4.	15 May 1996 – 8 August 1996	AR(3) EGARCH(0,1)	27.308
5.	9 August 1996 - 4 November 1996	AR(1) EGARCH(0,1)	43.920

According to the PEC selection method, the exponential GARCH(0,1) model describes best the conditional variance for the total examined period of 300 trading days. It is selected by the PEC selection method in each subperiod. Figure 1 shows the daily value of the ASE index and the one step ahead conditional standard deviation of its returns.

Figure 1. The ASE index and the one step ahead conditional standard deviation of its returns estimated by the EGARCH(0,1) models



Despite the fact that an asymmetric model is selected by the PEC algorithm, there are no asymmetries in the ASE index volatility. According to Figure 1, the major episodes of high volatility are not associated with market changes of the same sign. Figure 2 presents the values of the parameters a_1 and γ_1 of the 300 estimated EGARCH(0,1) models, while Figure 3 depicts the relevant standard errors for the parameters a_1 and γ_1. Obviously, the γ_1 parameter, which allows for the asymmetric effect, is positive but statistically insignificant. Therefore, the asymmetric relation between returns and changes in volatility does not characterize the examined period.

An interesting point is that the higher order of the conditional mean autoregressive process is chosen as adequate to produce more accurate predictions for the first and the fourth subperiods. As concerns the first subperiod, the AR(2)EGARCH(0,1) model

$$y_t = c_0 + c_1 y_{t-1} + c_2 y_{t-2} + \varepsilon_t$$
$$\ln\left(\sigma_t^2\right) = a_0 + a_1 \left|\frac{\varepsilon_{t-1}}{\sigma_{t-1}}\right| + \gamma_1 \left(\frac{\varepsilon_{t-1}}{\sigma_{t-1}}\right), \tag{5.10}$$

is the one with the lowest value of $\sum_{t=501}^{560}\left(\hat{z}_{t|t-1}^{2}\right)$ equal to 21.961. The hypothesis:

H$_0$: The model AR(2)EGARCH(0,1) has equivalent predictive ability to model X is tested versus

H$_1$: The model AR(2)EGARCH(0,1) produces “better” predictions than model X, with X denoting any one of the remainder models.

Figure 2. The parameters of the estimated EGARCH(0,1) models

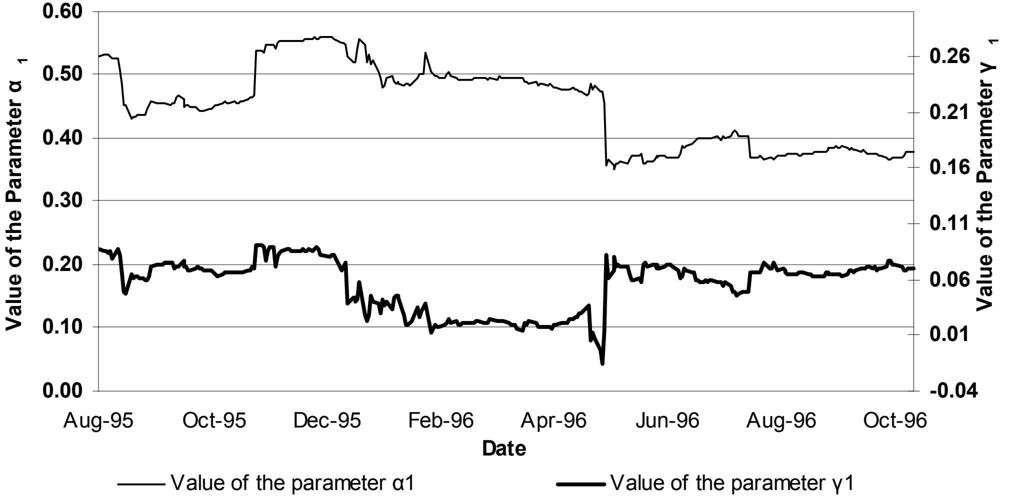
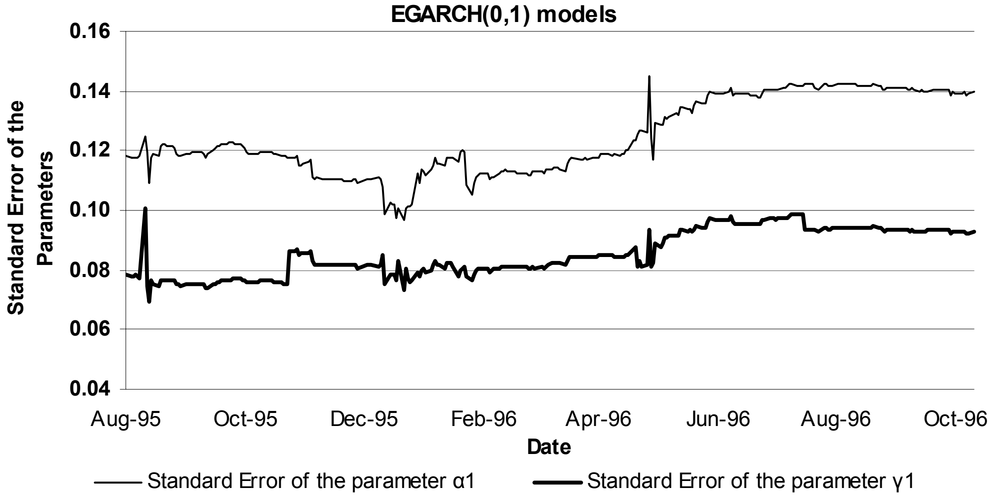


Figure 3. The standard error for the parameters of the estimated EGARCH(0,1) models



Note that the correlation between the standardized one step ahead prediction errors is greater than 0.9 in each case. If $Z_{60}^{AR(2)EGARCH(0,1),X} \equiv (21.96)^{-1}\sum_{t=501}^{560}\hat{z}_{t|t-1}^{(X)2} > CGR\left(\rho > 0.9, T/2 = 30, a\right)$, the null hypothesis of equivalent predictive ability of the models is rejected at $100a\%$ level of significance and the AR(2)EGARCH(0,1) model is regarded as “better” than model X. Table 4, in the Appendix, summarizes the results of the hypothesis tests, for each subperiod.

Figure 4, in the Appendix, depicts the one step ahead 95 per cent prediction intervals for the models with the lowest $\sum_{t=s+1}^{T+s}\left(\hat{z}_{t|t-1}^{2}\right)$ in each subperiod. The prediction intervals are constructed as the expected rate of return plus\minus 1.96 times the conditional standard deviation, both measurable to $t-1$ information set: $\hat{\mu}_{t|t-1} \pm 1.96\hat{\sigma}_{t|t-1}$. So, each time next day’s prediction interval is plotted, only information available at current day is used. Remark that around November 1995, a volatile period, the prediction interval in Figure 4 tracked the movement of the returns quite closely (seven outliers, or 2.33%, were observed).

6. An Alternative Approach

In this section an in-sample analysis is performed in order to select the appropriate models describing the data. Then, the selected models are used to estimate the one step ahead forecasts. Having assumed that the conditional mean of the returns follows a κ^{th}

order autoregressive process, as in (2.3), Richardson and Smith (1994) developed a test for autocorrelation. It is a robust version of the standard Box Pierce (Box and Pierce (1970)) procedure. For p_i denoting the estimated autocorrelation between the returns at time t and $t-i$, the test is formulated as:

$$RS(r) = n\sum_{i=1}^{r}\frac{p_i^2}{1+c_i}, \tag{6.1}$$

where n is the sample size and c_i is the adjustment factor for heteroscedasticity, which is calculated as:

$$c_i = \frac{Cov\left(\bar{y}_t^2, \bar{y}_{t-i}^2\right)}{Var(y_t)^2}, \tag{6.2}$$

where $\bar{y}_t = y_t - n^{-1}\sum_{t=1}^{n} y_t$. Under the null hypothesis of no autocorrelation, the statistic is asymptotically distributed as χ^2 with r degrees of freedom. If the null hypothesis of no autocorrelation cannot be rejected, then the returns' process is equal to a constant plus the residuals, ε_t. In other words, $\{y_t\}$ follows the AR(0) process. If the null of no autocorrelation is rejected, then $\{y_t\}$ follows the AR(1) process. In order to test for the existence of a higher order autocorrelation, the test is applied on the estimated residuals from the AR(1) model. In this case, the statistic, under the null hypothesis, is asymptotically distributed as χ^2 with $r-1$ degrees of freedom. The test is calculated on 7 autocorrelations $(r=7)$ for 800 observations yielding a value equal to $RS(7) = 14{,}86 > \chi^2_{7,0.05}$. As the null hypothesis of no autocorrelation is rejected the test is run on the estimated residuals from the AR(1) model that gives $RS(6) = 12{,}33 < \chi^2_{6,0.05}$. Thus, a first order autocorrelation is detected for the returns' process. Note that the AR(1) form allows for the autocorrelation imposed by discontinuous trading.

Having defined the conditional mean equation, the next step is the estimation of the conditional variance function. The AIC and the SBC criteria are used to select the appropriate conditional variance equation. Note that the AIC mainly chooses as best the less parsimonious model. Also, under certain regularity conditions, the SBC is consistent, in the sense that for large samples it leads to the correct model choice, assuming the "true" model does belong to the set of models examined. Thus, the SBC may be preferable to use. As concerns the specific dataset, both the AIC and SBC select the

GARCH(1,1) model as the most appropriate function to describe the conditional variance. So, performing an in-sample analysis the AR(1)GARCH(1,1) model is regarded as the most suitable, which is the model applied in most researches. Figure 5, in the Appendix, presents the in-sample 95 per cent confidence interval for the AR(1)GARCH(1,1) model. There are fourteen observations, or 4.66%, outside the confidence interval.

In order to compare the model selection methods, the choice of the models should be conducted at the same time points. Thus, the Richardson Smith test for autocorrelation detection and the information criteria for model selection are used in each subperiod separately. The models selected for in each subperiod are:

Subperiod	Richardson Smith Model selection	SBC Model Selection	AIC Model Selection
1.	AR(3)	GARCH(1,1)	EGARCH(1,2)
2.	AR(2)	GARCH(2,1)	GARCH(2,1)
3.	AR(0)	GARCH(1,1)	GARCH(1,1)
4.	AR(0)	GARCH(1,1)	GARCH(1,1)
5.	AR(0)	GARCH(1,1)	TARCH(1,1)

Based on Table 4, the hypothesis that the model selected by the in-sample analysis is equivalent to the model with minimum value of $\sum_{t=s+1}^{T+s}\left(\hat{z}_{t|t-1}^{2}\right)$ is rejected in the majority of the cases.

Proceeding as in the previous section, the one step ahead prediction intervals, for the models selected in each subperiod, are created. As in section 5, next day's prediction is based only on information available at current day. Figures 6 and 7, in the Appendix, present the one step ahead 95 per cent prediction intervals for the models selected by the SBC and AIC, respectively. There are thirteen observations, or 4.33%, outside the prediction interval for the models selected by the SBC, whereas there are fourteen outliers, or 4.66%, for the models selected by the AIC. Therefore, the importance of selecting a conditional variance model based on its ability to forecast and not on fitting the data gains a lead over. Of course, the construction of the prediction intervals is a naïve way to examine the accuracy of our method's predictability.

7. Conclusion

An alternative model selection approach, based on the CGR distribution, was introduced. Instead of being based on evaluating the ability of the models to describe the data (Akaike information and Schwarz Bayesian criteria), the proposed approach is based on evaluating the ability of the models to predict the conditional variance. The method was applied to 800 daily returns of the ASE index, a dataset covers the period from August 30th, 1993 to November 4th, 1996. The first s observations were used to estimate the one step ahead prediction of the conditional mean and variance at $s+1$. For $s=500$, a total of 300 one step ahead predictions of the conditional mean and variance were obtained. The out-of-sample data set were split to 5 subperiods and the PEC model selection algorithm were applied in each subperiod separately. Thus, the model selection was revised every 60 trading days.

The idea of "jumping" from one model to another, as stock market behavior alters, is introduced. The transition from one model to another is done according to the PEC model selection algorithm. Each time the model selection method is applied, the model is used to predict the conditional variance is revised. Of course, the idea of switching from one regime to another has been already applied to the class of switch regime ARCH models introduced by Cai (1994) and Hamilton and Susmel (1994) and extended by several authors such as Dueker (1997) and Hansen (1994). However, these models allow the parameters of a specific ARCH model to come from one of several different regimes, with transitions between regimes governed by an unobserved Markov chain.

Using an alternative approach, based on evaluating the ability of fitting the data, the conditional mean is first modeled and subsequently, an appropriate form for the conditional variance is chosen. Applying the PEC model selection algorithm, the null hypothesis, that the model selected by the in-sample analysis is equivalent to the model with minimum value of $\sum_{t=s+1}^{T+s}\left(\hat{z}_{t|t-1}^{2}\right)$, is rejected in the plurality of the cases at less than 5% level of significance. The in-sample model selection methods and the predictability-based method do not coincide in the sifting of the appropriate conditional variance model. Moreover, 2.33% and 4.33% of the data were outside the $\hat{\mu}_{t|t-1} \pm 1.96\hat{\sigma}_{t|t-1}$ prediction interval constructed based on the PEC and the SBC model selection methods, respectively.

The predictive ability of the PEC model selection algorithm has to be further investigated. Among the financial applications where this method could have a potential use are in the fields of portfolio analysis, risk management and trading option derivatives.

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Appendix

Table 3. Sum of squared standardized one step ahead prediction errors for each subperiod. The AR(κ)GARCH(p,q), AR(κ)EGARCH(p,q) and AR(κ)TARCH(p,q) models are applied, for κ=0,…,4, p=0,1,2 and q=1,2.

AR(κ) $$y_t = c_0 + \sum_{i=1}^{\kappa} \left(c_i y_{t-i}\right) + \varepsilon_t$$

GARCH(p,q) $$\sigma_t^2 = a_0 + \sum_{i=1}^{q} \left(a_i \varepsilon_{t-i}^2\right) + \sum_{j=1}^{p} \left(b_j \sigma_{t-j}^2\right)$$

EGARCH(p,q) $$\ln\left(\sigma_t^2\right) = a_0 + \sum_{i=1}^{q} \left(a_i \left| \frac{\varepsilon_{t-i}}{\sigma_{t-i}} \right| + \gamma_i \left(\frac{\varepsilon_{t-i}}{\sigma_{t-i}} \right) \right) + \sum_{j=1}^{p} \left(b_j \ln\left(\sigma_{t-j}^2\right)\right)$$

TARCH(p,q) $$\sigma_t^2 = a_0 + \sum_{i=1}^{q} \left(a_i \varepsilon_{t-i}^2\right) + \gamma \varepsilon_{t-1}^2 d_{t-1} + \sum_{j=1}^{p} \left(b_j \sigma_{t-j}^2\right)$$

	Table 3.a				
	25 August 1995 - 16 November 1995 (s=[501,560])				
	*κ=0**	*κ=1*	*κ=2*	*κ=3*	*κ=4*
GARCH(p,q)					
p=0, q=1	26,371	25,465	24,843	25,173	26,570
p=0, q=2	30,150	29,493	28,940	29,109	30,835
p=1, q=1	39,076	38,848	38,289	38,496	38,466
p=1, q=2	39,129	38,709	38,159	38,533	38,456
p=2, q=1	39,183	38,304	37,882	37,829	37,889
p=2, q=2	39,511	38,742	38,336	39,223	38,377
TARCH(p,q)					
p=0, q=1	26,795	25,892	25,270	25,683	27,300
p=0, q=2	31,151	30,981	30,442	30,619	32,125
p=1, q=1	39,070	38,624	38,146	38,506	38,550
p=1, q=2	39,016	38,667	38,185	38,660	38,482
p=2, q=1	39,279	37,836	37,422	38,005	38,290
p=2, q=2	40,975	38,732	38,180	38,755	38,398
EGARCH(p,q)					
p=0, q=1	23,770	22,644	21,961	22,047	22,722
p=0, q=2	27,289	27,340	26,731	26,896	28,312
p=1, q=1	44,281	43,555	43,131	43,321	41,934
p=1, q=2	43,754	42,427	41,360	42,235	41,231
p=2, q=1	44,620	43,216	43,138	43,142	42,077
p=2, q=2	43,926	42,915	42,231	42,645	41,138

	Table 3.b				
	17 November 1995 - 13 February 1996 (s=[561,620])				
	*κ=0**	*κ=1*	*κ=2*	*κ=3*	*κ=4*
GARCH(p,q)					
p=0, q=1	81,183	79,657	79,913	83,204	89,584
p=0, q=2	88,007	85,947	88,135	89,575	95,825
p=1, q=1	79,571	84,410	85,070	85,671	86,749
p=1, q=2	80,684	85,214	85,554	87,046	89,907
p=2, q=1	79,703	83,700	86,917	84,920	87,420
p=2, q=2	81,230	84,534	85,143	82,863	88,940
TARCH(p,q)					
p=0, q=1	81,505	80,810	81,158	84,704	90,674
p=0, q=2	88,977	88,465	91,004	92,734	98,915
p=1, q=1	81,296	85,321	86,339	87,601	88,412
p=1, q=2	86,517	87,338	88,246	92,729	98,976
p=2, q=1	81,609	86,085	85,458	84,975	90,097
p=2, q=2	89,614	86,608	87,364	91,126	98,289
EGARCH(p,q)					
p=0, q=1	76,315	78,689	78,342	78,551	84,422
p=0, q=2	87,867	91,361	92,862	93,526	101,216
p=1, q=1	88,246	96,778	98,579	99,805	99,650
p=1, q=2	98,798	103,714	105,834	107,774	108,783
p=2, q=1	90,043	98,056	99,570	101,509	101,531
p=2, q=2	93,750	102,953	112,441	105,882	**

	Table 3.c				
	14 February 1996 - 14 May 1996 (s=[621,680])				
	*κ=0**	*κ=1*	*κ=2*	*κ=3*	*κ=4*
GARCH(p,q)					
p=0, q=1	45,970	46,740	46,793	47,855	47,882
p=0, q=2	46,138	46,323	46,039	47,496	47,382
p=1, q=1	50,273	50,205	49,959	50,363	49,320
p=1, q=2	50,429	50,097	49,814	50,223	49,330
p=2, q=1	50,650	50,334	49,547	49,917	49,843
p=2, q=2	50,811	50,126	50,051	50,330	48,975
TARCH(p,q)					
p=0, q=1	45,947	46,731	46,749	47,769	47,806
p=0, q=2	46,114	46,311	46,001	47,422	47,263
p=1, q=1	50,461	50,262	50,006	50,396	49,368
p=1, q=2	50,677	50,145	49,830	50,229	49,512
p=2, q=1	50,769	49,491	48,737	50,231	49,613
p=2, q=2	51,664	49,794	50,262	50,548	50,133
EGARCH(p,q)					
p=0, q=1	42,176	42,724	42,688	43,561	43,383
p=0, q=2	43,712	44,279	44,178	45,395	44,838
p=1, q=1	49,382	48,836	48,837	49,369	48,644
p=1, q=2	49,140	48,716	48,592	49,065	48,608
p=2, q=1	49,422	48,384	48,301	48,452	48,380
p=2, q=2	51,970	49,555	**	48,992	**

*Regress the depedent variable on a constant.

** Model fails to converge at least once.

Table 3. Sum of squared standardized one step ahead prediction errors for each subperiod. The AR(κ)GARCH(p,q), AR(κ)EGARCH(p,q) and AR(κ)TARCH(p,q) models are applied, for κ=0,…,4, p=0,1,2 and q=1,2.

AR(κ) $$y_t = c_0 + \sum_{i=1}^{\kappa} \left(c_i y_{t-i} \right) + \varepsilon_t$$

GARCH(p,q) $$\sigma_t^2 = a_0 + \sum_{i=1}^{q} \left(a_i \varepsilon_{t-i}^2 \right) + \sum_{j=1}^{p} \left(b_j \sigma_{t-j}^2 \right)$$

EGARCH(p,q) $$\ln\left(\sigma_t^2\right) = a_0 + \sum_{i=1}^{q} \left(a_i \left| \frac{\varepsilon_{t-i}}{\sigma_{t-i}} \right| + \gamma_i \left(\frac{\varepsilon_{t-i}}{\sigma_{t-i}} \right) \right) + \sum_{j=1}^{p} \left(b_j \ln\left(\sigma_{t-j}^2\right) \right)$$

TARCH(p,q) $$\sigma_t^2 = a_0 + \sum_{i=1}^{q} \left(a_i \varepsilon_{t-i}^2 \right) + \gamma \varepsilon_{t-1}^2 d_{t-1} + \sum_{j=1}^{p} \left(b_j \sigma_{t-j}^2 \right)$$

Table 3.d

15 May 1996 - 8 August 1996 (s=[681,740])

	*κ=0**	*κ=1*	*κ=2*	*κ=3*	*κ=4*
GARCH(p,q)					
p=0, q=1	30,568	30,619	29,473	29,346	29,534
p=0, q=2	31,557	32,105	30,967	30,861	30,813
p=1, q=1	36,016	36,440	35,335	35,175	35,013
p=1, q=2	36,098	36,951	35,846	35,706	35,431
p=2, q=1	35,732	37,374	36,069	36,020	35,628
p=2, q=2	35,859	36,647	36,252	35,446	35,437
TARCH(p,q)					
p=0, q=1	30,747	30,605	29,419	29,352	29,593
p=0, q=2	31,821	31,978	30,804	30,785	30,811
p=1, q=1	36,029	36,326	35,157	35,147	35,075
p=1, q=2	36,117	36,636	35,489	35,482	35,298
p=2, q=1	36,279	37,214	35,789	36,224	35,946
p=2, q=2	35,945	37,646	35,776	36,005	36,030
EGARCH(p,q)					
p=0, q=1	29,252	28,733	27,428	27,308	27,330
p=0, q=2	30,310	30,109	28,772	28,644	28,563
p=1, q=1	35,972	36,142	34,806	34,716	34,754
p=1, q=2	36,251	36,923	35,548	35,477	35,460
p=2, q=1	35,706	37,371	36,176	36,190	36,266
p=2, q=2	35,562	35,109	34,329	34,210	34,777

Table 3.e

9 August 1996 - 4 November 1996 (s=[741,800])

	*κ=0**	*κ=1*	*κ=2*	*κ=3*	*κ=4*
GARCH(p,q)					
p=0, q=1	48,288	47,469	47,437	49,749	50,771
p=0, q=2	50,795	49,575	49,484	51,426	52,236
p=1, q=1	55,915	54,344	54,572	54,967	55,281
p=1, q=2	56,099	54,631	54,872	55,163	55,399
p=2, q=1	55,807	55,420	55,335	56,306	56,075
p=2, q=2	56,102	54,814	55,145	55,137	55,359
TARCH(p,q)					
p=0, q=1	47,179	47,143	47,101	49,494	50,529
p=0, q=2	49,483	49,131	49,030	51,031	51,935
p=1, q=1	53,866	53,341	53,616	53,897	54,272
p=1, q=2	54,065	53,684	53,835	54,075	54,327
p=2, q=1	53,925	54,199	53,999	54,245	56,211
p=2, q=2	54,181	54,482	54,725	55,039	54,846
EGARCH(p,q)					
p=0, q=1	44,260	43,920	44,047	45,908	46,528
p=0, q=2	46,453	45,986	46,035	47,513	47,990
p=1, q=1	52,752	53,271	53,285	53,801	53,944
p=1, q=2	53,233	54,767	54,191	54,450	54,617
p=2, q=1	53,922	55,703	55,410	55,596	55,726
p=2, q=2	52,438	54,052	53,963	**	54,716

*Regress the depedent variable on a constant.

** Model fails to converge at least once.

Table 4

Testing the null hypothesis that the model with the lowest sum of the squared standardized one step ahead prediction errors has equivalent predictive ability to model X, with X denoting any of the remainder models.

Table 4.a: 25 August 1995 - 16 November 1995 (1st subperiod)

H_0: The model AR(2)-EGARCH(0,1) is equivalent to model X
versus H_1: The model AR(2)-EGARCH(0,1) is "better" than model X.

		Model for Conditional Mean				
		AR(0)	AR(1)	AR(2)	AR(3)	AR(4)
Model for Conditional Variance	GARCH(0,1)	1,201	1,160	1,131	1,146	1,210
	p-value	<0.10	<0.10	<0.25	<0.25	<0.05
	GARCH(0,2)	1,373	1,343	1,318	1,326	1,404
	p-value	<0.01	<0.01	<0.01	<0.01	<0.01
	GARCH(1,1)	1,779	1,769	1,744	1,753	1,752
	p-value	<0.01	<0.01	<0.01	<0.01	<0.01
	GARCH(1,2)	1,782	1,763	1,738	1,755	1,751
	p-value	<0.01	<0.01	<0.01	<0.01	<0.01
	GARCH(2,1)	1,784	1,744	1,725	1,723	1,725
	p-value	<0.01	<0.01	<0.01	<0.01	<0.01
	GARCH(2,2)	1,799	1,764	1,746	1,786	1,748
	p-value	<0.01	<0.01	<0.01	<0.01	<0.01
	TARCH(0,1)	1,220	1,179	1,151	1,170	1,243
	p-value	<0.05	<0.10	<0.25	<0.10	<0.05
	TARCH(0,2)	1,418	1,411	1,386	1,394	1,463
	p-value	<0.01	<0.01	<0.01	<0.01	<0.01
	TARCH(1,1)	1,779	1,759	1,737	1,753	1,755
	p-value	<0.01	<0.01	<0.01	<0.01	<0.01
	TARCH(1,2)	1,777	1,761	1,739	1,760	1,752
	p-value	<0.01	<0.01	<0.01	<0.01	<0.01
	TARCH(2,1)	1,789	1,723	1,704	1,731	1,744
	p-value	<0.01	<0.01	<0.01	<0.01	<0.01
	TARCH(2,2)	1,866	1,764	1,739	1,765	1,748
	p-value	<0.01	<0.01	<0.01	<0.01	<0.01
	E-GARCH(0,1)	1,082	1,031		1,004	1,035
	p-value	<0.25	>0.25		>0.25	>0.25
	E-GARCH(0,2)	1,243	1,245	1,217	1,225	1,289
	p-value	<0.05	<0.05	<0.05	<0.05	<0.05
	E-GARCH(1,1)	2,016	1,983	1,964	1,973	1,909
	p-value	<0.01	<0.01	<0.01	<0.01	<0.01
	E-GARCH(1,2)	1,992	1,932	1,883	1,923	1,878
	p-value	<0.01	<0.01	<0.01	<0.01	<0.01
	E-GARCH(2,1)	2,032	1,968	1,964	1,965	1,916
	p-value	<0.01	<0.01	<0.01	<0.01	<0.01
	E-GARCH(2,2)	2,000	1,954	1,923	1,942	1,873
	p-value	<0.01	<0.01	<0.01	<0.01	<0.01

Table 4.b: 17 November 1995 - 13 February 1996 (2nd subperiod)

H_0: The model AR(0)-EGARCH(0,1) is equivalent to model X
versus H_1: The model AR(0)-EGARCH(0,1) is "better" than model X.

		Model for Conditional Mean				
		AR(0)	AR(1)	AR(2)	AR(3)	AR(4)
Model for Conditional Variance	GARCH(0,1)	1,064	1,044	1,047	1,090	1,174
	p-value	>0.25	>0.25	>0.25	<0.25	<0.1
	GARCH(0,2)	1,153	1,126	1,155	1,174	1,256
	p-value	<0.25	<0.25	<0.25	<0.1	<0.05
	GARCH(1,1)	1,043	1,106	1,115	1,123	1,137
	p-value	>0.25	<0.25	<0.25	<0.25	<0.25
	GARCH(1,2)	1,057	1,117	1,121	1,141	1,178
	p-value	>0.25	<0.25	<0.25	<0.25	<0.1
	GARCH(2,1)	1,044	1,097	1,139	1,113	1,146
	p-value	>0.25	<0.25	<0.25	<0.25	<0.25
	GARCH(2,2)	1,064	1,108	1,116	1,086	1,165
	p-value	>0.25	<0.25	<0.25	<0.25	<0.1
	TARCH(0,1)	1,068	1,059	1,063	1,110	1,188
	p-value	>0.25	>0.25	>0.25	<0.25	<0.1
	TARCH(0,2)	1,166	1,159	1,192	1,215	1,296
	p-value	<0.1	<0.1	<0.1	<0.05	<0.05
	TARCH(1,1)	1,065	1,118	1,131	1,148	1,159
	p-value	>0.25	<0.25	<0.25	<0.25	<0.1
	TARCH(1,2)	1,134	1,144	1,156	1,215	1,297
	p-value	<0.25	<0.25	<0.25	<0.05	<0.05
	TARCH(2,1)	1,069	1,128	1,120	1,113	1,181
	p-value	>0.25	<0.25	<0.25	<0.25	<0.1
	TARCH(2,2)	1,174	1,135	1,145	1,194	1,288
	p-value	<0.1	<0.25	<0.25	<0.1	<0.05
	E-GARCH(0,1)		1,031	1,027	1,029	1,106
	p-value		>0.25	>0.25	>0.25	<0.25
	E-GARCH(0,2)	1,151	1,197	1,217	1,226	1,326
	p-value	<0.25	<0.1	<0.05	<0.05	<0.01
	E-GARCH(1,1)	1,156	1,268	1,292	1,308	1,306
	p-value	<0.25	<0.05	<0.05	<0.05	<0.05
	E-GARCH(1,2)	1,295	1,359	1,387	1,412	1,425
	p-value	<0.05	<0.01	<0.01	<0.01	<0.01
	E-GARCH(2,1)	1,180	1,285	1,305	1,330	1,330
	p-value	<0.1	<0.05	<0.05	<0.01	<0.01
	E-GARCH(2,2)	1,228	1,349	1,473	1,387	**
	p-value	<0.05	<0.01	<0.01	<0.01	

** Model fails to converge at least once.

Table 4 (continued)

Testing the null hypothesis that the model with the lowest sum of the squared standardized one step ahead prediction errors has equivalent predictive ability to model X, with X denoting any of the remainder models.

Table 4.c: 14 February 1996 - 14 May 1996 (3rd subperiod)

H_0: The model AR(2)-EGARCH(0,1) is equivalent to model X
versus H_1: The model AR(2)-EGARCH(0,1) is "better" than model X.

		Model for Conditional Mean				
		AR(0)	AR(1)	AR(2)	AR(3)	AR(4)
Model for Conditional Variance	GARCH(0,1)	1,090	1,108	1,109	1,135	1,135
	p-value	<0.25	<0.25	<0.25	<0.25	<0.25
	GARCH(0,2)	1,094	1,098	1,092	1,126	1,123
	p-value	<0.25	<0.25	<0.25	<0.25	<0.25
	GARCH(1,1)	1,192	1,190	1,185	1,194	1,169
	p-value	<0.1	<0.1	<0.1	<0.1	<0.1
	GARCH(1,2)	1,196	1,188	1,181	1,191	1,170
	p-value	<0.1	<0.1	<0.1	<0.1	<0.1
	GARCH(2,1)	1,201	1,193	1,175	1,184	1,182
	p-value	<0.1	<0.1	<0.1	<0.1	<0.1
	GARCH(2,2)	1,205	1,188	1,187	1,193	1,161
	p-value	<0.1	<0.1	<0.1	<0.1	<0.1
	TARCH(0,1)	1,089	1,108	1,108	1,133	1,133
	p-value	<0.25	<0.25	<0.25	<0.25	<0.25
	TARCH(0,2)	1,093	1,098	1,091	1,124	1,121
	p-value	<0.25	<0.25	<0.25	<0.25	<0.25
	TARCH(1,1)	1,196	1,192	1,186	1,195	1,171
	p-value	<0.1	<0.1	<0.1	<0.1	<0.1
	TARCH(1,2)	1,202	1,189	1,181	1,191	1,174
	p-value	<0.1	<0.1	<0.1	<0.1	<0.1
	TARCH(2,1)	1,204	1,173	1,156	1,191	1,176
	p-value	<0.1	<0.1	<0.25	<0.1	<0.1
	TARCH(2,2)	1,225	1,181	1,192	1,199	1,189
	p-value	<0.05	<0.1	<0.1	<0.1	<0.1
	E-GARCH(0,1)		1,013	1,012	1,033	1,029
	p-value		>0.25	>0.25	>0.25	>0.25
	E-GARCH(0,2)	1,036	1,050	1,047	1,076	1,063
	p-value	>0.25	>0.25	>0.25	>0.25	>0.25
	E-GARCH(1,1)	1,171	1,158	1,158	1,171	1,153
	p-value	<0.1	<0.1	<0.1	<0.1	<0.25
	E-GARCH(1,2)	1,165	1,155	1,152	1,163	1,153
	p-value	<0.1	<0.25	<0.25	<0.1	<0.25
	E-GARCH(2,1)	1,172	1,147	1,145	1,149	1,147
	p-value	<0.1	<0.25	<0.25	<0.25	<0.25
	E-GARCH(2,2)	1,232	1,175	**	1,162	**
	p-value	<0.05	<0.1		<0.1	

Table 4.d: 15 May 1996 - 8 August 1996 (4th subperiod)

H_0: The model AR(3)-EGARCH(0,1) is equivalent to model X
versus H_1: The model AR(3)-EGARCH(0,1) is "better" than model X.

		Model for Conditional Mean				
		AR(0)	AR(1)	AR(2)	AR(3)	AR(4)
Model for Conditional Variance	GARCH(0,1)	1,119	1,121	1,079	1,075	1,081
	p-value	<0.25	<0.25	<0.25	>0.25	<0.25
	GARCH(0,2)	1,156	1,176	1,134	1,130	1,128
	p-value	<0.25	<0.1	<0.25	<0.25	<0.25
	GARCH(1,1)	1,319	1,334	1,294	1,288	1,282
	p-value	<0.01	<0.01	<0.05	<0.05	<0.05
	GARCH(1,2)	1,322	1,353	1,313	1,308	1,297
	p-value	<0.01	<0.01	<0.01	<0.05	<0.05
	GARCH(2,1)	1,308	1,369	1,321	1,319	1,305
	p-value	<0.01	<0.01	<0.01	<0.01	<0.05
	GARCH(2,2)	1,313	1,342	1,328	1,298	1,298
	p-value	<0.01	<0.01	<0.01	<0.05	<0.05
	TARCH(0,1)	1,126	1,121	1,077	1,075	1,084
	p-value	<0.25	<0.25	>0.25	>0.25	<0.25
	TARCH(0,2)	1,165	1,171	1,128	1,127	1,128
	p-value	<0.1	<0.1	<0.25	<0.25	<0.25
	TARCH(1,1)	1,319	1,330	1,287	1,287	1,284
	p-value	<0.01	<0.01	<0.05	<0.05	<0.05
	TARCH(1,2)	1,323	1,342	1,300	1,299	1,293
	p-value	<0.01	<0.01	<0.05	<0.05	<0.05
	TARCH(2,1)	1,329	1,363	1,311	1,327	1,316
	p-value	<0.01	<0.01	<0.01	<0.01	<0.01
	TARCH(2,2)	1,316	1,379	1,310	1,318	1,319
	p-value	<0.01	<0.01	<0.01	<0.01	<0.01
	E-GARCH(0,1)	1,071	1,052	1,004		1,001
	p-value	>0.25	>0.25	>0.25		>0.25
	E-GARCH(0,2)	1,110	1,103	1,054	1,049	1,046
	p-value	<0.25	<0.25	>0.25	>0.25	>0.25
	E-GARCH(1,1)	1,317	1,323	1,275	1,271	1,273
	p-value	<0.01	<0.01	<0.05	<0.05	<0.05
	E-GARCH(1,2)	1,327	1,352	1,302	1,299	1,299
	p-value	<0.01	<0.01	<0.05	<0.05	<0.05
	E-GARCH(2,1)	1,308	1,368	1,325	1,325	1,328
	p-value	<0.05	<0.01	<0.01	<0.01	<0.01
	E-GARCH(2,2)	1,302	1,286	1,257	1,253	1,274
	p-value	<0.05	<0.05	<0.05	<0.05	<0.05

** Model fails to converge at least once.

Table 4 (continued)

Testing the null hypothesis that the model with the lowest sum of the squared standardized one step ahead prediction errors has equivalent predictive ability to model X, with X denoting any of the remainder models.

Table 4.e: 9 August 1996 - 4 November 1996 (5th subperiod)

H_0: The model AR(1)-EGARCH(0,1) is equivalent to model X
versus H_1: The model AR(1)-EGARCH(0,1) is "better" than model X.

		Model for Conditional Mean				
		AR(0)	AR(1)	AR(2)	AR(3)	AR(4)
Model for Conditional Variance	GARCH(0,1)	1,099	1,081	1,080	1,133	1,156
	p-value	<0.25	<0.25	<0.25	<0.25	<0.25
	GARCH(0,2)	1,157	1,129	1,127	1,171	1,189
	p-value	<0.25	<0.25	<0.25	<0.1	<0.1
	GARCH(1,1)	1,273	1,237	1,243	1,252	1,259
	p-value	<0.05	<0.05	<0.05	<0.05	<0.05
	GARCH(1,2)	1,277	1,244	1,249	1,256	1,261
	p-value	<0.05	<0.05	<0.05	<0.05	<0.05
	GARCH(2,1)	1,271	1,262	1,260	1,282	1,277
	p-value	<0.05	<0.05	<0.05	<0.05	<0.05
	GARCH(2,2)	1,277	1,248	1,256	1,255	1,260
	p-value	<0.05	<0.05	<0.05	<0.05	<0.05
	TARCH(0,1)	1,074	1,073	1,072	1,127	1,150
	p-value	>0.25	>0.25	>0.25	<0.25	<0.25
	TARCH(0,2)	1,127	1,119	1,116	1,162	1,183
	p-value	<0.25	<0.25	<0.25	<0.1	<0.1
	TARCH(1,1)	1,226	1,215	1,221	1,227	1,236
	p-value	<0.05	<0.05	<0.05	<0.05	<0.05
	TARCH(1,2)	1,231	1,222	1,226	1,231	1,237
	p-value	<0.05	<0.05	<0.05	<0.05	<0.05
	TARCH(2,1)	1,228	1,234	1,230	1,235	1,280
	p-value	<0.05	<0.05	<0.05	<0.05	<0.05
	TARCH(2,2)	1,234	1,240	1,246	1,253	1,249
	p-value	<0.05	<0.05	<0.05	<0.05	<0.05
	E-GARCH(0,1)	1,008		1,003	1,045	1,059
	p-value	>0.25		>0.25	>0.25	>0.25
	E-GARCH(0,2)	1,058	1,047	1,048	1,082	1,093
	p-value	>0.25	>0.25	>0.25	<0.25	<0.25
	E-GARCH(1,1)	1,201	1,213	1,213	1,225	1,228
	p-value	<0.1	<0.05	<0.05	<0.05	<0.05
	E-GARCH(1,2)	1,212	1,247	1,234	1,240	1,244
	p-value	<0.05	<0.05	<0.05	<0.05	<0.05
	E-GARCH(2,1)	1,228	1,268	1,262	1,266	1,269
	p-value	<0.05	<0.05	<0.05	<0.05	<0.05
	E-GARCH(2,2)	1,194	1,231	1,229	**	1,246
	p-value	<0.1	<0.05	<0.05		<0.05

** Model fails to converge at least once.

Figure 4

One Step Ahead 95% Forecasted Interval for the Models with the Lowest Sum of the Squared Standardized One Step Ahead Prediction Errors

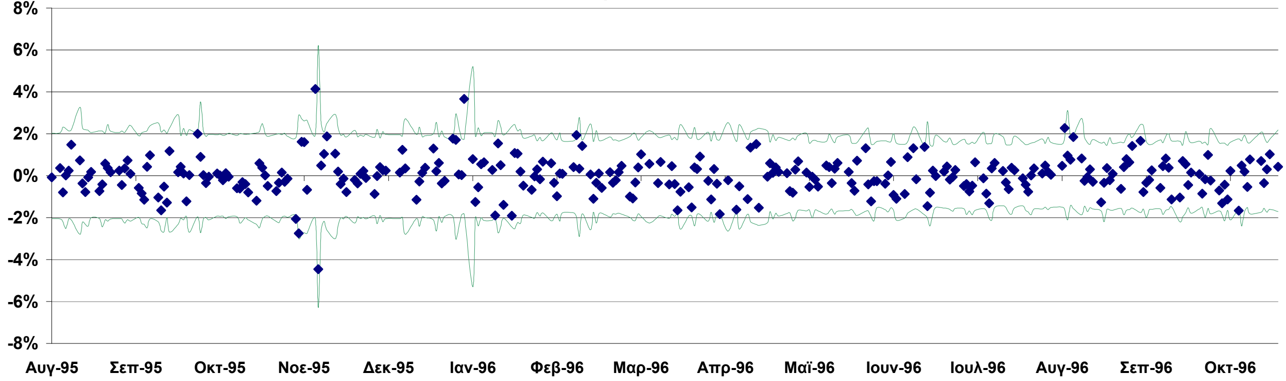


Figure 5

In-Sample 95% Confidence Interval for the AR(1) GARCH(1,1) Model

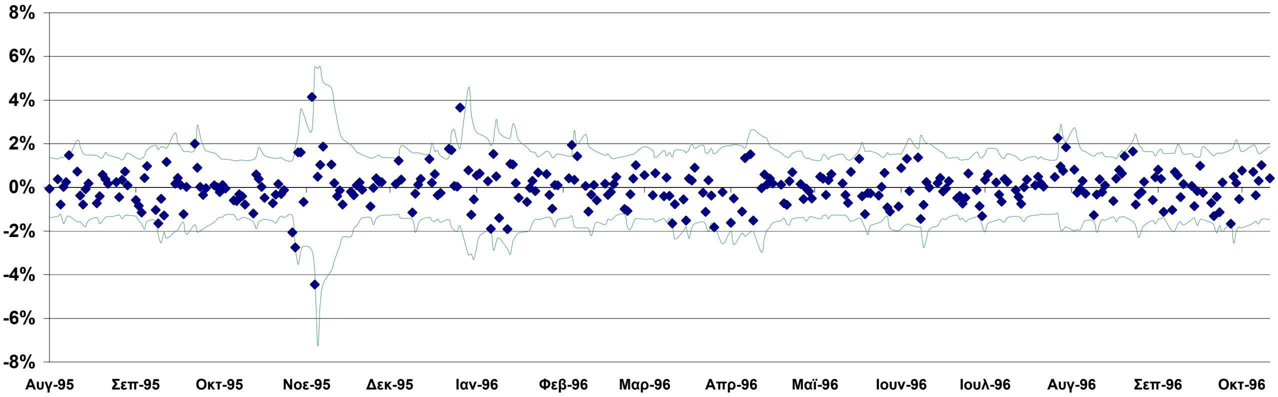


Figure 6

One Step Ahead 95% Forecasted Intervals for the Models Selected by the SBC

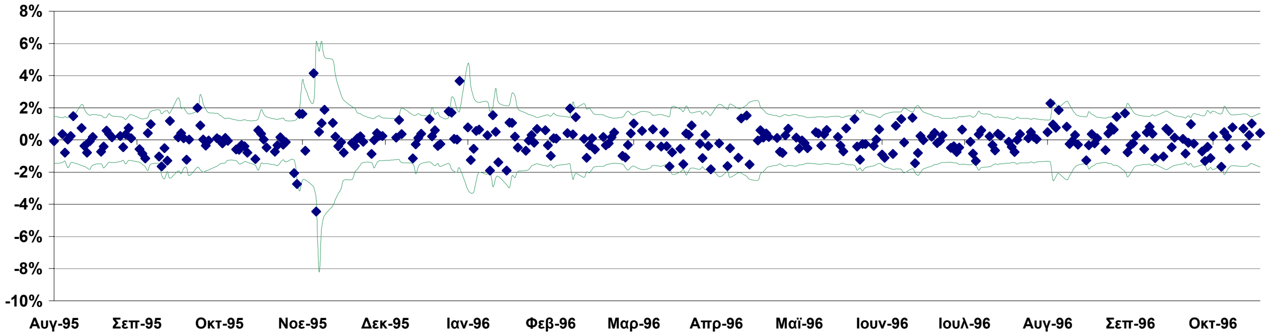


Figure 7

One Step Ahead 95% Forecasted Intervals for the Models Selected by the AIC

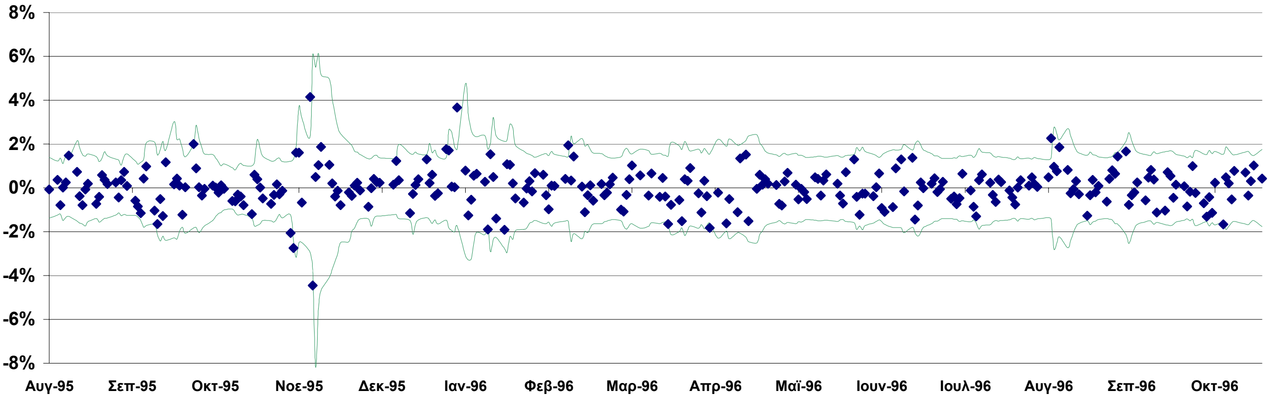
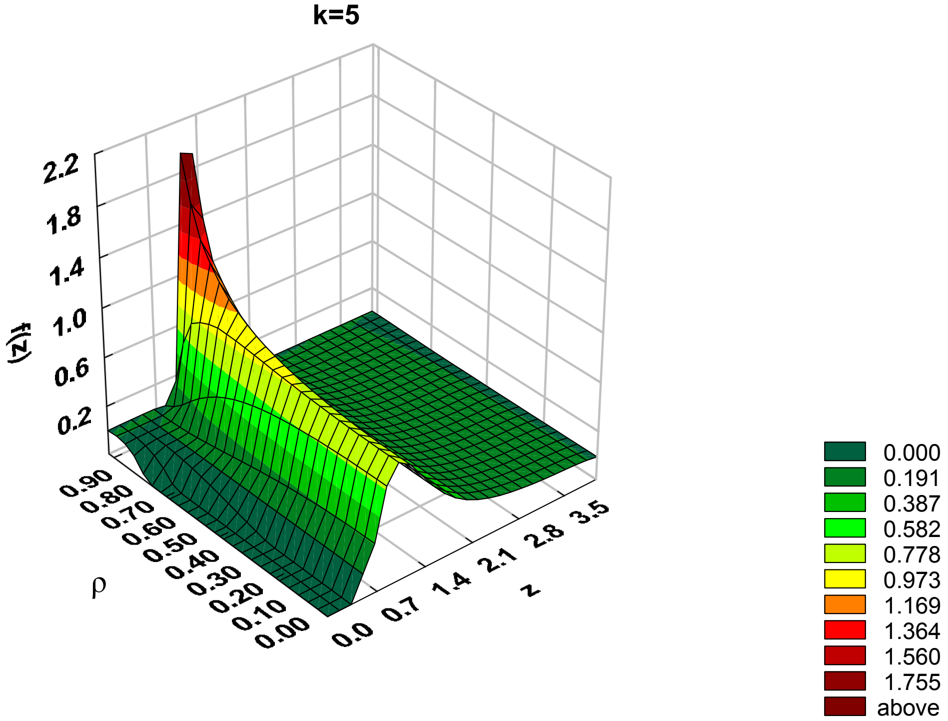


Figure 8. The probability density function of the Correlated Gamma Ratio Distribution

$$f(z)=\frac{\left(1-\rho^2\right)^k}{B(k,k)}z^{k-1}(1+z)^{-2k}\left[1-\left(\frac{2\rho}{z+1}\right)^2 z\right]^{-\frac{2k+1}{2}}$$, for $z\geq 0$, $0\leq\rho<1$, $k=5$



The probability density function of the Correlated Gamma Ratio Distribution for $k=5$ and $\rho=0.1,0.5,0.7,0.9$

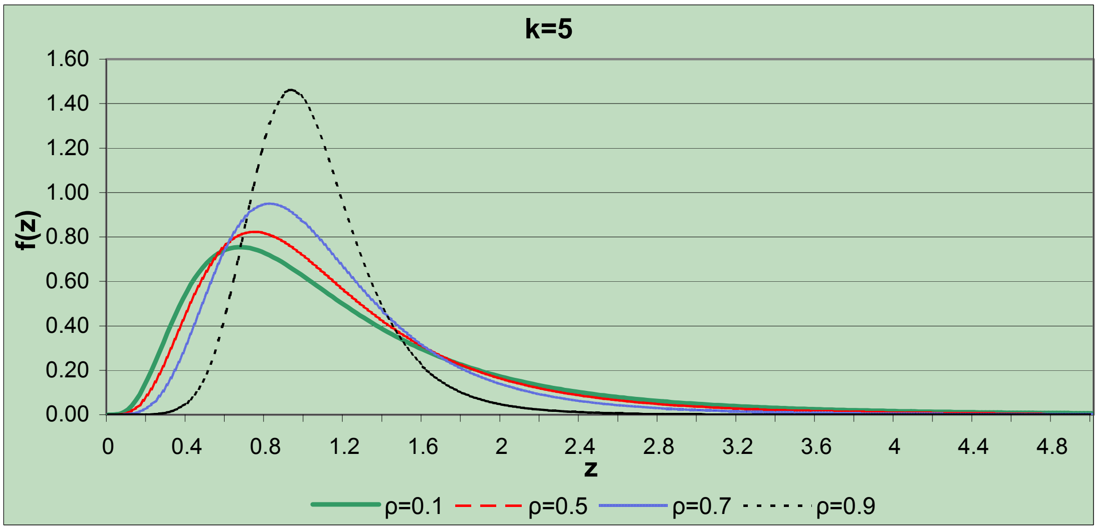
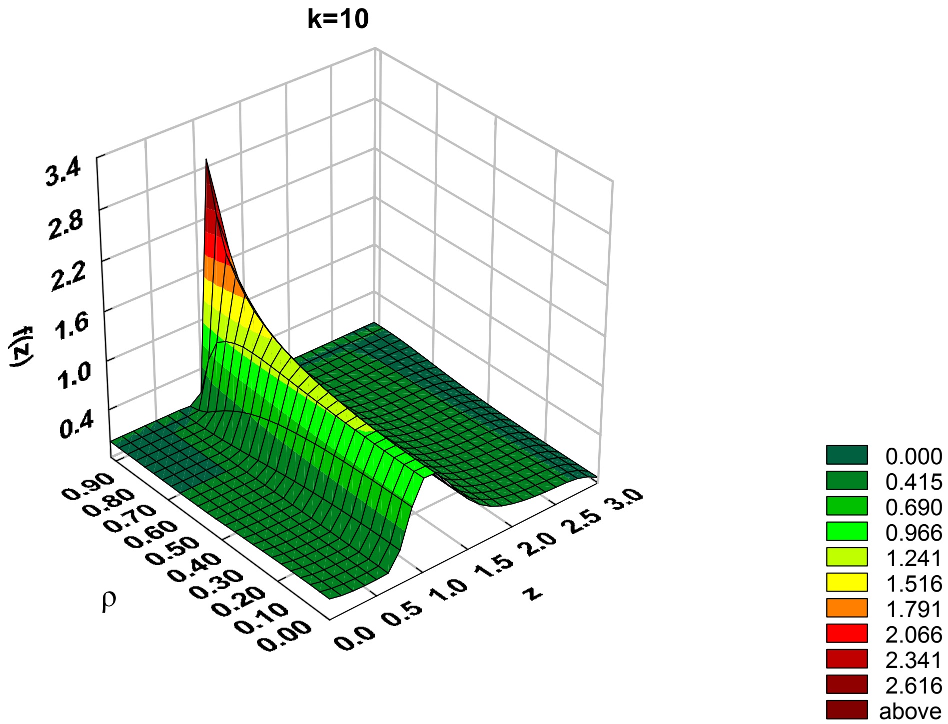


Figure 9. The probability density function of the Correlated Gamma Ratio Distribution

$$f(z)=\frac{\left(1-\rho^2\right)^k}{B(k,k)}z^{k-1}(1+z)^{-2k}\left[1-\left(\frac{2\rho}{z+1}\right)^2 z\right]^{-\frac{2k+1}{2}}$$, for $z\geq 0$, $0\leq\rho<1$, $k=10$



The probability density function of the Correlated Gamma Ratio Distribution for $k=10$ and $\rho=0.1,0.5,0.7,0.9$

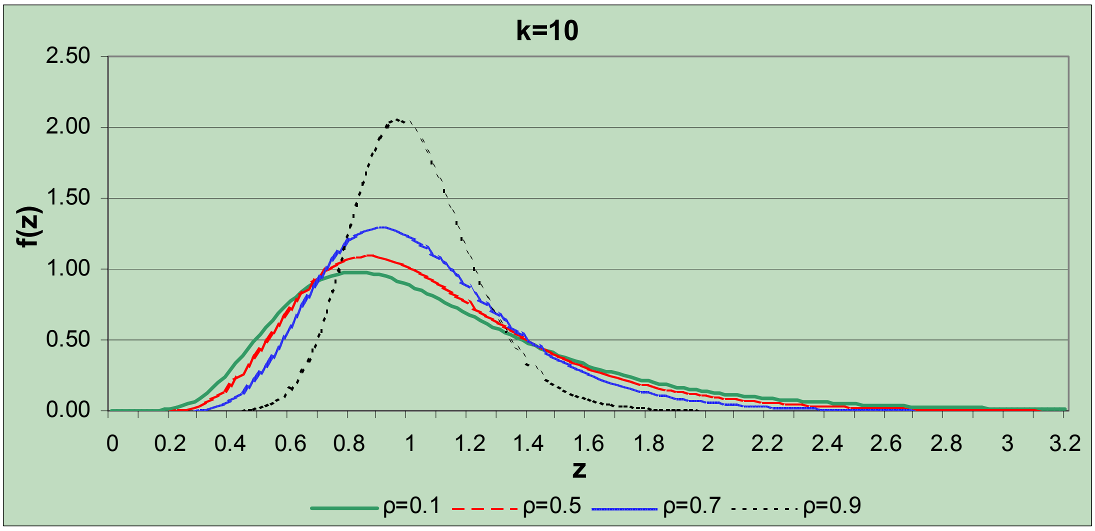
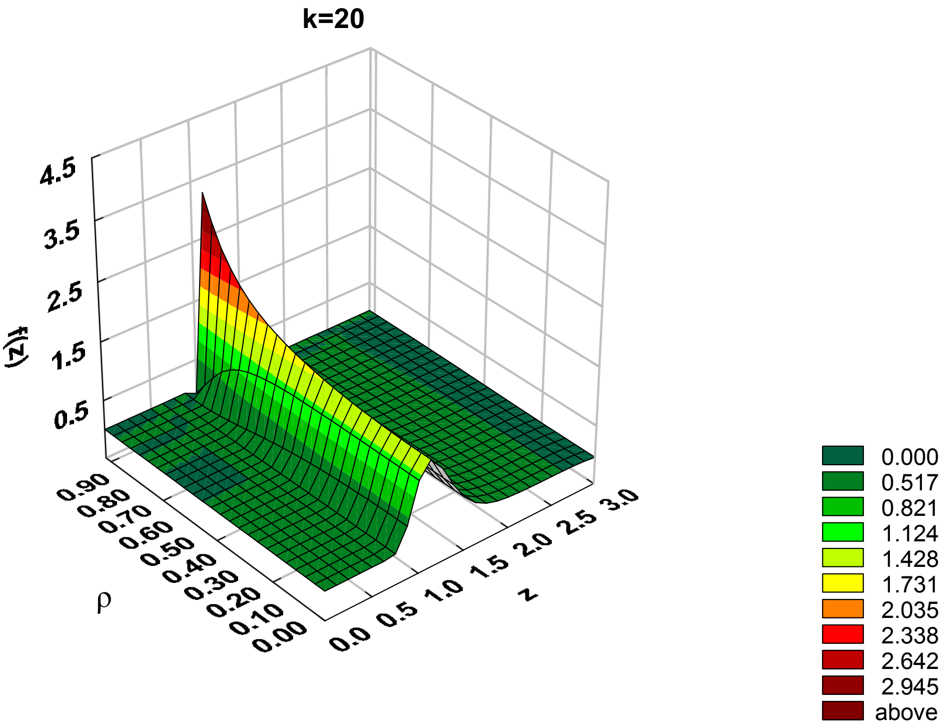


Figure 10. The probability density function of the Correlated Gamma Ratio Distribution

$$f(z)=\frac{\left(1-\rho^2\right)^k}{B(k,k)}z^{k-1}(1+z)^{-2k}\left[1-\left(\frac{2\rho}{z+1}\right)^2 z\right]^{-\frac{2k+1}{2}}\text{, for } z\geq 0\text{, } 0\leq\rho<1\text{, } k=20$$



The probability density function of the Correlated Gamma Ratio Distribution for $k=20$ and $\rho=0.1,0.5,0.7,0.9$

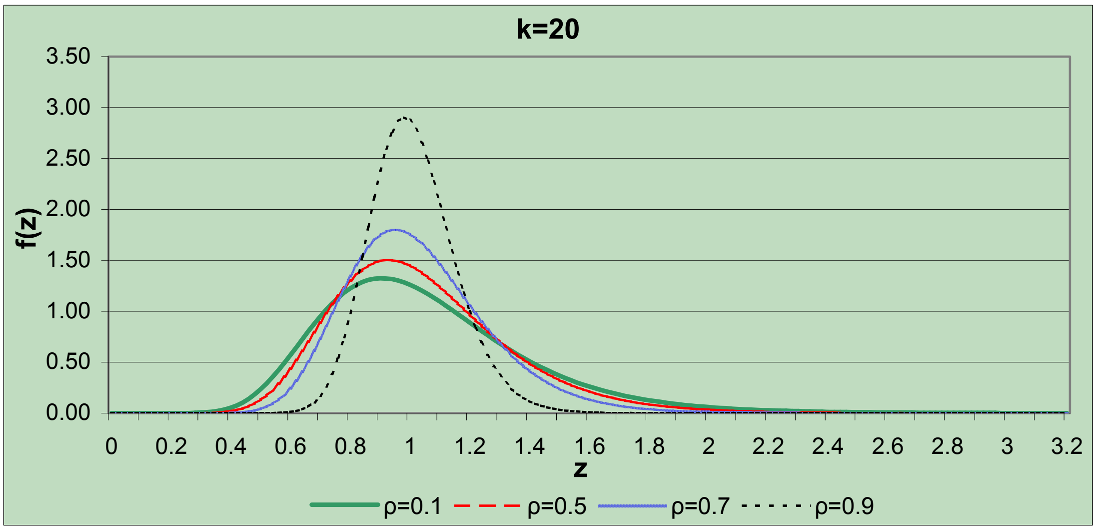
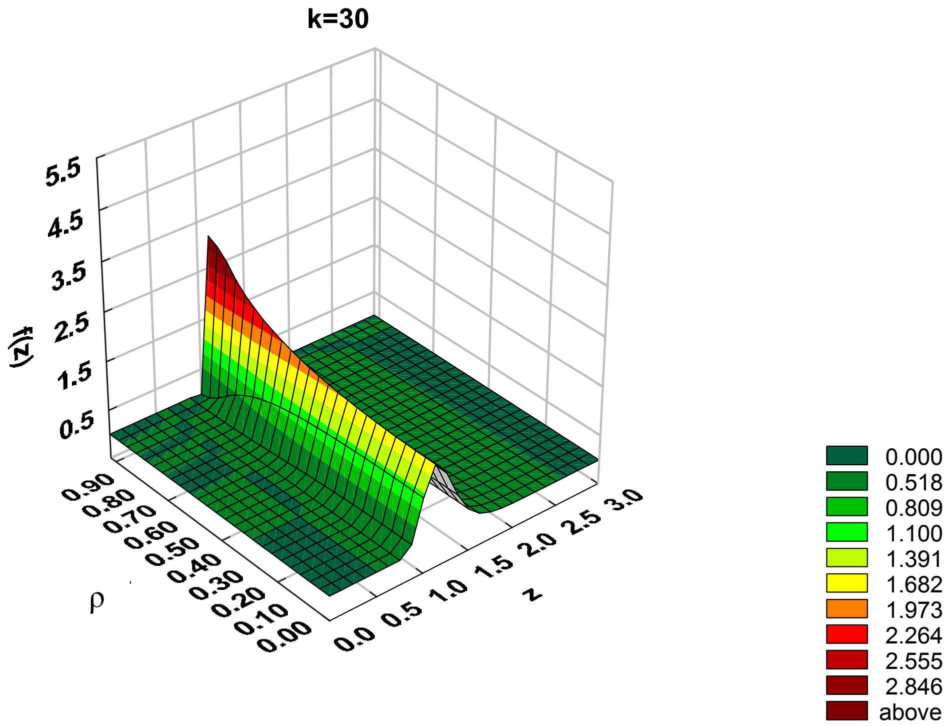


Figure 11. The probability density function of the Correlated Gamma Ratio Distribution

$$f(z)=\frac{\left(1-\rho^2\right)^k}{B(k,k)}z^{k-1}(1+z)^{-2k}\left[1-\left(\frac{2\rho}{z+1}\right)^2 z\right]^{-\frac{2k+1}{2}}$$, for $z \geq 0$, $0 \leq \rho < 1$, $k = 30$



The probability density function of the Correlated Gamma Ratio Distribution for $k = 30$ and $\rho = 0.1, 0.5, 0.7, 0.9$

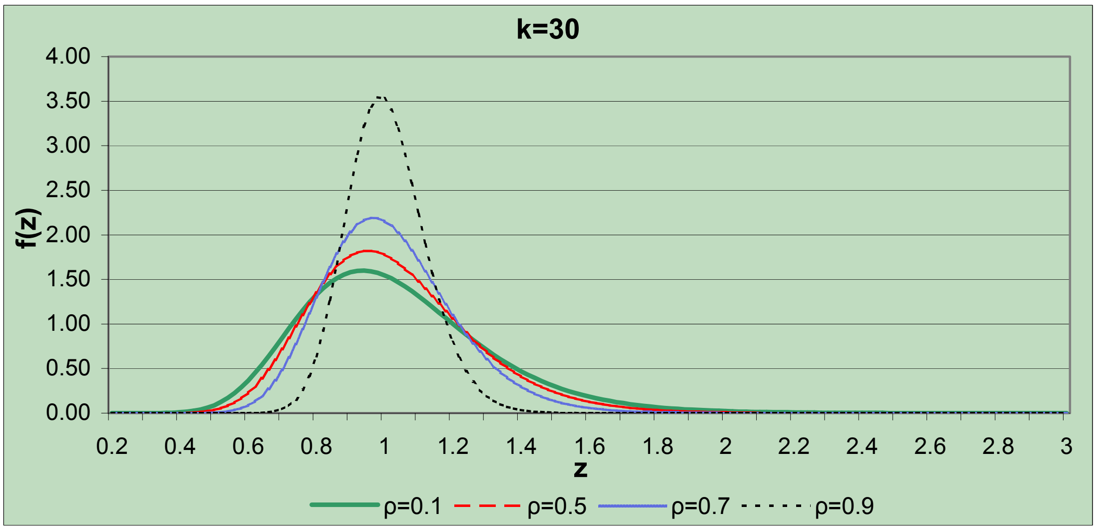
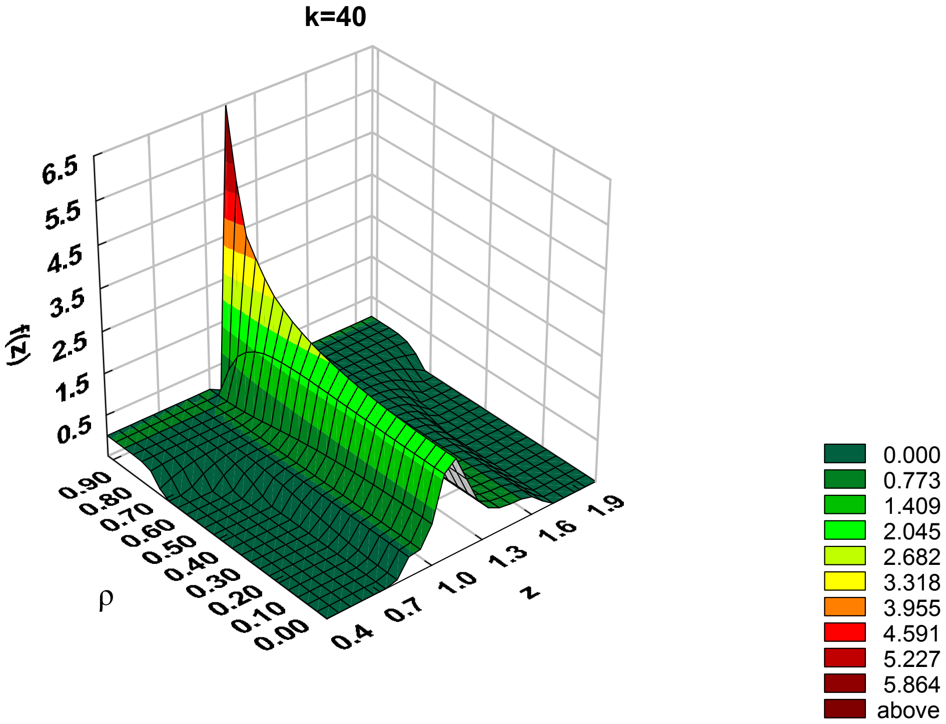


Figure 12. The probability density function of the Correlated Gamma Ratio Distribution

$$f(z)=\frac{\left(1-\rho^2\right)^k}{B(k,k)}z^{k-1}(1+z)^{-2k}\left[1-\left(\frac{2\rho}{z+1}\right)^2 z\right]^{-\frac{2k+1}{2}}$$, for $z \geq 0$, $0 \leq \rho < 1$, $k = 40$



The probability density function of the Correlated Gamma Ratio Distribution for $k = 40$ and $\rho = 0.1, 0.5, 0.7, 0.9$

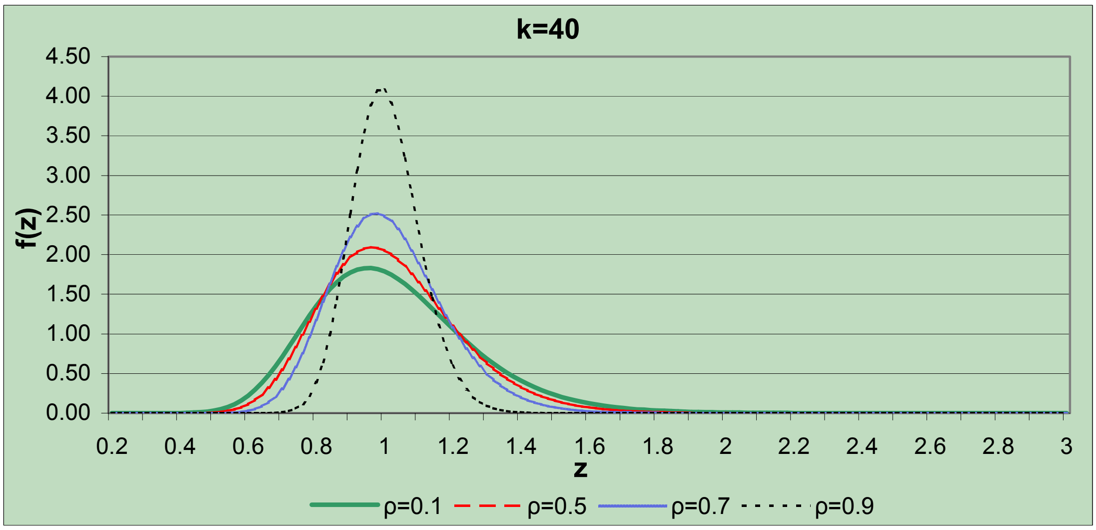
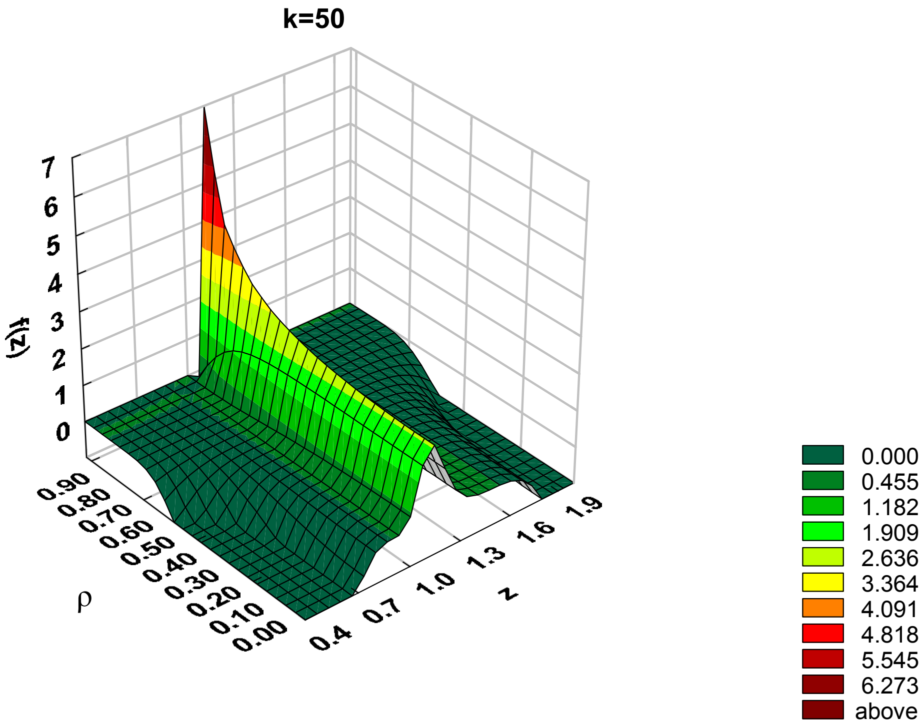


Figure 13. The probability density function of the Correlated Gamma Ratio Distribution

$$f(z)=\frac{\left(1-\rho^2\right)^k}{B(k,k)}z^{k-1}(1+z)^{-2k}\left[1-\left(\frac{2\rho}{z+1}\right)^2 z\right]^{-\frac{2k+1}{2}}$$, for $z \geq 0$, $0 \leq \rho < 1$, $k = 50$



The probability density function of the Correlated Gamma Ratio Distribution for $k = 50$ and $\rho = 0.1, 0.5, 0.7, 0.9$

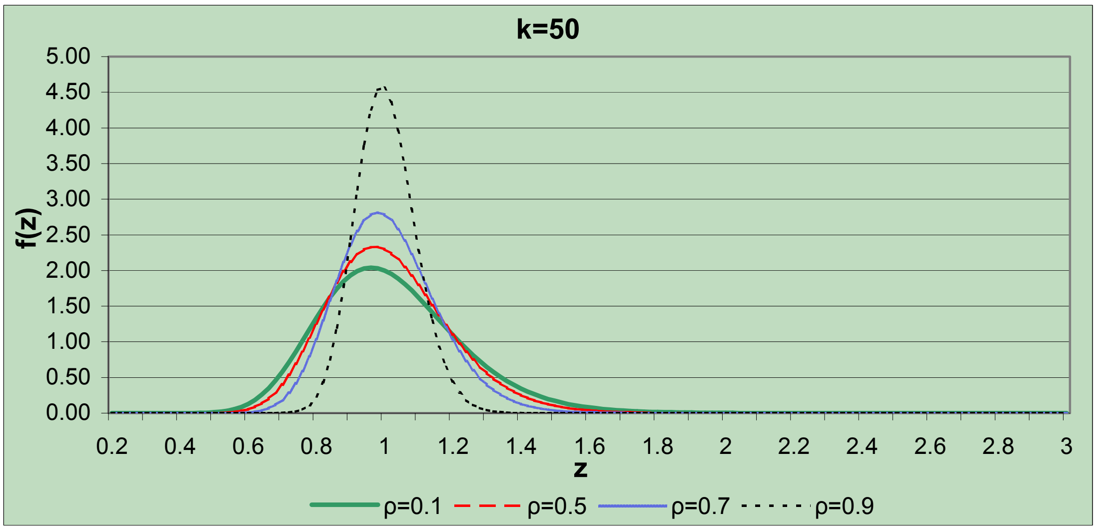
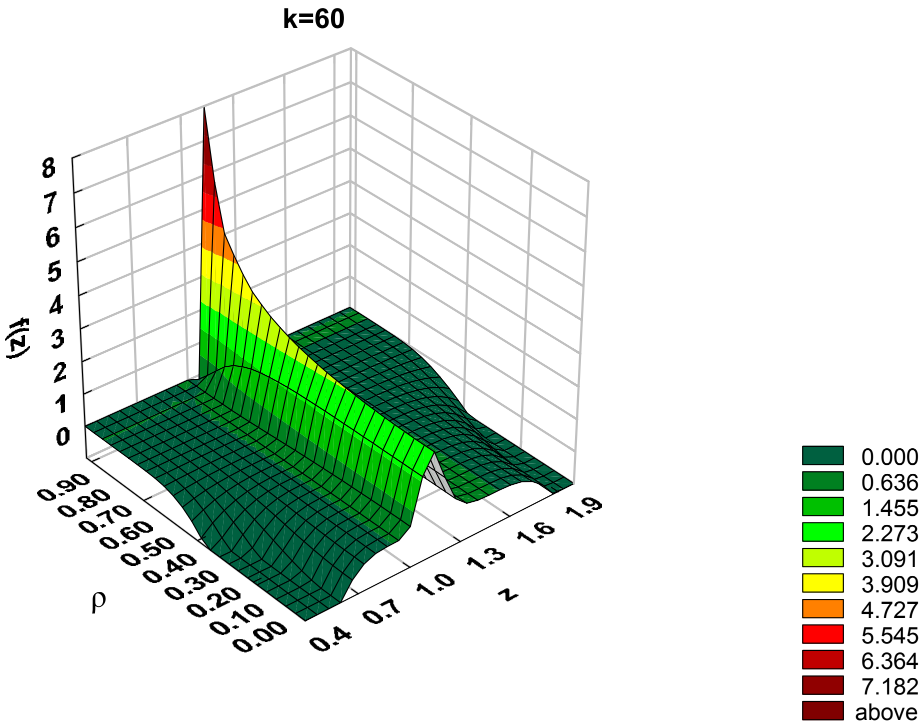
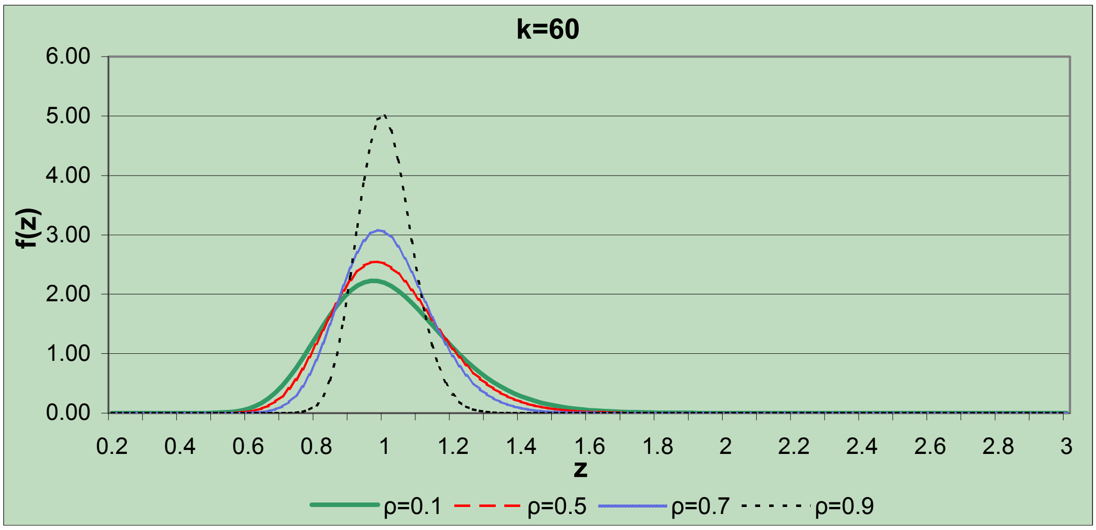


Figure 14. The probability density function of the Correlated Gamma Ratio Distribution

$$f(z)=\frac{\left(1-\rho^2\right)^k}{B(k,k)}z^{k-1}(1+z)^{-2k}\left[1-\left(\frac{2\rho}{z+1}\right)^2 z\right]^{-\frac{2k+1}{2}}$$, for $z \geq 0$, $0 \leq \rho < 1$, $k = 60$

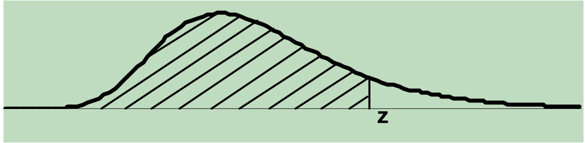


The probability density function of the Correlated Gamma Ratio Distribution for $k = 60$ and $\rho = 0.1, 0.5, 0.7, 0.9$



Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.25$

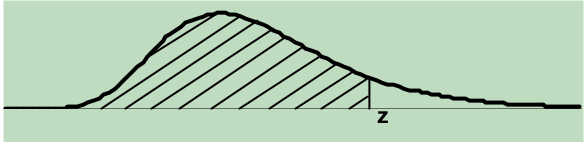
$$\Phi(z) = \int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
1	3.008	3.004	2.993	2.974	2.947	2.913	2.871	2.821	2.763	2.697
2	2.064	2.062	2.057	2.048	2.035	2.019	1.999	1.975	1.947	1.915
3	1.782	1.781	1.777	1.771	1.762	1.751	1.736	1.72	1.7	1.678
4	1.64	1.639	1.636	1.631	1.624	1.615	1.603	1.59	1.574	1.557
5	1.551	1.55	1.548	1.544	1.538	1.53	1.521	1.509	1.496	1.481
6	1.49	1.489	1.487	1.484	1.478	1.472	1.463	1.453	1.442	1.429
7	1.445	1.444	1.442	1.439	1.434	1.428	1.421	1.412	1.402	1.39
8	1.41	1.409	1.407	1.404	1.4	1.395	1.388	1.38	1.37	1.359
9	1.381	1.381	1.379	1.376	1.372	1.367	1.361	1.354	1.345	1.335
10	1.358	1.357	1.356	1.353	1.35	1.345	1.339	1.332	1.324	1.315
11	1.338	1.338	1.336	1.334	1.33	1.326	1.321	1.314	1.306	1.297
12	1.321	1.321	1.32	1.317	1.314	1.31	1.305	1.298	1.291	1.283
13	1.307	1.306	1.305	1.303	1.3	1.296	1.291	1.285	1.278	1.27
14	1.294	1.293	1.292	1.29	1.287	1.283	1.279	1.273	1.266	1.259
15	1.282	1.282	1.281	1.279	1.276	1.272	1.268	1.262	1.256	1.249
16	1.272	1.272	1.271	1.269	1.266	1.262	1.258	1.253	1.247	1.24
17	1.263	1.262	1.261	1.259	1.257	1.254	1.249	1.244	1.239	1.232
18	1.254	1.254	1.253	1.251	1.249	1.245	1.241	1.237	1.231	1.224
19	1.247	1.246	1.245	1.244	1.241	1.238	1.234	1.229	1.224	1.218
20	1.24	1.239	1.238	1.237	1.234	1.231	1.227	1.223	1.218	1.212
21	1.233	1.233	1.232	1.23	1.228	1.225	1.221	1.217	1.212	1.206
22	1.227	1.227	1.226	1.224	1.222	1.219	1.216	1.211	1.206	1.201
23	1.222	1.221	1.22	1.219	1.217	1.214	1.21	1.206	1.201	1.196
24	1.216	1.216	1.215	1.214	1.212	1.209	1.205	1.201	1.197	1.191
25	1.212	1.211	1.21	1.209	1.207	1.204	1.201	1.197	1.192	1.187
26	1.207	1.207	1.206	1.204	1.202	1.2	1.197	1.193	1.188	1.183
27	1.203	1.202	1.202	1.2	1.198	1.196	1.193	1.189	1.184	1.179
28	1.199	1.198	1.198	1.196	1.194	1.192	1.189	1.185	1.181	1.176
29	1.195	1.195	1.194	1.192	1.191	1.188	1.185	1.181	1.177	1.172
30	1.191	1.191	1.19	1.189	1.187	1.185	1.182	1.178	1.174	1.169

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.25$

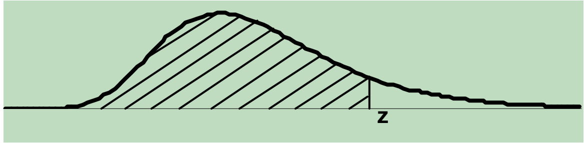
$$\Phi(z) = \int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
31	1.188	1.188	1.187	1.186	1.184	1.181	1.178	1.175	1.171	1.166
32	1.185	1.184	1.184	1.182	1.181	1.178	1.175	1.172	1.168	1.163
33	1.181	1.181	1.181	1.179	1.178	1.175	1.172	1.169	1.165	1.161
34	1.179	1.178	1.178	1.176	1.175	1.172	1.17	1.166	1.163	1.158
35	1.176	1.176	1.175	1.174	1.172	1.17	1.167	1.164	1.16	1.156
36	1.173	1.173	1.172	1.171	1.169	1.167	1.164	1.161	1.158	1.153
37	1.17	1.17	1.17	1.168	1.167	1.165	1.162	1.159	1.155	1.151
38	1.168	1.168	1.167	1.166	1.164	1.162	1.16	1.157	1.153	1.149
39	1.166	1.165	1.165	1.164	1.162	1.16	1.157	1.154	1.151	1.147
40	1.163	1.163	1.163	1.161	1.16	1.158	1.155	1.152	1.149	1.145
41	1.161	1.161	1.16	1.159	1.158	1.156	1.153	1.15	1.147	1.143
42	1.159	1.159	1.158	1.157	1.156	1.154	1.151	1.148	1.145	1.141
43	1.157	1.157	1.156	1.155	1.154	1.152	1.149	1.147	1.143	1.139
44	1.155	1.155	1.154	1.153	1.152	1.15	1.148	1.145	1.141	1.138
45	1.153	1.153	1.153	1.151	1.15	1.148	1.146	1.143	1.14	1.136
46	1.152	1.151	1.151	1.15	1.148	1.146	1.144	1.141	1.138	1.134
47	1.15	1.15	1.149	1.148	1.147	1.145	1.142	1.14	1.136	1.133
48	1.148	1.148	1.147	1.146	1.145	1.143	1.141	1.138	1.135	1.131
49	1.146	1.146	1.146	1.145	1.143	1.141	1.139	1.137	1.133	1.13
50	1.145	1.145	1.144	1.143	1.142	1.14	1.138	1.135	1.132	1.128
51	1.143	1.143	1.143	1.142	1.14	1.138	1.136	1.134	1.131	1.127
52	1.142	1.142	1.141	1.14	1.139	1.137	1.135	1.132	1.129	1.126
53	1.14	1.14	1.14	1.139	1.137	1.136	1.134	1.131	1.128	1.125
54	1.139	1.139	1.138	1.137	1.136	1.134	1.132	1.13	1.127	1.123
55	1.138	1.137	1.137	1.136	1.135	1.133	1.131	1.128	1.125	1.122
56	1.136	1.136	1.136	1.135	1.133	1.132	1.13	1.127	1.124	1.121
57	1.135	1.135	1.134	1.133	1.132	1.13	1.128	1.126	1.123	1.12
58	1.134	1.134	1.133	1.132	1.131	1.129	1.127	1.125	1.122	1.119
59	1.133	1.132	1.132	1.131	1.13	1.128	1.126	1.124	1.121	1.118
60	1.131	1.131	1.131	1.13	1.129	1.127	1.125	1.123	1.12	1.117

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.25$

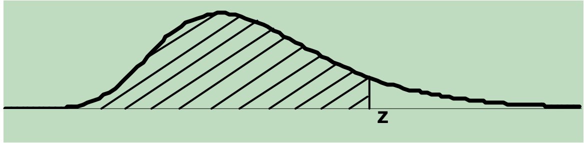
$$\Phi(z)=\int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)}\, x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
1	2.623	2.54	2.448	2.346	2.234	2.111	1.974	1.822	1.646	1.431
2	1.879	1.839	1.794	1.743	1.687	1.624	1.554	1.474	1.379	1.26
3	1.652	1.623	1.591	1.555	1.515	1.47	1.419	1.36	1.29	1.2
4	1.536	1.513	1.487	1.458	1.426	1.389	1.348	1.3	1.243	1.169
5	1.464	1.444	1.422	1.398	1.37	1.339	1.303	1.262	1.212	1.148
6	1.413	1.396	1.377	1.355	1.331	1.303	1.272	1.235	1.191	1.133
7	1.376	1.361	1.343	1.324	1.302	1.277	1.248	1.215	1.175	1.122
8	1.347	1.333	1.317	1.299	1.279	1.256	1.23	1.199	1.162	1.114
9	1.323	1.31	1.295	1.279	1.26	1.239	1.215	1.186	1.152	1.107
10	1.304	1.292	1.278	1.262	1.245	1.225	1.202	1.175	1.143	1.101
11	1.287	1.276	1.263	1.248	1.232	1.213	1.192	1.166	1.136	1.096
12	1.273	1.262	1.25	1.236	1.221	1.203	1.182	1.159	1.13	1.091
13	1.261	1.251	1.239	1.226	1.211	1.194	1.174	1.152	1.124	1.087
14	1.25	1.24	1.229	1.216	1.202	1.186	1.167	1.146	1.119	1.084
15	1.24	1.231	1.22	1.208	1.195	1.179	1.161	1.14	1.115	1.081
16	1.232	1.223	1.212	1.201	1.188	1.173	1.156	1.135	1.111	1.078
17	1.224	1.215	1.205	1.194	1.182	1.167	1.15	1.131	1.107	1.076
18	1.217	1.209	1.199	1.188	1.176	1.162	1.146	1.127	1.104	1.073
19	1.211	1.202	1.193	1.183	1.171	1.157	1.142	1.123	1.101	1.071
20	1.205	1.197	1.188	1.177	1.166	1.153	1.138	1.12	1.098	1.07
21	1.199	1.191	1.183	1.173	1.162	1.149	1.134	1.117	1.096	1.068
22	1.194	1.187	1.178	1.168	1.158	1.145	1.131	1.114	1.093	1.066
23	1.189	1.182	1.174	1.164	1.154	1.142	1.128	1.111	1.091	1.065
24	1.185	1.178	1.17	1.161	1.15	1.138	1.125	1.109	1.089	1.063
25	1.181	1.174	1.166	1.157	1.147	1.135	1.122	1.106	1.087	1.062
26	1.177	1.17	1.162	1.154	1.144	1.133	1.12	1.104	1.086	1.061
27	1.173	1.167	1.159	1.151	1.141	1.13	1.117	1.102	1.084	1.059
28	1.17	1.163	1.156	1.148	1.138	1.127	1.115	1.1	1.082	1.058
29	1.167	1.16	1.153	1.145	1.136	1.125	1.113	1.098	1.081	1.057
30	1.164	1.157	1.15	1.142	1.133	1.123	1.111	1.097	1.079	1.056

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.25$

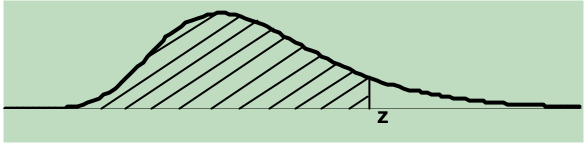
$$\Phi(z)=\int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
31	1.161	1.155	1.148	1.14	1.131	1.121	1.109	1.095	1.078	1.055
32	1.158	1.152	1.145	1.137	1.129	1.119	1.107	1.093	1.077	1.054
33	1.155	1.15	1.143	1.135	1.127	1.117	1.105	1.092	1.075	1.054
34	1.153	1.147	1.141	1.133	1.125	1.115	1.104	1.09	1.074	1.053
35	1.151	1.145	1.138	1.131	1.123	1.113	1.102	1.089	1.073	1.052
36	1.148	1.143	1.136	1.129	1.121	1.111	1.101	1.088	1.072	1.051
37	1.146	1.141	1.134	1.127	1.119	1.11	1.099	1.087	1.071	1.05
38	1.144	1.139	1.132	1.125	1.117	1.108	1.098	1.085	1.07	1.05
39	1.142	1.137	1.131	1.124	1.116	1.107	1.096	1.084	1.069	1.049
40	1.14	1.135	1.129	1.122	1.114	1.105	1.095	1.083	1.068	1.048
41	1.138	1.133	1.127	1.12	1.113	1.104	1.094	1.082	1.067	1.048
42	1.136	1.131	1.125	1.119	1.111	1.103	1.093	1.081	1.067	1.047
43	1.135	1.13	1.124	1.117	1.11	1.101	1.092	1.08	1.066	1.047
44	1.133	1.128	1.122	1.116	1.109	1.1	1.09	1.079	1.065	1.046
45	1.132	1.127	1.121	1.115	1.107	1.099	1.089	1.078	1.064	1.046
46	1.13	1.125	1.12	1.113	1.106	1.098	1.088	1.077	1.063	1.045
47	1.129	1.124	1.118	1.112	1.105	1.097	1.087	1.076	1.063	1.045
48	1.127	1.122	1.117	1.111	1.104	1.096	1.086	1.076	1.062	1.044
49	1.126	1.121	1.116	1.109	1.103	1.095	1.086	1.075	1.061	1.044
50	1.124	1.12	1.114	1.108	1.101	1.094	1.085	1.074	1.061	1.043
51	1.123	1.118	1.113	1.107	1.1	1.093	1.084	1.073	1.06	1.043
52	1.122	1.117	1.112	1.106	1.099	1.092	1.083	1.072	1.06	1.042
53	1.121	1.116	1.111	1.105	1.098	1.091	1.082	1.072	1.059	1.042
54	1.119	1.115	1.11	1.104	1.097	1.09	1.081	1.071	1.058	1.042
55	1.118	1.114	1.109	1.103	1.097	1.089	1.08	1.07	1.058	1.041
56	1.117	1.113	1.108	1.102	1.096	1.088	1.08	1.07	1.057	1.041
57	1.116	1.112	1.107	1.101	1.095	1.087	1.079	1.069	1.057	1.04
58	1.115	1.111	1.106	1.1	1.094	1.087	1.078	1.068	1.056	1.04
59	1.114	1.11	1.105	1.099	1.093	1.086	1.078	1.068	1.056	1.04
60	1.113	1.109	1.104	1.098	1.092	1.085	1.077	1.067	1.055	1.039

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.20$

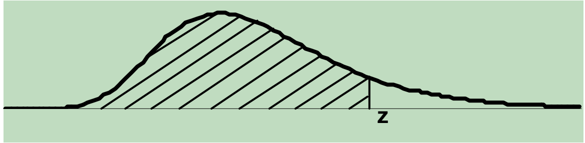
$$\Phi(z)=\int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)}\, x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
1	4.013	4.006	3.988	3.958	3.915	3.861	3.794	3.714	3.622	3.518
2	2.483	2.48	2.472	2.459	2.44	2.416	2.387	2.352	2.311	2.265
3	2.062	2.06	2.055	2.046	2.033	2.017	1.997	1.973	1.945	1.914
4	1.856	1.855	1.851	1.844	1.834	1.821	1.806	1.787	1.765	1.741
5	1.732	1.73	1.727	1.721	1.713	1.702	1.689	1.674	1.656	1.635
6	1.646	1.645	1.642	1.637	1.63	1.621	1.61	1.596	1.58	1.562
7	1.584	1.583	1.58	1.576	1.57	1.561	1.551	1.539	1.525	1.509
8	1.536	1.535	1.533	1.528	1.523	1.515	1.506	1.495	1.482	1.468
9	1.497	1.497	1.494	1.491	1.485	1.478	1.47	1.46	1.448	1.435
10	1.466	1.465	1.463	1.459	1.454	1.448	1.44	1.431	1.42	1.407
11	1.439	1.438	1.436	1.433	1.429	1.423	1.415	1.407	1.396	1.385
12	1.416	1.416	1.414	1.411	1.406	1.401	1.394	1.386	1.376	1.365
13	1.397	1.396	1.394	1.391	1.387	1.382	1.375	1.368	1.359	1.348
14	1.379	1.379	1.377	1.374	1.371	1.365	1.359	1.352	1.343	1.333
15	1.364	1.364	1.362	1.359	1.356	1.351	1.345	1.338	1.329	1.32
16	1.35	1.35	1.348	1.346	1.342	1.338	1.332	1.325	1.317	1.308
17	1.338	1.338	1.336	1.334	1.33	1.326	1.32	1.314	1.306	1.297
18	1.327	1.327	1.325	1.323	1.32	1.315	1.31	1.304	1.296	1.288
19	1.317	1.316	1.315	1.313	1.31	1.306	1.3	1.294	1.287	1.279
20	1.308	1.307	1.306	1.304	1.301	1.297	1.292	1.286	1.279	1.271
21	1.299	1.299	1.297	1.295	1.292	1.288	1.284	1.278	1.271	1.263
22	1.291	1.291	1.29	1.287	1.285	1.281	1.276	1.271	1.264	1.257
23	1.284	1.283	1.282	1.28	1.277	1.274	1.269	1.264	1.257	1.25
24	1.277	1.277	1.275	1.274	1.271	1.267	1.263	1.258	1.251	1.244
25	1.271	1.27	1.269	1.267	1.265	1.261	1.257	1.252	1.246	1.239
26	1.265	1.264	1.263	1.261	1.259	1.255	1.251	1.246	1.24	1.233
27	1.259	1.259	1.258	1.256	1.253	1.25	1.246	1.241	1.235	1.229
28	1.254	1.253	1.252	1.251	1.248	1.245	1.241	1.236	1.23	1.224
29	1.249	1.248	1.247	1.246	1.243	1.24	1.236	1.231	1.226	1.22
30	1.244	1.244	1.243	1.241	1.239	1.236	1.232	1.227	1.222	1.215

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.20$

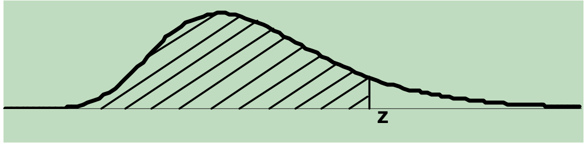
$$\Phi(z) = \int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1} (1+x)^{-2k} \left[1 - \left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1 - a$$



	ρ									
k	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
31	1.24	1.239	1.238	1.237	1.234	1.231	1.228	1.223	1.218	1.212
32	1.235	1.235	1.234	1.232	1.23	1.227	1.224	1.219	1.214	1.208
33	1.231	1.231	1.23	1.229	1.226	1.223	1.22	1.215	1.21	1.204
34	1.228	1.227	1.226	1.225	1.223	1.22	1.216	1.212	1.207	1.201
35	1.224	1.224	1.223	1.221	1.219	1.216	1.213	1.208	1.204	1.198
36	1.22	1.22	1.219	1.218	1.216	1.213	1.209	1.205	1.2	1.195
37	1.217	1.217	1.216	1.214	1.212	1.21	1.206	1.202	1.197	1.192
38	1.214	1.214	1.213	1.211	1.209	1.207	1.203	1.199	1.194	1.189
39	1.211	1.211	1.21	1.208	1.206	1.204	1.2	1.196	1.192	1.186
40	1.208	1.208	1.207	1.205	1.203	1.201	1.198	1.194	1.189	1.184
41	1.205	1.205	1.204	1.203	1.201	1.198	1.195	1.191	1.187	1.181
42	1.202	1.202	1.201	1.2	1.198	1.195	1.192	1.189	1.184	1.179
43	1.2	1.2	1.199	1.197	1.195	1.193	1.19	1.186	1.182	1.177
44	1.197	1.197	1.196	1.195	1.193	1.19	1.187	1.184	1.179	1.175
45	1.195	1.195	1.194	1.192	1.191	1.188	1.185	1.182	1.177	1.172
46	1.193	1.192	1.191	1.19	1.188	1.186	1.183	1.179	1.175	1.17
47	1.19	1.19	1.189	1.188	1.186	1.184	1.181	1.177	1.173	1.168
48	1.188	1.188	1.187	1.186	1.184	1.182	1.179	1.175	1.171	1.166
49	1.186	1.186	1.185	1.184	1.182	1.18	1.177	1.173	1.169	1.165
50	1.184	1.184	1.183	1.182	1.18	1.178	1.175	1.171	1.167	1.163
51	1.182	1.182	1.181	1.18	1.178	1.176	1.173	1.17	1.166	1.161
52	1.18	1.18	1.179	1.178	1.176	1.174	1.171	1.168	1.164	1.159
53	1.178	1.178	1.177	1.176	1.174	1.172	1.169	1.166	1.162	1.158
54	1.176	1.176	1.175	1.174	1.173	1.17	1.168	1.164	1.161	1.156
55	1.175	1.174	1.174	1.172	1.171	1.169	1.166	1.163	1.159	1.155
56	1.173	1.173	1.172	1.171	1.169	1.167	1.164	1.161	1.157	1.153
57	1.171	1.171	1.17	1.169	1.168	1.165	1.163	1.16	1.156	1.152
58	1.17	1.169	1.169	1.168	1.166	1.164	1.161	1.158	1.154	1.15
59	1.168	1.168	1.167	1.166	1.164	1.162	1.16	1.157	1.153	1.149
60	1.167	1.166	1.166	1.165	1.163	1.161	1.158	1.155	1.152	1.148

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.20$

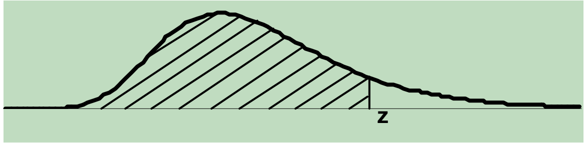
$$\Phi(z)=\int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
1	3.4	3.269	3.125	2.966	2.792	2.602	2.394	2.163	1.902	1.591
2	2.212	2.153	2.088	2.015	1.935	1.846	1.746	1.634	1.503	1.34
3	1.878	1.837	1.792	1.742	1.686	1.623	1.553	1.473	1.378	1.259
4	1.712	1.68	1.645	1.605	1.561	1.511	1.455	1.39	1.314	1.216
5	1.611	1.584	1.555	1.521	1.484	1.442	1.394	1.339	1.273	1.189
6	1.542	1.518	1.492	1.463	1.43	1.393	1.351	1.303	1.245	1.17
7	1.491	1.47	1.446	1.42	1.391	1.358	1.32	1.276	1.224	1.156
8	1.451	1.432	1.411	1.387	1.36	1.33	1.295	1.255	1.207	1.144
9	1.419	1.402	1.382	1.36	1.335	1.307	1.275	1.238	1.194	1.135
10	1.393	1.377	1.359	1.338	1.315	1.289	1.259	1.224	1.182	1.128
11	1.371	1.356	1.339	1.319	1.298	1.273	1.245	1.212	1.173	1.121
12	1.352	1.338	1.322	1.304	1.283	1.26	1.233	1.202	1.165	1.115
13	1.336	1.322	1.307	1.29	1.27	1.248	1.223	1.193	1.157	1.11
14	1.322	1.309	1.294	1.277	1.259	1.238	1.214	1.185	1.151	1.106
15	1.309	1.296	1.282	1.267	1.249	1.229	1.205	1.178	1.145	1.102
16	1.298	1.286	1.272	1.257	1.24	1.22	1.198	1.172	1.14	1.099
17	1.287	1.276	1.263	1.248	1.232	1.213	1.191	1.166	1.136	1.096
18	1.278	1.267	1.254	1.24	1.224	1.206	1.185	1.161	1.132	1.093
19	1.269	1.259	1.247	1.233	1.218	1.2	1.18	1.156	1.128	1.09
20	1.262	1.251	1.24	1.226	1.211	1.194	1.175	1.152	1.124	1.088
21	1.255	1.244	1.233	1.22	1.206	1.189	1.17	1.148	1.121	1.085
22	1.248	1.238	1.227	1.215	1.201	1.184	1.166	1.144	1.118	1.083
23	1.242	1.232	1.221	1.209	1.196	1.18	1.162	1.141	1.115	1.081
24	1.236	1.227	1.216	1.204	1.191	1.176	1.158	1.138	1.113	1.08
25	1.231	1.222	1.211	1.2	1.187	1.172	1.155	1.135	1.11	1.078
26	1.226	1.217	1.207	1.196	1.183	1.168	1.152	1.132	1.108	1.076
27	1.221	1.212	1.203	1.192	1.179	1.165	1.148	1.129	1.106	1.075
28	1.217	1.208	1.199	1.188	1.175	1.162	1.146	1.127	1.104	1.073
29	1.212	1.204	1.195	1.184	1.172	1.159	1.143	1.124	1.102	1.072
30	1.208	1.2	1.191	1.181	1.169	1.156	1.14	1.122	1.1	1.071

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.20$

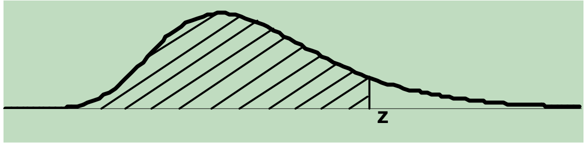
$$\Phi(z)=\int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)}\, x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
31	1.205	1.197	1.188	1.178	1.166	1.153	1.138	1.12	1.098	1.07
32	1.201	1.193	1.184	1.174	1.163	1.15	1.135	1.118	1.097	1.068
33	1.198	1.19	1.181	1.172	1.16	1.148	1.133	1.116	1.095	1.067
34	1.194	1.187	1.178	1.169	1.158	1.145	1.131	1.114	1.094	1.066
35	1.191	1.184	1.176	1.166	1.155	1.143	1.129	1.112	1.092	1.065
36	1.188	1.181	1.173	1.164	1.153	1.141	1.127	1.111	1.091	1.064
37	1.186	1.178	1.17	1.161	1.151	1.139	1.125	1.109	1.09	1.063
38	1.183	1.176	1.168	1.159	1.149	1.137	1.123	1.108	1.088	1.062
39	1.18	1.173	1.166	1.157	1.147	1.135	1.122	1.106	1.087	1.062
40	1.178	1.171	1.163	1.154	1.145	1.133	1.12	1.105	1.086	1.061
41	1.175	1.169	1.161	1.152	1.143	1.131	1.119	1.103	1.085	1.06
42	1.173	1.167	1.159	1.15	1.141	1.13	1.117	1.102	1.084	1.059
43	1.171	1.164	1.157	1.149	1.139	1.128	1.116	1.101	1.083	1.059
44	1.169	1.162	1.155	1.147	1.137	1.127	1.114	1.1	1.082	1.058
45	1.167	1.16	1.153	1.145	1.136	1.125	1.113	1.098	1.081	1.057
46	1.165	1.158	1.151	1.143	1.134	1.124	1.112	1.097	1.08	1.057
47	1.163	1.157	1.15	1.142	1.133	1.122	1.11	1.096	1.079	1.056
48	1.161	1.155	1.148	1.14	1.131	1.121	1.109	1.095	1.078	1.055
49	1.159	1.153	1.146	1.138	1.13	1.12	1.108	1.094	1.077	1.055
50	1.158	1.151	1.145	1.137	1.128	1.118	1.107	1.093	1.076	1.054
51	1.156	1.15	1.143	1.136	1.127	1.117	1.106	1.092	1.076	1.054
52	1.154	1.148	1.142	1.134	1.126	1.116	1.105	1.091	1.075	1.053
53	1.153	1.147	1.14	1.133	1.124	1.115	1.103	1.09	1.074	1.053
54	1.151	1.145	1.139	1.131	1.123	1.113	1.102	1.089	1.073	1.052
55	1.15	1.144	1.137	1.13	1.122	1.112	1.101	1.089	1.073	1.052
56	1.148	1.143	1.136	1.129	1.121	1.111	1.1	1.088	1.072	1.051
57	1.147	1.141	1.135	1.128	1.12	1.11	1.1	1.087	1.071	1.051
58	1.145	1.14	1.134	1.127	1.118	1.109	1.099	1.086	1.071	1.05
59	1.144	1.139	1.132	1.125	1.117	1.108	1.098	1.085	1.07	1.05
60	1.143	1.137	1.131	1.124	1.116	1.107	1.097	1.085	1.07	1.049

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.15$

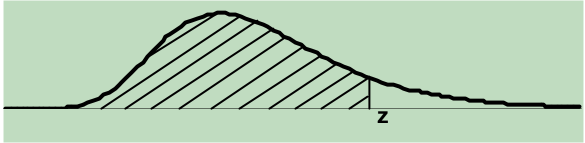
$$\Phi(z)=\int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
1	5.689	5.679	5.649	5.599	5.528	5.438	5.327	5.197	5.045	4.873
2	3.092	3.088	3.076	3.056	3.028	2.992	2.949	2.896	2.836	2.767
3	2.449	2.447	2.439	2.426	2.408	2.385	2.356	2.322	2.282	2.237
4	2.149	2.147	2.141	2.131	2.117	2.099	2.077	2.051	2.021	1.986
5	1.97	1.969	1.964	1.956	1.945	1.93	1.912	1.89	1.865	1.837
6	1.851	1.849	1.845	1.838	1.829	1.816	1.801	1.782	1.761	1.736
7	1.764	1.763	1.759	1.753	1.744	1.733	1.72	1.703	1.684	1.662
8	1.698	1.697	1.693	1.688	1.68	1.67	1.658	1.643	1.626	1.606
9	1.645	1.644	1.641	1.636	1.629	1.62	1.609	1.595	1.579	1.561
10	1.602	1.601	1.599	1.594	1.587	1.579	1.568	1.556	1.541	1.525
11	1.566	1.566	1.563	1.559	1.553	1.545	1.535	1.523	1.51	1.494
12	1.536	1.535	1.533	1.529	1.523	1.515	1.506	1.495	1.483	1.468
13	1.51	1.509	1.507	1.503	1.497	1.49	1.482	1.471	1.459	1.445
14	1.487	1.486	1.484	1.48	1.475	1.468	1.46	1.45	1.439	1.426
15	1.466	1.466	1.464	1.46	1.455	1.449	1.441	1.432	1.421	1.408
16	1.448	1.448	1.446	1.442	1.438	1.431	1.424	1.415	1.405	1.393
17	1.432	1.431	1.429	1.426	1.422	1.416	1.409	1.4	1.39	1.379
18	1.417	1.417	1.415	1.412	1.407	1.402	1.395	1.387	1.377	1.366
19	1.404	1.403	1.402	1.399	1.394	1.389	1.382	1.374	1.365	1.354
20	1.392	1.391	1.389	1.387	1.383	1.377	1.371	1.363	1.354	1.344
21	1.381	1.38	1.378	1.376	1.372	1.367	1.36	1.353	1.344	1.334
22	1.37	1.37	1.368	1.365	1.362	1.357	1.351	1.343	1.335	1.325
23	1.361	1.36	1.358	1.356	1.352	1.347	1.342	1.335	1.326	1.317
24	1.352	1.351	1.35	1.347	1.343	1.339	1.333	1.326	1.318	1.309
25	1.343	1.343	1.341	1.339	1.335	1.331	1.325	1.319	1.311	1.302
26	1.336	1.335	1.334	1.331	1.328	1.323	1.318	1.311	1.304	1.295
27	1.328	1.328	1.326	1.324	1.321	1.316	1.311	1.305	1.297	1.289
28	1.321	1.321	1.32	1.317	1.314	1.31	1.305	1.298	1.291	1.283
29	1.315	1.314	1.313	1.311	1.308	1.304	1.299	1.292	1.285	1.277
30	1.309	1.308	1.307	1.305	1.302	1.298	1.293	1.287	1.28	1.272

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.15$

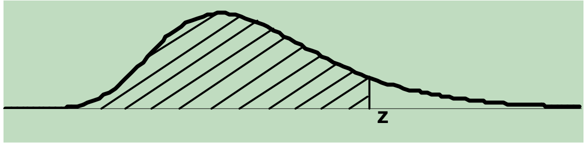
$$\Phi(z)=\int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)}\, x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
31	1.303	1.303	1.301	1.299	1.296	1.292	1.287	1.282	1.275	1.267
32	1.298	1.297	1.296	1.294	1.291	1.287	1.282	1.276	1.27	1.262
33	1.292	1.292	1.291	1.289	1.286	1.282	1.277	1.272	1.265	1.258
34	1.287	1.287	1.286	1.284	1.281	1.277	1.273	1.267	1.261	1.253
35	1.283	1.282	1.281	1.279	1.276	1.273	1.268	1.263	1.256	1.249
36	1.278	1.278	1.277	1.275	1.272	1.268	1.264	1.259	1.252	1.245
37	1.274	1.274	1.272	1.27	1.268	1.264	1.26	1.255	1.249	1.241
38	1.27	1.269	1.268	1.266	1.264	1.26	1.256	1.251	1.245	1.238
39	1.266	1.265	1.264	1.262	1.26	1.256	1.252	1.247	1.241	1.234
40	1.262	1.262	1.261	1.259	1.256	1.253	1.249	1.244	1.238	1.231
41	1.258	1.258	1.257	1.255	1.253	1.249	1.245	1.24	1.235	1.228
42	1.255	1.255	1.254	1.252	1.249	1.246	1.242	1.237	1.231	1.225
43	1.252	1.251	1.25	1.248	1.246	1.243	1.239	1.234	1.228	1.222
44	1.248	1.248	1.247	1.245	1.243	1.24	1.236	1.231	1.226	1.219
45	1.245	1.245	1.244	1.242	1.24	1.237	1.233	1.228	1.223	1.216
46	1.242	1.242	1.241	1.239	1.237	1.234	1.23	1.225	1.22	1.214
47	1.239	1.239	1.238	1.236	1.234	1.231	1.227	1.223	1.217	1.211
48	1.237	1.236	1.235	1.234	1.231	1.228	1.225	1.22	1.215	1.209
49	1.234	1.234	1.233	1.231	1.229	1.226	1.222	1.218	1.212	1.206
50	1.231	1.231	1.23	1.228	1.226	1.223	1.22	1.215	1.21	1.204
51	1.229	1.228	1.227	1.226	1.224	1.221	1.217	1.213	1.208	1.202
52	1.226	1.226	1.225	1.223	1.221	1.218	1.215	1.211	1.206	1.2
53	1.224	1.224	1.223	1.221	1.219	1.216	1.213	1.208	1.203	1.198
54	1.222	1.221	1.22	1.219	1.217	1.214	1.21	1.206	1.201	1.196
55	1.219	1.219	1.218	1.217	1.214	1.212	1.208	1.204	1.199	1.194
56	1.217	1.217	1.216	1.214	1.212	1.21	1.206	1.202	1.197	1.192
57	1.215	1.215	1.214	1.212	1.21	1.208	1.204	1.2	1.195	1.19
58	1.213	1.213	1.212	1.21	1.208	1.206	1.202	1.198	1.194	1.188
59	1.211	1.211	1.21	1.208	1.206	1.204	1.2	1.196	1.192	1.186
60	1.209	1.209	1.208	1.206	1.204	1.202	1.198	1.195	1.19	1.185

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.15$

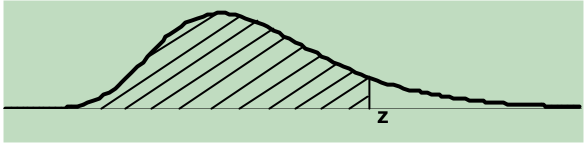
$$\Phi(z)=\int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)}\, x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
1	4.681	4.467	4.232	3.975	3.695	3.391	3.059	2.697	2.295	1.828
2	2.689	2.603	2.507	2.401	2.284	2.155	2.013	1.853	1.67	1.447
3	2.186	2.128	2.064	1.994	1.915	1.828	1.731	1.621	1.493	1.334
4	1.947	1.903	1.853	1.799	1.738	1.67	1.593	1.507	1.405	1.276
5	1.804	1.768	1.727	1.681	1.63	1.574	1.51	1.436	1.35	1.24
6	1.708	1.676	1.641	1.601	1.557	1.508	1.452	1.388	1.312	1.215
7	1.638	1.61	1.578	1.543	1.504	1.46	1.41	1.352	1.284	1.196
8	1.584	1.558	1.53	1.498	1.463	1.423	1.377	1.325	1.262	1.182
9	1.541	1.518	1.492	1.462	1.43	1.393	1.351	1.302	1.245	1.17
10	1.506	1.484	1.46	1.433	1.402	1.368	1.329	1.284	1.23	1.16
11	1.476	1.456	1.434	1.408	1.38	1.348	1.311	1.269	1.218	1.152
12	1.451	1.432	1.411	1.387	1.36	1.33	1.295	1.255	1.207	1.144
13	1.429	1.412	1.391	1.369	1.343	1.315	1.282	1.244	1.198	1.138
14	1.411	1.393	1.374	1.353	1.329	1.301	1.27	1.234	1.19	1.133
15	1.394	1.377	1.359	1.339	1.315	1.289	1.259	1.224	1.183	1.128
16	1.379	1.363	1.346	1.326	1.304	1.279	1.25	1.216	1.176	1.123
17	1.365	1.35	1.333	1.315	1.293	1.269	1.241	1.209	1.17	1.119
18	1.353	1.339	1.322	1.304	1.284	1.26	1.234	1.202	1.165	1.116
19	1.342	1.328	1.312	1.295	1.275	1.252	1.227	1.196	1.16	1.112
20	1.332	1.318	1.303	1.286	1.267	1.245	1.22	1.191	1.156	1.109
21	1.323	1.31	1.295	1.278	1.26	1.238	1.214	1.186	1.151	1.106
22	1.314	1.301	1.287	1.271	1.253	1.232	1.209	1.181	1.148	1.104
23	1.306	1.294	1.28	1.264	1.246	1.226	1.203	1.177	1.144	1.101
24	1.298	1.287	1.273	1.258	1.241	1.221	1.199	1.173	1.141	1.099
25	1.292	1.28	1.267	1.252	1.235	1.216	1.194	1.169	1.138	1.097
26	1.285	1.274	1.261	1.246	1.23	1.211	1.19	1.165	1.135	1.095
27	1.279	1.268	1.255	1.241	1.225	1.207	1.186	1.162	1.132	1.093
28	1.273	1.262	1.25	1.236	1.221	1.203	1.182	1.159	1.13	1.091
29	1.268	1.257	1.245	1.232	1.216	1.199	1.179	1.156	1.127	1.09
30	1.263	1.252	1.241	1.227	1.212	1.195	1.176	1.153	1.125	1.088

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.15$

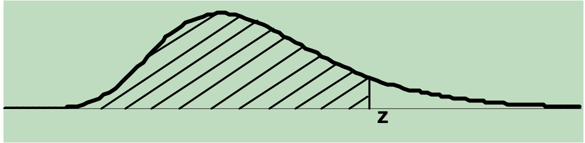
$$\Phi(z)=\int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)}\, x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
31	1.258	1.248	1.236	1.223	1.208	1.192	1.172	1.15	1.123	1.086
32	1.253	1.243	1.232	1.219	1.205	1.188	1.169	1.147	1.121	1.085
33	1.249	1.239	1.228	1.215	1.201	1.185	1.167	1.145	1.119	1.084
34	1.245	1.235	1.224	1.212	1.198	1.182	1.164	1.143	1.117	1.082
35	1.241	1.231	1.221	1.209	1.195	1.179	1.161	1.14	1.115	1.081
36	1.237	1.228	1.217	1.205	1.192	1.177	1.159	1.138	1.113	1.08
37	1.233	1.224	1.214	1.202	1.189	1.174	1.157	1.136	1.112	1.079
38	1.23	1.221	1.211	1.199	1.186	1.171	1.154	1.134	1.11	1.078
39	1.227	1.218	1.208	1.196	1.184	1.169	1.152	1.132	1.108	1.077
40	1.223	1.215	1.205	1.194	1.181	1.167	1.15	1.131	1.107	1.076
41	1.22	1.212	1.202	1.191	1.179	1.164	1.148	1.129	1.106	1.075
42	1.217	1.209	1.199	1.189	1.176	1.162	1.146	1.127	1.104	1.074
43	1.215	1.206	1.197	1.186	1.174	1.16	1.144	1.126	1.103	1.073
44	1.212	1.204	1.194	1.184	1.172	1.158	1.143	1.124	1.102	1.072
45	1.209	1.201	1.192	1.182	1.17	1.156	1.141	1.123	1.1	1.071
46	1.207	1.199	1.19	1.179	1.168	1.154	1.139	1.121	1.099	1.07
47	1.204	1.196	1.187	1.177	1.166	1.153	1.138	1.12	1.098	1.069
48	1.202	1.194	1.185	1.175	1.164	1.151	1.136	1.119	1.097	1.069
49	1.2	1.192	1.183	1.173	1.162	1.149	1.135	1.117	1.096	1.068
50	1.197	1.19	1.181	1.171	1.16	1.148	1.133	1.116	1.095	1.067
51	1.195	1.188	1.179	1.17	1.159	1.146	1.132	1.115	1.094	1.067
52	1.193	1.186	1.177	1.168	1.157	1.145	1.13	1.114	1.093	1.066
53	1.191	1.184	1.175	1.166	1.155	1.143	1.129	1.112	1.092	1.065
54	1.189	1.182	1.174	1.164	1.154	1.142	1.128	1.111	1.091	1.065
55	1.187	1.18	1.172	1.163	1.152	1.14	1.126	1.11	1.09	1.064
56	1.186	1.178	1.17	1.161	1.151	1.139	1.125	1.109	1.09	1.063
57	1.184	1.177	1.169	1.16	1.149	1.138	1.124	1.108	1.089	1.063
58	1.182	1.175	1.167	1.158	1.148	1.136	1.123	1.107	1.088	1.062
59	1.18	1.173	1.166	1.157	1.147	1.135	1.122	1.106	1.087	1.062
60	1.179	1.172	1.164	1.155	1.145	1.134	1.121	1.105	1.086	1.061

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.10$

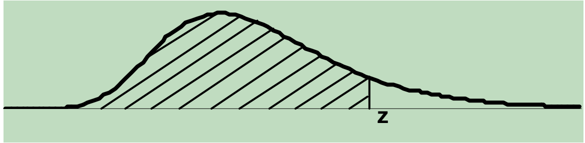
$$\Phi(z) = \int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
1	9.05	9.032	8.977	8.886	8.758	8.594	8.393	8.156	7.881	7.571
2	4.107	4.101	4.082	4.051	4.007	3.951	3.882	3.8	3.705	3.597
3	3.055	3.051	3.039	3.02	2.992	2.957	2.914	2.863	2.804	2.737
4	2.589	2.586	2.578	2.564	2.543	2.517	2.485	2.447	2.403	2.353
5	2.323	2.32	2.313	2.302	2.285	2.264	2.239	2.208	2.172	2.131
6	2.147	2.145	2.14	2.13	2.116	2.098	2.076	2.05	2.02	1.985
7	2.022	2.021	2.016	2.007	1.995	1.979	1.96	1.937	1.911	1.88
8	1.928	1.927	1.922	1.914	1.904	1.89	1.872	1.852	1.828	1.801
9	1.854	1.853	1.848	1.841	1.832	1.819	1.803	1.785	1.763	1.738
10	1.794	1.793	1.789	1.782	1.773	1.762	1.747	1.73	1.71	1.688
11	1.744	1.743	1.739	1.733	1.725	1.714	1.701	1.685	1.667	1.645
12	1.702	1.701	1.697	1.692	1.684	1.674	1.662	1.647	1.629	1.61
13	1.666	1.665	1.661	1.656	1.649	1.639	1.628	1.614	1.597	1.579
14	1.634	1.633	1.63	1.625	1.618	1.609	1.598	1.585	1.57	1.552
15	1.606	1.606	1.603	1.598	1.591	1.583	1.572	1.56	1.545	1.528
16	1.582	1.581	1.578	1.574	1.568	1.559	1.549	1.537	1.523	1.507
17	1.56	1.559	1.556	1.552	1.546	1.538	1.529	1.517	1.504	1.488
18	1.54	1.539	1.537	1.533	1.527	1.519	1.51	1.499	1.486	1.471
19	1.522	1.521	1.519	1.515	1.509	1.502	1.493	1.483	1.47	1.456
20	1.506	1.505	1.503	1.499	1.493	1.486	1.478	1.468	1.456	1.442
21	1.491	1.49	1.488	1.484	1.479	1.472	1.464	1.454	1.442	1.429
22	1.477	1.476	1.474	1.47	1.465	1.459	1.451	1.441	1.43	1.417
23	1.464	1.463	1.461	1.458	1.453	1.446	1.439	1.429	1.419	1.406
24	1.452	1.451	1.449	1.446	1.441	1.435	1.428	1.419	1.408	1.396
25	1.441	1.44	1.438	1.435	1.43	1.424	1.417	1.408	1.398	1.386
26	1.431	1.43	1.428	1.425	1.42	1.415	1.407	1.399	1.389	1.377
27	1.421	1.42	1.418	1.415	1.411	1.405	1.398	1.39	1.38	1.369
28	1.412	1.411	1.409	1.406	1.402	1.397	1.39	1.382	1.372	1.361
29	1.403	1.403	1.401	1.398	1.394	1.388	1.382	1.374	1.364	1.354
30	1.395	1.395	1.393	1.39	1.386	1.381	1.374	1.366	1.357	1.347

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.10$

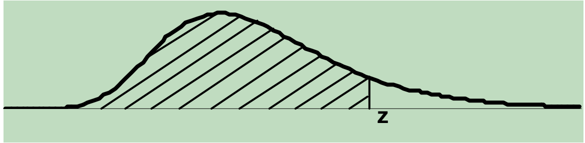
$$\Phi(z)=\int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)}\, x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
31	1.388	1.387	1.385	1.382	1.378	1.373	1.367	1.359	1.35	1.34
32	1.38	1.38	1.378	1.375	1.371	1.366	1.36	1.353	1.344	1.334
33	1.373	1.373	1.371	1.369	1.365	1.36	1.354	1.346	1.338	1.328
34	1.367	1.366	1.365	1.362	1.358	1.354	1.348	1.34	1.332	1.322
35	1.361	1.36	1.359	1.356	1.352	1.348	1.342	1.335	1.326	1.317
36	1.355	1.354	1.353	1.35	1.347	1.342	1.336	1.329	1.321	1.312
37	1.349	1.349	1.347	1.345	1.341	1.337	1.331	1.324	1.316	1.307
38	1.344	1.343	1.342	1.339	1.336	1.331	1.326	1.319	1.311	1.302
39	1.339	1.338	1.337	1.334	1.331	1.326	1.321	1.314	1.307	1.298
40	1.334	1.333	1.332	1.329	1.326	1.322	1.316	1.31	1.302	1.293
41	1.329	1.328	1.327	1.325	1.321	1.317	1.312	1.305	1.298	1.289
42	1.324	1.324	1.323	1.32	1.317	1.313	1.307	1.301	1.294	1.285
43	1.32	1.32	1.318	1.316	1.313	1.308	1.303	1.297	1.29	1.282
44	1.316	1.315	1.314	1.312	1.309	1.304	1.299	1.293	1.286	1.278
45	1.312	1.311	1.31	1.308	1.305	1.301	1.296	1.29	1.282	1.274
46	1.308	1.307	1.306	1.304	1.301	1.297	1.292	1.286	1.279	1.271
47	1.304	1.304	1.302	1.3	1.297	1.293	1.288	1.282	1.276	1.268
48	1.3	1.3	1.299	1.297	1.294	1.29	1.285	1.279	1.272	1.265
49	1.297	1.296	1.295	1.293	1.29	1.286	1.281	1.276	1.269	1.261
50	1.293	1.293	1.292	1.29	1.287	1.283	1.278	1.273	1.266	1.258
51	1.29	1.29	1.289	1.286	1.284	1.28	1.275	1.27	1.263	1.256
52	1.287	1.287	1.285	1.283	1.28	1.277	1.272	1.267	1.26	1.253
53	1.284	1.283	1.282	1.28	1.277	1.274	1.269	1.264	1.257	1.25
54	1.281	1.28	1.279	1.277	1.274	1.271	1.266	1.261	1.255	1.248
55	1.278	1.278	1.276	1.274	1.272	1.268	1.264	1.258	1.252	1.245
56	1.275	1.275	1.274	1.272	1.269	1.265	1.261	1.256	1.25	1.243
57	1.272	1.272	1.271	1.269	1.266	1.263	1.258	1.253	1.247	1.24
58	1.27	1.269	1.268	1.266	1.264	1.26	1.256	1.251	1.245	1.238
59	1.267	1.267	1.266	1.264	1.261	1.258	1.253	1.248	1.242	1.236
60	1.265	1.264	1.263	1.261	1.259	1.255	1.251	1.246	1.24	1.233

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.10$

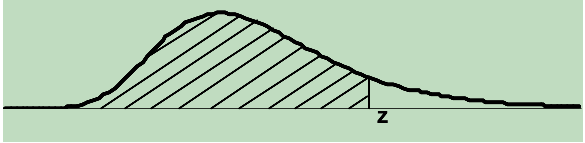
$$\Phi(z)=\int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)}\, x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
1	7.223	6.838	6.416	5.955	5.456	4.917	4.336	3.707	3.022	2.249
2	3.475	3.34	3.191	3.027	2.848	2.651	2.436	2.197	1.928	1.607
3	2.661	2.576	2.481	2.377	2.263	2.137	1.997	1.84	1.66	1.441
4	2.296	2.233	2.162	2.083	1.997	1.901	1.794	1.673	1.533	1.36
5	2.085	2.034	1.976	1.912	1.841	1.762	1.674	1.573	1.457	1.31
6	1.946	1.902	1.853	1.798	1.737	1.669	1.593	1.506	1.404	1.276
7	1.846	1.807	1.764	1.716	1.662	1.602	1.534	1.457	1.366	1.251
8	1.77	1.735	1.696	1.653	1.605	1.55	1.489	1.419	1.337	1.232
9	1.71	1.679	1.643	1.604	1.559	1.51	1.454	1.389	1.313	1.216
10	1.662	1.632	1.6	1.563	1.522	1.476	1.424	1.365	1.294	1.203
11	1.621	1.594	1.564	1.529	1.491	1.448	1.4	1.344	1.278	1.192
12	1.587	1.561	1.533	1.501	1.465	1.425	1.379	1.326	1.264	1.183
13	1.557	1.533	1.506	1.476	1.442	1.404	1.361	1.311	1.252	1.175
14	1.532	1.509	1.483	1.455	1.423	1.386	1.345	1.298	1.241	1.167
15	1.509	1.487	1.463	1.436	1.405	1.371	1.331	1.286	1.231	1.161
16	1.489	1.468	1.445	1.419	1.389	1.356	1.319	1.275	1.223	1.155
17	1.471	1.451	1.429	1.404	1.375	1.344	1.308	1.266	1.215	1.15
18	1.455	1.435	1.414	1.39	1.363	1.332	1.297	1.257	1.209	1.145
19	1.44	1.421	1.401	1.377	1.351	1.322	1.288	1.249	1.202	1.141
20	1.426	1.408	1.388	1.366	1.341	1.312	1.28	1.242	1.196	1.137
21	1.414	1.397	1.377	1.356	1.331	1.303	1.272	1.235	1.191	1.134
22	1.402	1.386	1.367	1.346	1.322	1.295	1.265	1.229	1.186	1.13
23	1.392	1.376	1.357	1.337	1.314	1.288	1.258	1.223	1.182	1.127
24	1.382	1.366	1.348	1.329	1.306	1.281	1.252	1.218	1.177	1.124
25	1.373	1.357	1.34	1.321	1.299	1.274	1.246	1.213	1.173	1.121
26	1.364	1.349	1.332	1.314	1.292	1.268	1.241	1.208	1.17	1.119
27	1.356	1.342	1.325	1.307	1.286	1.262	1.235	1.204	1.166	1.117
28	1.349	1.334	1.318	1.3	1.28	1.257	1.231	1.2	1.163	1.114
29	1.341	1.328	1.312	1.294	1.274	1.252	1.226	1.196	1.16	1.112
30	1.335	1.321	1.306	1.289	1.269	1.247	1.222	1.192	1.157	1.11

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.10$

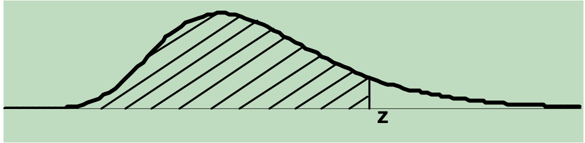
$$\Phi(z) = \int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
31	1.328	1.315	1.3	1.283	1.264	1.243	1.218	1.189	1.154	1.108
32	1.322	1.309	1.295	1.278	1.259	1.238	1.214	1.186	1.151	1.106
33	1.317	1.304	1.29	1.273	1.255	1.234	1.21	1.183	1.149	1.105
34	1.311	1.299	1.285	1.269	1.251	1.23	1.207	1.18	1.146	1.103
35	1.306	1.294	1.28	1.264	1.247	1.227	1.204	1.177	1.144	1.101
36	1.301	1.289	1.275	1.26	1.243	1.223	1.2	1.174	1.142	1.1
37	1.296	1.285	1.271	1.256	1.239	1.22	1.197	1.171	1.14	1.098
38	1.292	1.28	1.267	1.252	1.235	1.216	1.194	1.169	1.138	1.097
39	1.288	1.276	1.263	1.249	1.232	1.213	1.192	1.167	1.136	1.096
40	1.283	1.272	1.259	1.245	1.229	1.21	1.189	1.164	1.134	1.094
41	1.28	1.268	1.256	1.242	1.226	1.207	1.186	1.162	1.132	1.093
42	1.276	1.265	1.252	1.238	1.223	1.205	1.184	1.16	1.131	1.092
43	1.272	1.261	1.249	1.235	1.22	1.202	1.182	1.158	1.129	1.091
44	1.269	1.258	1.246	1.232	1.217	1.199	1.179	1.156	1.127	1.09
45	1.265	1.255	1.243	1.229	1.214	1.197	1.177	1.154	1.126	1.089
46	1.262	1.251	1.24	1.227	1.212	1.195	1.175	1.152	1.124	1.088
47	1.259	1.248	1.237	1.224	1.209	1.192	1.173	1.15	1.123	1.087
48	1.256	1.246	1.234	1.221	1.207	1.19	1.171	1.149	1.122	1.086
49	1.253	1.243	1.231	1.219	1.204	1.188	1.169	1.147	1.12	1.085
50	1.25	1.24	1.229	1.216	1.202	1.186	1.167	1.145	1.119	1.084
51	1.247	1.237	1.226	1.214	1.2	1.184	1.165	1.144	1.118	1.083
52	1.244	1.235	1.224	1.212	1.198	1.182	1.164	1.142	1.116	1.082
53	1.242	1.232	1.221	1.209	1.196	1.18	1.162	1.141	1.115	1.081
54	1.239	1.23	1.219	1.207	1.194	1.178	1.16	1.14	1.114	1.081
55	1.237	1.227	1.217	1.205	1.192	1.176	1.159	1.138	1.113	1.08
56	1.234	1.225	1.215	1.203	1.19	1.175	1.157	1.137	1.112	1.079
57	1.232	1.223	1.213	1.201	1.188	1.173	1.156	1.136	1.111	1.078
58	1.23	1.221	1.211	1.199	1.186	1.171	1.154	1.134	1.11	1.078
59	1.228	1.219	1.209	1.197	1.184	1.17	1.153	1.133	1.109	1.077
60	1.226	1.217	1.207	1.195	1.183	1.168	1.151	1.132	1.108	1.076

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.05$

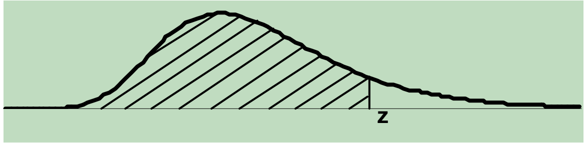
$$\Phi(z)=\int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
1	19.202	19.158	19.02	18.808	18.50	18.109	17.62	17.06	16.40	15.663
2	6.388	6.377	6.342	6.283	6.202	6.097	5.968	5.816	5.64	5.441
3	4.284	4.277	4.257	4.224	4.177	4.117	4.043	3.956	3.855	3.74
4	3.438	3.433	3.419	3.396	3.362	3.32	3.267	3.205	3.133	3.051
5	2.978	2.975	2.963	2.945	2.919	2.885	2.844	2.795	2.739	2.674
6	2.687	2.684	2.674	2.659	2.637	2.609	2.575	2.535	2.487	2.434
7	2.484	2.481	2.473	2.46	2.441	2.417	2.388	2.353	2.312	2.265
8	2.333	2.331	2.324	2.312	2.296	2.275	2.249	2.218	2.182	2.141
9	2.217	2.215	2.209	2.198	2.184	2.164	2.141	2.113	2.081	2.044
10	2.124	2.122	2.117	2.107	2.093	2.076	2.055	2.029	2	1.966
11	2.048	2.046	2.041	2.032	2.019	2.003	1.984	1.96	1.933	1.902
12	1.984	1.982	1.977	1.969	1.957	1.943	1.924	1.902	1.877	1.848
13	1.929	1.928	1.923	1.915	1.905	1.891	1.874	1.853	1.829	1.802
14	1.882	1.881	1.876	1.869	1.859	1.846	1.83	1.81	1.788	1.762
15	1.841	1.84	1.835	1.829	1.819	1.807	1.791	1.773	1.752	1.727
16	1.804	1.803	1.799	1.793	1.784	1.772	1.757	1.74	1.72	1.697
17	1.772	1.771	1.767	1.761	1.752	1.741	1.727	1.711	1.691	1.669
18	1.743	1.742	1.738	1.732	1.724	1.713	1.7	1.684	1.666	1.644
19	1.717	1.716	1.712	1.706	1.698	1.688	1.675	1.66	1.643	1.622
20	1.693	1.692	1.688	1.683	1.675	1.665	1.653	1.638	1.621	1.602
21	1.671	1.67	1.667	1.661	1.654	1.644	1.633	1.619	1.602	1.583
22	1.651	1.65	1.647	1.642	1.635	1.625	1.614	1.6	1.584	1.566
23	1.632	1.631	1.629	1.624	1.617	1.608	1.597	1.584	1.568	1.55
24	1.615	1.614	1.612	1.607	1.6	1.591	1.581	1.568	1.553	1.536
25	1.599	1.599	1.596	1.591	1.585	1.576	1.566	1.553	1.539	1.522
26	1.585	1.584	1.581	1.577	1.57	1.562	1.552	1.54	1.526	1.51
27	1.571	1.57	1.567	1.563	1.557	1.549	1.539	1.527	1.514	1.498
28	1.558	1.557	1.555	1.55	1.544	1.536	1.527	1.515	1.502	1.487
29	1.546	1.545	1.542	1.538	1.532	1.525	1.516	1.504	1.491	1.476
30	1.534	1.534	1.531	1.527	1.521	1.514	1.505	1.494	1.481	1.466

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.05$

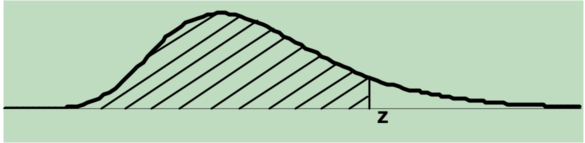
$$\Phi(z)=\int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)}\, x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
31	1.524	1.523	1.52	1.516	1.511	1.504	1.495	1.484	1.472	1.457
32	1.513	1.513	1.51	1.506	1.501	1.494	1.485	1.475	1.462	1.448
33	1.504	1.503	1.501	1.497	1.491	1.485	1.476	1.466	1.454	1.44
34	1.494	1.494	1.491	1.488	1.482	1.476	1.467	1.457	1.446	1.432
35	1.486	1.485	1.483	1.479	1.474	1.467	1.459	1.449	1.438	1.425
36	1.477	1.477	1.475	1.471	1.466	1.459	1.451	1.442	1.431	1.418
37	1.469	1.469	1.467	1.463	1.458	1.452	1.444	1.434	1.423	1.411
38	1.462	1.461	1.459	1.456	1.451	1.445	1.437	1.428	1.417	1.404
39	1.455	1.454	1.452	1.449	1.444	1.438	1.43	1.421	1.41	1.398
40	1.448	1.447	1.445	1.442	1.437	1.431	1.424	1.415	1.404	1.392
41	1.441	1.44	1.438	1.435	1.431	1.425	1.417	1.408	1.398	1.386
42	1.435	1.434	1.432	1.429	1.424	1.419	1.411	1.403	1.393	1.381
43	1.429	1.428	1.426	1.423	1.418	1.413	1.406	1.397	1.387	1.376
44	1.423	1.422	1.42	1.417	1.413	1.407	1.4	1.392	1.382	1.371
45	1.417	1.416	1.415	1.412	1.407	1.402	1.395	1.386	1.377	1.366
46	1.412	1.411	1.409	1.406	1.402	1.396	1.39	1.381	1.372	1.361
47	1.406	1.406	1.404	1.401	1.397	1.391	1.385	1.377	1.367	1.356
48	1.401	1.401	1.399	1.396	1.392	1.387	1.38	1.372	1.363	1.352
49	1.396	1.396	1.394	1.391	1.387	1.382	1.375	1.367	1.358	1.348
50	1.392	1.391	1.389	1.387	1.383	1.377	1.371	1.363	1.354	1.344
51	1.387	1.387	1.385	1.382	1.378	1.373	1.367	1.359	1.35	1.34
52	1.383	1.382	1.381	1.378	1.374	1.369	1.362	1.355	1.346	1.336
53	1.378	1.378	1.376	1.373	1.37	1.365	1.358	1.351	1.342	1.332
54	1.374	1.374	1.372	1.369	1.366	1.361	1.354	1.347	1.339	1.329
55	1.37	1.37	1.368	1.365	1.362	1.357	1.351	1.343	1.335	1.325
56	1.366	1.366	1.364	1.362	1.358	1.353	1.347	1.34	1.331	1.322
57	1.363	1.362	1.361	1.358	1.354	1.349	1.343	1.336	1.328	1.319
58	1.359	1.358	1.357	1.354	1.351	1.346	1.34	1.333	1.325	1.315
59	1.355	1.355	1.353	1.351	1.347	1.342	1.337	1.33	1.322	1.312
60	1.352	1.351	1.35	1.347	1.344	1.339	1.333	1.327	1.319	1.309

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.05$

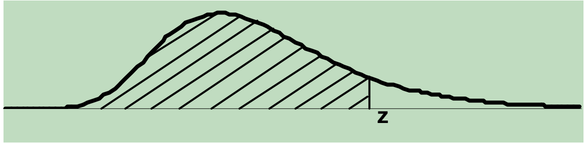
$$\Phi(z)=\int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)}\, x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
1	14.835	13.92	12.91	11.83	10.65	9.392	8.041	6.596	5.049	3.368
2	5.217	4.969	4.696	4.397	4.072	3.719	3.336	2.919	2.456	1.923
3	3.611	3.467	3.309	3.135	2.944	2.736	2.507	2.255	1.971	1.633
4	2.959	2.856	2.742	2.616	2.478	2.327	2.159	1.973	1.76	1.503
5	2.601	2.52	2.429	2.33	2.22	2.098	1.964	1.813	1.64	1.428
6	2.373	2.305	2.229	2.145	2.053	1.951	1.837	1.709	1.56	1.377
7	2.213	2.154	2.088	2.016	1.935	1.846	1.747	1.634	1.503	1.34
8	2.094	2.042	1.984	1.919	1.847	1.768	1.679	1.578	1.46	1.312
9	2.002	1.954	1.902	1.843	1.779	1.706	1.625	1.533	1.425	1.29
10	1.927	1.884	1.836	1.783	1.723	1.657	1.582	1.497	1.397	1.272
11	1.866	1.826	1.782	1.732	1.677	1.616	1.546	1.467	1.374	1.256
12	1.815	1.778	1.736	1.69	1.638	1.581	1.516	1.442	1.354	1.243
13	1.771	1.736	1.697	1.654	1.605	1.551	1.49	1.42	1.337	1.232
14	1.733	1.7	1.663	1.622	1.577	1.525	1.467	1.401	1.322	1.222
15	1.7	1.669	1.634	1.595	1.551	1.502	1.447	1.384	1.309	1.213
16	1.67	1.641	1.607	1.57	1.529	1.482	1.43	1.369	1.297	1.205
17	1.644	1.616	1.584	1.548	1.509	1.464	1.414	1.356	1.287	1.198
18	1.62	1.593	1.563	1.529	1.491	1.448	1.399	1.344	1.277	1.192
19	1.599	1.573	1.544	1.511	1.474	1.433	1.386	1.333	1.269	1.186
20	1.58	1.554	1.526	1.495	1.459	1.42	1.375	1.323	1.261	1.181
21	1.562	1.538	1.51	1.48	1.446	1.407	1.364	1.313	1.253	1.176
22	1.545	1.522	1.496	1.466	1.433	1.396	1.354	1.305	1.247	1.171
23	1.53	1.508	1.482	1.454	1.421	1.385	1.344	1.297	1.24	1.167
24	1.516	1.494	1.47	1.442	1.411	1.376	1.336	1.29	1.234	1.163
25	1.503	1.482	1.458	1.431	1.401	1.367	1.328	1.283	1.229	1.159
26	1.491	1.47	1.447	1.421	1.391	1.358	1.32	1.276	1.224	1.156
27	1.48	1.46	1.437	1.411	1.382	1.35	1.313	1.27	1.219	1.153
28	1.469	1.449	1.427	1.402	1.374	1.343	1.307	1.265	1.215	1.15
29	1.459	1.44	1.418	1.394	1.366	1.336	1.3	1.26	1.211	1.147
30	1.45	1.431	1.41	1.386	1.359	1.329	1.294	1.255	1.207	1.144

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.05$

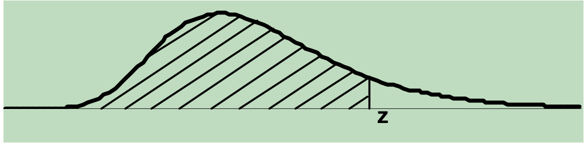
$$\Phi(z)=\int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
31	1.441	1.422	1.402	1.378	1.352	1.323	1.289	1.25	1.203	1.141
32	1.432	1.414	1.394	1.371	1.346	1.317	1.284	1.245	1.199	1.139
33	1.425	1.407	1.387	1.365	1.339	1.311	1.279	1.241	1.196	1.137
34	1.417	1.4	1.38	1.358	1.334	1.306	1.274	1.237	1.193	1.135
35	1.41	1.393	1.374	1.352	1.328	1.301	1.269	1.233	1.189	1.132
36	1.403	1.386	1.367	1.346	1.323	1.296	1.265	1.229	1.186	1.13
37	1.396	1.38	1.362	1.341	1.317	1.291	1.261	1.226	1.184	1.128
38	1.39	1.374	1.356	1.335	1.313	1.287	1.257	1.223	1.181	1.127
39	1.384	1.368	1.35	1.33	1.308	1.282	1.253	1.219	1.178	1.125
40	1.378	1.363	1.345	1.326	1.303	1.278	1.25	1.216	1.176	1.123
41	1.373	1.358	1.34	1.321	1.299	1.274	1.246	1.213	1.174	1.122
42	1.368	1.353	1.336	1.316	1.295	1.271	1.243	1.21	1.171	1.12
43	1.363	1.348	1.331	1.312	1.291	1.267	1.24	1.208	1.169	1.118
44	1.358	1.343	1.327	1.308	1.287	1.264	1.236	1.205	1.167	1.117
45	1.353	1.339	1.322	1.304	1.283	1.26	1.233	1.202	1.165	1.116
46	1.348	1.334	1.318	1.3	1.28	1.257	1.231	1.2	1.163	1.114
47	1.344	1.33	1.314	1.297	1.276	1.254	1.228	1.198	1.161	1.113
48	1.34	1.326	1.31	1.293	1.273	1.251	1.225	1.195	1.159	1.112
49	1.336	1.322	1.307	1.29	1.27	1.248	1.223	1.193	1.157	1.11
50	1.332	1.318	1.303	1.286	1.267	1.245	1.22	1.191	1.156	1.109
51	1.328	1.315	1.3	1.283	1.264	1.242	1.218	1.189	1.154	1.108
52	1.324	1.311	1.296	1.28	1.261	1.24	1.215	1.187	1.152	1.107
53	1.321	1.308	1.293	1.277	1.258	1.237	1.213	1.185	1.151	1.106
54	1.317	1.305	1.29	1.274	1.255	1.235	1.211	1.183	1.149	1.105
55	1.314	1.301	1.287	1.271	1.253	1.232	1.209	1.181	1.148	1.104
56	1.311	1.298	1.284	1.268	1.25	1.23	1.207	1.179	1.146	1.103
57	1.308	1.295	1.281	1.266	1.248	1.228	1.205	1.178	1.145	1.102
58	1.305	1.292	1.279	1.263	1.245	1.225	1.203	1.176	1.144	1.101
59	1.302	1.289	1.276	1.26	1.243	1.223	1.201	1.174	1.142	1.1
60	1.299	1.287	1.273	1.258	1.241	1.221	1.199	1.173	1.141	1.099

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.01$

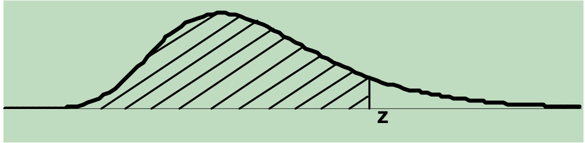
$$\Phi(z) = \int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
1	100	100	100	100	99.95	97.546	94.60	91.141	87.15	82.665
2	15.977	15.942	15.83	15.66	15.41	15.096	14.70	14.25	13.72	13.121
3	8.466	8.449	8.399	8.316	8.199	8.048	7.865	7.647	7.396	7.11
4	6.029	6.018	5.986	5.932	5.856	5.759	5.64	5.499	5.336	5.151
5	4.849	4.841	4.817	4.777	4.721	4.649	4.561	4.457	4.336	4.198
6	4.155	4.149	4.13	4.098	4.053	3.996	3.926	3.842	3.746	3.636
7	3.698	3.692	3.676	3.65	3.612	3.564	3.506	3.436	3.355	3.263
8	3.372	3.367	3.354	3.331	3.299	3.257	3.207	3.146	3.077	2.997
9	3.128	3.124	3.112	3.092	3.063	3.027	2.982	2.929	2.867	2.797
10	2.938	2.934	2.923	2.905	2.88	2.847	2.807	2.759	2.704	2.641
11	2.785	2.782	2.772	2.755	2.732	2.702	2.666	2.622	2.572	2.515
12	2.659	2.656	2.647	2.632	2.611	2.583	2.55	2.51	2.464	2.411
13	2.554	2.551	2.542	2.528	2.509	2.483	2.452	2.415	2.372	2.323
14	2.464	2.461	2.453	2.44	2.422	2.398	2.369	2.335	2.295	2.249
15	2.386	2.384	2.376	2.364	2.347	2.325	2.297	2.265	2.227	2.184
16	2.318	2.316	2.309	2.297	2.281	2.26	2.235	2.204	2.168	2.128
17	2.258	2.256	2.25	2.239	2.223	2.203	2.179	2.15	2.116	2.078
18	2.205	2.203	2.197	2.186	2.172	2.153	2.13	2.102	2.07	2.033
19	2.157	2.155	2.149	2.139	2.126	2.108	2.085	2.059	2.029	1.993
20	2.114	2.112	2.107	2.097	2.084	2.067	2.045	2.02	1.991	1.957
21	2.075	2.073	2.068	2.059	2.046	2.029	2.009	1.985	1.957	1.925
22	2.04	2.038	2.033	2.024	2.011	1.996	1.976	1.953	1.926	1.895
23	2.007	2.005	2	1.992	1.98	1.965	1.946	1.923	1.897	1.867
24	1.977	1.975	1.97	1.962	1.951	1.936	1.918	1.896	1.871	1.842
25	1.949	1.947	1.943	1.935	1.924	1.909	1.892	1.871	1.847	1.819
26	1.923	1.922	1.917	1.909	1.899	1.885	1.868	1.848	1.824	1.797
27	1.899	1.898	1.893	1.886	1.876	1.862	1.846	1.826	1.803	1.777
28	1.877	1.875	1.871	1.864	1.854	1.841	1.825	1.806	1.783	1.758
29	1.856	1.855	1.85	1.843	1.834	1.821	1.805	1.787	1.765	1.74
30	1.836	1.835	1.831	1.824	1.814	1.802	1.787	1.769	1.748	1.724

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.01$

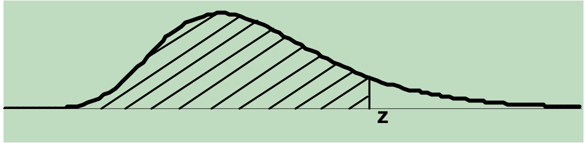
$$\Phi(z)=\int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
31	1.818	1.816	1.813	1.806	1.797	1.785	1.77	1.752	1.732	1.708
32	1.8	1.799	1.795	1.789	1.78	1.768	1.754	1.736	1.716	1.693
33	1.784	1.783	1.779	1.773	1.764	1.752	1.738	1.721	1.702	1.679
34	1.768	1.767	1.763	1.757	1.749	1.737	1.724	1.707	1.688	1.666
35	1.754	1.752	1.749	1.743	1.734	1.723	1.71	1.694	1.675	1.654
36	1.74	1.738	1.735	1.729	1.721	1.71	1.697	1.681	1.663	1.642
37	1.726	1.725	1.722	1.716	1.708	1.697	1.684	1.669	1.651	1.63
38	1.714	1.713	1.709	1.704	1.696	1.685	1.673	1.658	1.64	1.62
39	1.702	1.701	1.697	1.692	1.684	1.674	1.661	1.647	1.629	1.609
40	1.69	1.689	1.686	1.68	1.673	1.663	1.651	1.636	1.619	1.6
41	1.679	1.678	1.675	1.669	1.662	1.652	1.64	1.626	1.609	1.59
42	1.668	1.667	1.664	1.659	1.652	1.642	1.63	1.616	1.6	1.581
43	1.658	1.657	1.654	1.649	1.642	1.632	1.621	1.607	1.591	1.573
44	1.649	1.648	1.645	1.64	1.632	1.623	1.612	1.598	1.582	1.564
45	1.639	1.638	1.635	1.63	1.623	1.614	1.603	1.59	1.574	1.556
46	1.63	1.629	1.626	1.622	1.615	1.606	1.595	1.582	1.566	1.549
47	1.622	1.621	1.618	1.613	1.606	1.597	1.587	1.574	1.559	1.541
48	1.613	1.612	1.61	1.605	1.598	1.59	1.579	1.566	1.551	1.534
49	1.605	1.604	1.602	1.597	1.59	1.582	1.571	1.559	1.544	1.527
50	1.598	1.597	1.594	1.589	1.583	1.574	1.564	1.552	1.537	1.521
51	1.59	1.589	1.587	1.582	1.576	1.567	1.557	1.545	1.531	1.514
52	1.583	1.582	1.579	1.575	1.569	1.56	1.55	1.538	1.524	1.508
53	1.576	1.575	1.573	1.568	1.562	1.554	1.544	1.532	1.518	1.502
54	1.569	1.568	1.566	1.561	1.555	1.547	1.538	1.526	1.512	1.496
55	1.563	1.562	1.559	1.555	1.549	1.541	1.531	1.52	1.506	1.491
56	1.556	1.556	1.553	1.549	1.543	1.535	1.526	1.514	1.501	1.485
57	1.55	1.549	1.547	1.543	1.537	1.529	1.52	1.509	1.495	1.48
58	1.544	1.544	1.541	1.537	1.531	1.524	1.514	1.503	1.49	1.475
59	1.539	1.538	1.535	1.531	1.525	1.518	1.509	1.498	1.485	1.47
60	1.533	1.532	1.53	1.526	1.52	1.513	1.504	1.493	1.48	1.465

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.01$

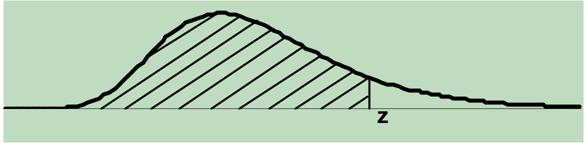
$$\Phi(z)=\int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)}\, x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
1	77.666	72.171	66.18	59.724	52.79	45.398	37.55	29.277	20.566	11.419
2	12.45	11.707	10.894	10.008	9.05	8.018	6.91	5.721	4.442	3.04
3	6.791	6.437	6.049	5.625	5.164	4.666	4.128	3.545	2.907	2.184
4	4.944	4.714	4.46	4.183	3.882	3.554	3.197	2.808	2.376	1.876
5	4.044	3.873	3.684	3.477	3.251	3.004	2.734	2.438	2.106	1.715
6	3.512	3.375	3.223	3.056	2.874	2.674	2.455	2.213	1.94	1.614
7	3.159	3.044	2.917	2.776	2.622	2.453	2.267	2.061	1.826	1.544
8	2.908	2.808	2.697	2.575	2.441	2.294	2.132	1.95	1.743	1.493
9	2.718	2.629	2.532	2.423	2.304	2.173	2.028	1.866	1.68	1.453
10	2.569	2.49	2.402	2.304	2.197	2.078	1.946	1.799	1.629	1.421
11	2.45	2.377	2.297	2.208	2.109	2.001	1.88	1.744	1.587	1.394
12	2.351	2.284	2.21	2.128	2.037	1.936	1.825	1.698	1.553	1.372
13	2.268	2.206	2.137	2.061	1.976	1.882	1.778	1.66	1.523	1.353
14	2.197	2.139	2.074	2.003	1.924	1.836	1.737	1.626	1.497	1.337
15	2.136	2.081	2.02	1.953	1.878	1.795	1.702	1.597	1.475	1.322
16	2.082	2.03	1.973	1.909	1.838	1.76	1.672	1.572	1.455	1.309
17	2.034	1.985	1.931	1.87	1.803	1.728	1.644	1.549	1.438	1.298
18	1.992	1.945	1.893	1.835	1.771	1.7	1.62	1.529	1.422	1.288
19	1.954	1.909	1.86	1.804	1.743	1.674	1.598	1.51	1.407	1.278
20	1.919	1.877	1.829	1.776	1.717	1.651	1.577	1.493	1.394	1.27
21	1.888	1.847	1.801	1.75	1.694	1.63	1.559	1.478	1.383	1.262
22	1.86	1.82	1.776	1.727	1.672	1.611	1.542	1.464	1.372	1.255
23	1.833	1.795	1.753	1.705	1.652	1.593	1.527	1.451	1.361	1.248
24	1.809	1.772	1.731	1.685	1.634	1.577	1.513	1.439	1.352	1.242
25	1.787	1.751	1.711	1.667	1.617	1.562	1.499	1.428	1.343	1.236
26	1.766	1.732	1.693	1.65	1.602	1.548	1.487	1.418	1.335	1.231
27	1.747	1.713	1.676	1.634	1.587	1.535	1.476	1.408	1.328	1.226
28	1.729	1.696	1.66	1.619	1.573	1.522	1.465	1.399	1.321	1.221
29	1.712	1.68	1.645	1.605	1.56	1.511	1.455	1.39	1.314	1.216
30	1.696	1.665	1.63	1.592	1.548	1.5	1.445	1.382	1.308	1.212

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.01$

$$\Phi(z)=\int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
31	1.681	1.651	1.617	1.579	1.537	1.49	1.436	1.375	1.302	1.208
32	1.667	1.637	1.604	1.568	1.526	1.48	1.428	1.367	1.296	1.204
33	1.654	1.625	1.593	1.556	1.516	1.471	1.42	1.361	1.291	1.201
34	1.641	1.613	1.581	1.546	1.506	1.462	1.412	1.354	1.286	1.197
35	1.629	1.602	1.571	1.536	1.497	1.454	1.405	1.348	1.281	1.194
36	1.618	1.591	1.56	1.527	1.489	1.446	1.398	1.342	1.276	1.191
37	1.607	1.58	1.551	1.518	1.48	1.439	1.391	1.337	1.272	1.188
38	1.597	1.571	1.542	1.509	1.472	1.431	1.385	1.331	1.268	1.185
39	1.587	1.561	1.533	1.501	1.465	1.425	1.379	1.326	1.264	1.183
40	1.577	1.552	1.524	1.493	1.458	1.418	1.373	1.321	1.26	1.18
41	1.568	1.544	1.516	1.485	1.451	1.412	1.368	1.317	1.256	1.178
42	1.56	1.536	1.508	1.478	1.444	1.406	1.362	1.312	1.252	1.175
43	1.551	1.528	1.501	1.471	1.438	1.4	1.357	1.308	1.249	1.173
44	1.544	1.52	1.494	1.465	1.432	1.395	1.353	1.304	1.246	1.171
45	1.536	1.513	1.487	1.458	1.426	1.389	1.348	1.3	1.243	1.169
46	1.529	1.506	1.481	1.452	1.42	1.384	1.343	1.296	1.24	1.166
47	1.522	1.499	1.474	1.446	1.415	1.379	1.339	1.292	1.237	1.165
48	1.515	1.493	1.468	1.44	1.409	1.374	1.335	1.289	1.234	1.163
49	1.508	1.487	1.462	1.435	1.404	1.37	1.331	1.285	1.231	1.161
50	1.502	1.481	1.457	1.43	1.399	1.365	1.327	1.282	1.228	1.159
51	1.496	1.475	1.451	1.425	1.395	1.361	1.323	1.279	1.226	1.157
52	1.49	1.469	1.446	1.42	1.39	1.357	1.319	1.276	1.223	1.156
53	1.484	1.464	1.441	1.415	1.386	1.353	1.316	1.273	1.221	1.154
54	1.479	1.458	1.436	1.41	1.381	1.349	1.312	1.27	1.219	1.152
55	1.473	1.453	1.431	1.406	1.377	1.345	1.309	1.267	1.216	1.151
56	1.468	1.448	1.426	1.401	1.373	1.342	1.306	1.264	1.214	1.149
57	1.463	1.443	1.422	1.397	1.369	1.338	1.303	1.262	1.212	1.148
58	1.458	1.439	1.417	1.393	1.366	1.335	1.3	1.259	1.21	1.146
59	1.453	1.434	1.413	1.389	1.362	1.331	1.297	1.256	1.208	1.145
60	1.449	1.43	1.409	1.385	1.358	1.328	1.294	1.254	1.206	1.144