DEMAND FORECAST AND INVENTORY PLANNING

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ABSTRACT

The paper proposes an approach to the two period inventory problem for items that have heterogeneous Poisson demands. A model is constructed whose appealing features reveal aspects of the nature of the optimal stocking problem that enable the manager to assess the degree to which demand is affected by factors such as the adopted promotional policy or the utility and inherent appeal of the item. The forecasts obtained by the model are utilized in the derivation of the optimal inventory stocking policy from a profit maximization standpoint.

1. INTRODUCTION

In the context of inventory decision models, the problem of deriving the optimal inventory stocking policy that will maximize the expected profit during a period has received much attention. The questions of when to place an order for additional material and that of how much to order being of equal importance have induced a considerable amount of research as reflected by a vast number of articles in the management science, operational research and statistical literature (see, for example, Xekalaki (1983, 1986, 1990), Bradford and Sugrue (1990) and the references therein).

As naturally expected, both aspects of the inventory problem have been looked upon in terms of the distribution of demand whose fluctuations affect at each point in time the amount of the on hand stock and hence the determination of the time-to-order and the amount-to-order.

This paper focuses attention on the latter aspect. In particular, it presents a model for the inventory planning and control of a warehouse or a firm based on demand forecast. The model uses a Bayesian procedure for forecasting the demand and updating model parameters which further allows us to get a deeper insight into the mechanism that generates demand. More specifically, the paper introduces a model for the so called two-period style-goods inventory problem that assumes heterogeneous Poisson distributed demand in a rather more general sense than that of Bradford and Sugrue (1990) in that it assumes that heterogeneity in the aggregate demand is the result of differences in the individual item marketing as well as differences in the individual item's inherent appeal to the consumer. So, apart from enabling us to forecast the demand it allows us to assess the degree to which the above types of differences across items affect a given situation. The model is introduced and studied in sections 2 and 3. Section 4 goes on to examine this model in relation to inventory planning by focusing on the determination of the optimal policy for stocking which

maximizes the expected profit over the period of observation. It turns out that in certain cases the algorithm involved can be considerably simplified.

2. PREDICTING DEMAND DURING A PERIOD ON THE BASIS OF DEMAND DURING THE IMMEDIATELY PRECEDING PERIOD

As mentioned above, Bradford and Sugrue (1990) considered a two-period style-goods inventory problem on the hypothesis of heterogeneous Poisson demand. They assumed in other words that items have constant but unequal probabilities to be ordered. Averaging out this heterogeneity in terms of a gamma distribution they obtained a negative binomial distribution for the aggregate demand.

This heterogeneity hypothesis has been known in the statistical literature as the "apparent contagion hypothesis" since 1919 when it was introduced by Greenwood and Woods (1919), though not in the context of inventory decision models. Making such a hypothesis reflects the need to recognize the existence of factors other than pure chance affecting the placement of an order for the item and a gamma distribution for an appropriately chosen parameter is usually assumed in an attempt to describe the way is in which heterogeneity manifests itself.

It is obvious that much could be gained if one were to have some insight into the extent in which various types of non-random factors contribute to the observed demand. However, the effects of the various types of these non-random factors are confounded and unless extra information is available the mathematics alone cannot lead us to a sound conclusion.

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In the sequel, all non-random factors are assumed to be further split into internal and external factors. The first type encompasses factors that have to do with the item's specific features and qualities that appeal to the buyer and predispose the consumer to buy it. (All these item characteristics that predispose it to be a good or a poor seller). The second type on the other hand refers to the item's exposure in the market (e.g. to factors relating to advertising, marketing, etc.).

In what follows we use the term "proneness" to refer to the set of all internal factors that predispose the item to success and "liability" to refer to the set of all external factors that affect the demand of the particular item and pertain to promotional activities, space allocation in the store etc.

Consider items of proneness v and liability $\lambda_1 | v$ over the first half of a selling period $[0,2t]$ and assume that the demand X for these items (in number of units ordered) follows a Poisson distribution with mean λt and probability generating function

$$G_{X | (\lambda_1, v)}(s) = e^{(\lambda t | v)(s-1)} , \quad \lambda > 0 . \tag{2.1}$$

Let the liability parameter $\lambda_1 | v$ for these items be distributed as gamma $(k;v)$ i.e. let the distribution of λ_1 for given v be defined by the density function

$$f_{\lambda_1 | v}(\lambda) = \frac{v^{-k}}{\Gamma(k)} e^{-\lambda/v} \lambda^{k-1} , \quad k, v > 0 . \tag{2.2}$$

Then, for items with the same proneness v but varying liabilities the distribution of demand over period $[0,t]$ will have probability generating function given by

$$G_{X | v}(s) = \int_0^\infty \frac{v^{-k}}{\Gamma(k)} e^{-\lambda(\frac{1}{v} + t(1-s))} \lambda^{k-1} d\lambda = [1 + vt(1-s)]^{-k} . \tag{2.3}$$

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So assuming that period 1 is of arbitrary unit length, the probability function of the demand for items of the same proneness v is

$$P(X=x \mid v) = \binom{k+x-1}{x} \left(\frac{1}{1+v}\right)^k \left(\frac{v}{1+v}\right)^x , \quad x = 0,1,\dots \qquad (2.4)$$

Suppose further that the proneness parameter v has a beta distribution of the second kind (Pearson type VI) with parameters α and ϱ defined by the density function

$$f_v(v) = \frac{\Gamma(\alpha+\varrho)}{\Gamma(\alpha)\Gamma(\varrho)} v^{\alpha-1}(1+v)^{-(\alpha+\varrho)} , \quad \alpha,\varrho > 0, \quad v > 0 . \qquad (2.5)$$

Then the unconditional distribution of demand for period 1 has probability generating function given by

$$G_X(s) = \frac{\Gamma(\alpha+\varrho)}{\Gamma(\alpha)\Gamma(\varrho)} \int_0^\infty v^{\alpha-1}(1+v)^{-(\alpha+\varrho)}(1+v(1-s))^{-k} \, dv$$

$$= \frac{\varrho_{(k)}}{(\alpha+\varrho)_{(k)}} {}_2F_1(\alpha,k; \alpha+k+\varrho; s) . \qquad (2.6)$$

Here

$$_2F_1(a,b; c; z) = \sum_{r=0}^{\infty} \frac{a_{(r)} b_{(r)}}{c_{(r)}} \frac{z^r}{r!}$$

is the Gauss hypergeometric function where $\alpha_{(r)} = \Gamma(\alpha+r)/\Gamma(\alpha)$, $r = 0,1,2,\dots$.

Therefore, the probability function of the aggregate demand X is given by

$$P(X=x) = \frac{\varrho_{(k)}}{(\alpha+\varrho)_{(k)}} \frac{\alpha_{(x)} k_{(x)}}{(\alpha+k+\varrho)_{(x)}} \frac{1}{x!} , \quad x = 0,1,2,\dots \qquad (2.7)$$

The distribution of X is in other words the generalized, Waring distribution with parameters α,k and ϱ (UGWD(α,k; ϱ)). (For more

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information about this distribution the reader is referred to Xekalaki (1981, 1983a,b,c, 1984a,b) and the references therein). Then, utilizing this demand distribution through a Bayesian type of argument one can predict future demand conditional on the realized demand of a previous period.

Bradford and Sugrue (1990) working along these lines considered a Bayesian model which predicted the demand for an item with respect to a time period based on the distribution of the heterogeneity parameter λ as this evolved through information acquired on it in the form of the realized demand $X=x$ during an immediately preceding period. Starting thus, with a Poisson distributed demand, whose parameter λ fluctuated as a result of heterogeneity according to a gamma distribution with parameters α and ϱ, they ended up with a negative binomial distribution for the aggregate demand in the first period of observation. They then derived the conditional period 2 demand distribution given the demand X of period 1 as a mixture of a Poisson distribution whose mean λ conditional on the period 1 realized demand had a gamma distribution with updated parameters $\alpha+1$ and $r+x$. The result was again a negative binomial distribution leading to a linear regression of the period 2 demand on the period 1 demand.

Both the unconditional (period 1) and conditional (period 2) negative binomial distributions appear to adequately describe aggregate demand and the authors incorporate this result into an optimal stocking policy model that they suggest. It seems however, that the demand model suggested by the above authors is somewhat restrictive as it implies identical unconditional period 1 and period 2 distributions. Moreover, it implicitly assumes that heterogeneity in the demand is manifested through the fluctuations of λ in both periods. Analogous are the implications of the model considered in the present paper if one decides to adopt a similar approach in predicting conditional future demand.

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Indeed, as suggested by model (2.6), at the beginning of period 1 when no previous knowledge on the demand $X|(\lambda_1|v)$ of an item exists, the best estimate of its prior mean would be the mean of λ_1. This is given by $E(E(\lambda_1|v))=kE(v)=\alpha k/(\varrho-1)$ as the distribution of λ_1 is assumed to be a scale mixture of the gamma distribution where the scale parameter v has a Pearson type VI distribution (beta of the second type) as defined by (2.5). (The choice of the latter form for the distribution of v was based on the fact that it implies a beta distribution of the first kind (Pearson type I) on $(0,1)$ for the parameter $q = v/(1+v)$ of the negative binomial distribution of $X|v$).

At the end of period 1 the realized aggregate demand $X=x$ provides further information on the distribution of v and hence of λ as one can study their distribution among items whose demand was x in period 1.

So, the predicted distribution of v for period 2 has probability density function

$$f_{v|(X=x)}(v) = \frac{P(X=x|v) \cdot f_v(v)}{P(X=x)}$$

$$= \frac{\dfrac{k(x)}{x!}\left(\dfrac{v}{1+v}\right)^x \left(\dfrac{1}{1+v}\right)^k \cdot \dfrac{\Gamma(\alpha+\varrho)}{\Gamma(\alpha)\Gamma(\varrho)}v^{\alpha-1}(1+v)^{-(\alpha+\varrho)}}{\dfrac{\varrho_{(k)}}{(\alpha+\varrho)_{(k)}}\dfrac{\alpha_{(x)}k_{(x)}}{(\alpha+k+\varrho)_{(k)}}\dfrac{1}{x!}}$$

$$= \frac{\Gamma(\alpha+k+\varrho+x)}{\Gamma(\varrho+k)\Gamma(\alpha+x)}v^{\alpha+x-1}(1+v)^{-(\alpha+k+\varrho+x)} .$$

This implies that the posterior distribution of $v|(X=x)$ is a Pearson type VI (Beta of the second type) distribution with parameters

$\alpha+x$ and $\rho+k$. This, in turn, implies that $\lambda\,|\,(v,X=x)$ has a posterior distribution defined by the probability density function

$$f_{\lambda\,|\,(v,X=x)} = \frac{P(X=x\,|\,v,\lambda)\;\cdot\;f_{\lambda\,|\,v}(\lambda)\;\cdot\;f_v(v)}{P(X=x\,|\,v)\;\cdot\;f_v(v)}$$

$$= \frac{e^{-\lambda}\dfrac{\lambda^x}{x!}\;v^{-k}\;e^{-\lambda/v}\;\lambda^{k-1}}{\dfrac{k_{(x)}}{x!}\left(\dfrac{v}{1+v}\right)^x\left(\dfrac{1}{1+v}\right)^k}$$

$$= \frac{\left(\dfrac{v}{1+v}\right)^{-(k+x)}}{\Gamma(k+x)}\;e^{-\lambda\left(\frac{v}{1+v}\right)}\;\lambda^{k+x-1}$$

i.e. $\lambda\,|\,(v,X=x)$ has a gamma distribution with parameters $k+x$ and $1+\dfrac{1}{v}$.

So, the posterior distribution of $\lambda\,|\,(X=x)$ is again a scale mixture of the gamma distribution whose scale parameter has a Pearson type VI distribution implying that the conditional expected demand for period 2 given the realized demand for period 1 is $(k+x)(\alpha+x)/(k+\rho-1)$.

However, although a distinction is considered between endogenous and exogenous factors influencing demand, no such distinction is implied between period 1 and period 2 exogenous or endogenous factors. So, again the model implicitly assumes no differences in "proneness" or "liability" between the two periods. Taking into account the fact that the term proneness reflects the item's inherent appeal to the buyer, such an implication may not be unreasonable at least for a limited period of time. A constancy implication on the exogenous factors can, however, be regarded as restrictive. Thus, a model that can be more flexible by allowing for

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differences in the external factors from period to period might be preferable. Next section deals with this problem by developing a model that looks into the manner in which period 1 and period 2 demands fluctuate jointly.

3. PREDICTING DEMAND ON THE BASIS OF A TWO PERIOD MODEL

Consider items of "proneness" v and "liability" $\lambda_i \mid v$ for a period i of observation. Assume that over two non-overlapping time periods the numbers X, Y of units demanded follow a double Poisson distribution with probability generating function

$$G_{(X,Y)\mid \lambda_1,\lambda_2,v}(s,t) = exp\left\{(\lambda_1 \mid v)(s\text{-}1)+(\lambda_2 \mid v)(t\text{-}1)\right\}$$ (3.1)

$$\lambda_1,\lambda_2 > 0 .$$

Let the liability parameters $\lambda_1 \mid v$, $\lambda_2 \mid v$ be independently distributed as gamma $(k;v)$ and gamma $(m;v)$ respectively, i.e.,

$$f_{\lambda_1 \mid v}(\lambda_1) = \frac{v^{-k}}{\Gamma(k)} e^{-\lambda_1/v} \lambda_1^{k-1} , \quad k,v > 0$$ (3.2)

$$f_{\lambda_2 \mid v}(\lambda_2) = \frac{v^{-m}}{\Gamma(m)} e^{-\lambda_2/v} \lambda_2^{m-1} , \quad m,v > 0 .$$ (3.3)

Then for items with the same "proneness" but varying "liabilities" the joint distribution of demand over the two periods considered has probability generating function given by

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$$G_{(X,Y)|\nu}(s,t) =$$

$$\frac{\nu^{-k-m}}{\Gamma(k)\Gamma(m)} \int_0^\infty \int_0^\infty e^{\lambda_1(s-1)+\lambda_2(t-1)-\lambda_1/\nu-\lambda_2/\nu} \lambda_1^{k-1} \lambda_2^{m-1} d\lambda_1 \, d\lambda_2$$

$$= \frac{\nu^{-k}}{\Gamma(k)} \int_0^\infty e^{-(\lambda_1/\nu)(1+\nu(1-s))} \lambda_1^{k-1} d\lambda_1 \times$$

$$\frac{\nu^{-m}}{\Gamma(m)} \int_0^\infty e^{-(\lambda_2/\nu)(1+\nu(1-t))} \lambda_2^{m-1} d\lambda_1$$

i.e.

$$G_{(X,Y)|\nu}(s,t) = \left[1+\nu(1-s)\right]^{-k} \left[1+\nu(1-t)\right]^{-m}. \qquad (3.4)$$

Therefore, items with the same proneness are jointly demanded according to a double negative binomial distribution. If we now let the "proneness" parameter ν vary from item to item according to a beta distribution of the second type as defined by (2.5), the probability generating function of the joint distribution of demand over the two periods is

$$G_{(X,Y)}(s,t) = \frac{\Gamma(\alpha+\varrho)}{\Gamma(\alpha)\Gamma(\varrho)} \int_0^\infty \nu^{\alpha-1}(1+\nu)^{-(\alpha+\varrho)} \left[1+\nu(1-s)\right]^{-k} \left[1+\nu(1-t)\right]^{-m} d\nu$$

$$= \frac{\Gamma(\alpha+\varrho)}{\Gamma(\alpha)\Gamma(\varrho)} \sum_{r=0}^\infty \sum_{l=0}^\infty \frac{k_{(r)}}{r!} \frac{m_{(l)}}{l!} \int_0^\infty \nu^{\alpha+r+l-1} (1+\nu)^{-(\alpha+\varrho+k+m+r+l)} d\nu$$

i.e.

$$G_{(X,Y)}(s,t) = \frac{\varrho_{(k+m)}}{(\alpha+\varrho)_{(k+m)}} F_1(\alpha; k,m; \alpha+k+\mu+\varrho; s,t) \qquad (3.5)$$

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where

$$F_1(\alpha;\beta,\beta';\gamma;u,v) = \sum_{r=0}^{\infty} \sum_{l=0}^{\infty} \frac{\alpha_{(r+l)}\beta_{(r)}\beta'_{(l)}}{\gamma_{(r+l)}} \frac{u^r}{r!} \frac{v^l}{l!} \qquad (3.6)$$

is Appell's hypergeometric function of the first kind.

Then, the joint probability function of demand over the two periods is given by

$$P(X=x,\ Y=y) = \frac{\varrho_{(k+m)}}{(\alpha+\varrho)_{(k+m)}} \frac{\alpha_{(x+y)}\ k_{(x)}m_{(y)}}{(\alpha+k+m+\varrho)_{(x+y)}} \frac{1}{x!} \frac{1}{y!}$$

$$\alpha,k,m,\varrho > 0\ ;\quad x = 0,1,2,,...\ ;\quad y = 0,1,2,... \qquad (3.7)$$

This is the probability function of the bivariate generalized Waring distribution with parameters α,k,m and ϱ (BGWD($\alpha;k,m;\varrho$)) introduced by Xekalaki (1984a) in a different context.

By its definition, this model allows not only for differences in the exogenous (external) factors from item to item within each subperiod, but also for differences in the exogenous factors affecting the same item from period to period. The item's "proneness" is regarded as constant throughout out the entire period of observation. By no means of course does the model imply that the item's inherent appeal to the buyers will remain constant for ever as the buyers' preferences are expected to change as their perception of handiness and/or beauty changes within very long periods. Moreover, the choice of a gamma form for the distribution of each of the liability parameters and of a beta form for the distribution of the "proneness" parameter is analogous to that of the univariate case.

Note that the period 1 and period 2 demand distributions of X and Y respectively as well as the distribution of total demand X+Y over the entire period are of the univariate generalized Waring type as given by (2.7). In particular, X has the UGWD ($\alpha,k;\ \varrho$), Y has the UGWD ($\alpha,m;\ \varrho$) while X+Y has the UGWD ($\alpha,k+m;\ \varrho$). Moreover,

the conditional period 2 distribution given the observed period 1 demand is UGWD $(a+x,m; \varrho+k)$. In other words

$$P(Y=y \mid X=x) = \frac{(\varrho+k)_{(m)}}{(a+\varrho+k)_{(m)}} \frac{(a+x)_{(y)} \, m_{(y)}}{(a+\varrho+k+m)_{(y)}} \frac{1}{y!}, \quad y = 0,1,2,\ldots \quad (3.8)$$

from which it follows that the period 2 conditional expected demand given the aggregate demand $X=x$ at the end of period 1 is a linear function of x. In particular,

$$E(Y \mid X=x) = \frac{(a+x) \cdot m}{\varrho+k-1}. \quad (3.9)$$

Applying (3.7) to the data set used by Bradford and Sugrue (1990) on the demand for framed and unframed poster art sold by a small retail firm yields the results summarized in Table 1. As the above authors mention, the data were split into two consecutive and non overlapping 4-month periods. However, they only provided the period 1 observed frequency distribution. So, the model (3.7) cannot be judged otherwise but only indirectly through the closeness of the calculated period 2 average conditional demand to the corresponding observed one.

The bivariate generalized Waring distribution has been fitted by a moment method that utilizes the first and second order moments of the distribution whose sample estimates are provided by Bradford and Sugrue. In particular, the estimating equations used are

$$\bar{X} = \frac{\hat{a}\,\hat{k}}{\hat{\varrho}-1}$$

$$\bar{Y} = \frac{\hat{a}\,\hat{m}}{\hat{\varrho}-1}$$

$$S_X^2 + S_Y^2 = \frac{\hat{\alpha}(\hat{\alpha}+\hat{\varrho}-1)[\hat{k}(\hat{\varrho}+\hat{k}-1)+\hat{m}(\hat{\varrho}+\hat{m}-1)]}{(\hat{\varrho}-1)^2(\hat{\varrho}-2)}$$

$$S_{X+Y}^2 = \frac{\hat{\alpha}(\hat{k}+\hat{m})(\hat{\varrho}+\hat{k}+\hat{m}-1)(\hat{\varrho}+\hat{\alpha}-1)]}{(\hat{\varrho}-1)^2(\hat{\varrho}-2)}$$

where S_W^2 denotes the sample estimate of the variance of the random variable W.

The estimates obtained from this system are

$$\hat{\alpha} = \frac{2\,\bar{X}\,\bar{Y}\,(S_X^2+S_Y^2)}{(S_{X+Y}^2 - S_X^2 - S_Y^2)\,(\bar{X}+\bar{Y})} - \frac{\bar{X}^2 + \bar{Y}^2}{\bar{X}+\bar{Y}}$$

$$\hat{\varrho} = \frac{2\,\hat{\alpha}\,S_{X+Y}^2 + (\hat{\alpha}-1)\,(\bar{X}+\bar{Y})\,(\hat{\alpha}+\bar{X}+\bar{Y})}{\hat{\alpha}\,S_{X+Y}^2 - (\bar{X}+\bar{Y})\,(\hat{\alpha}+\bar{X}+\bar{Y})}$$

$$\hat{k} = \frac{(\hat{\varrho}-1)\bar{X}}{\hat{\alpha}}$$

$$\hat{m} = \frac{\bar{Y}}{\bar{X}}\,\hat{k}\,.$$

On the basis of the fact that the average demand for the total season was 3.015 units and the standard deviation 3.236 units while the corresponding sample values for periods 1 and 2 were 1.517 units with standard deviation 1.803 units and 1.498 units with standard deviation 1.924 units respectively, the above equations yield the following parameter estimates:

$$\hat{\alpha} = 1.471 \qquad \hat{\varrho} = 19.812 \qquad \hat{k} = 19.405 \qquad \hat{m} = 19.162$$

Hence the predicted period 2 expected demands can be calculated through (3.9). Table 1 summarizes the results. Also included in table 1 are the period 2 calculated forecasts for the negative binomial model (NBD) of Bradford and Sugrue (1990). For comparison purposes the generalized Waring model forecasts have been multiplied by the correction factor 0.9872 by which Bradford and Sugrue multiplied their estimate to reflect the observed change in aggregate demand from 1012 units in period 1 to 999 units in period 2.

Table 1

Expected period 2 demand given period 1 demand

Actual demand x (period 1)	Observed period 2 conditional average demand	Estimated period 2 conditional average demand	
		NBD	UGWD
0	0.688	0.728	0.699
1	1.182	1.223	1.225
2	1.840	1.718	1.752
3	2.424	2.213	2.278
4	2.907	2.708	2.805
5	2.773	3.203	3.331
6	3.176	3.698	3.858
≥7	5.909	-	-

As seen from table 1 the generalized Waring distribution produced reasonably accurate forecasts which are not appreciably different from those provided by the negative binomial model. The null hypothesis that the joint distribution over the two periods is the bivariate generarlized Waring distribution cannot be tested as the observed joint frequencies are not provided. However, the closeness of the period 1 observed frequency distribution to the corresponding expected marginal frequency distribution of demand as calculated through the probabilities of the $UGWD(\alpha, \hat{k}, ; \hat{\varrho})$ (the period 1 marginal of the $BGWD(\alpha; k,m; \varrho)$ for $\hat{\alpha} = 1.471$, $\hat{k} = 19.405$,

and $\hat{\rho} = 19.812$) is indicative of the adequacy of description provided by the bivariate model (see table 2).

Table 2
Period 1 demand distribution

Actual demand	0	1	2	3	4	5	6	≥ 7
Observed frequency	260	154	94	66	43	22	17	11
Expected frequency	246	173	105	61	35	20	11	16

If we disregard the fact that the parameter estimates of the bivariate model were utilized in calculating the expected frequencies in table 2 and calculate the value of the chi-squared goodness of fit criterion we obtain $\chi^2 = 10.9994$. Following, for comparison purposes, Bradford and Sugrue (1990) in overlooking the fact that we have estimated parameters and considering thus having 7 degrees of freedom yields a p-value of 0.1386. The corresponding χ^2 value given by Bradford and Sugrue (1990) for their NBD model is 9.21 with 7 degrees of freedom yielding a p-value of 0.238. Hence, the results are far from indicating an appreciable difference between the two models considered as descriptors of the frequency of demand and of the expected conditional period 2 demand for the given data.

The advantage, however, of the generalized Waring model in this situation is not to be sought only in the fit the model provides compared to that of another model. It is to be sought in the features of the model that allow us to have an insight into the extent in which the various factors affect the demand for an item. This can be achieved by making use of the fact that, as Xekalaki (1984a) has pointed out, under the bivariate generalized Waring model the total variance of the observations for the total period and for each of the two subperiods can be written in the form of a sum of three components: one due to randomness (σ_R^2) one due to the exogenous factors (σ_λ^2) and one due to the endogenous

factors (σ_v^2). In particular, the total variance for period i can be written in the form

$$\sigma_i^2 = \sigma_{R_i}^2 + \sigma_{\lambda_i}^2 + c_i\sigma_v^2$$

where c_i is a positive constant expressed in terms of the parameters of the model. Table 3 demonstrates this potential.

Table 3

The components of the variance of the bivariate generalized Waring model

Component due to	Marginal variance of X (*period 1*)	Marginal variance of Y (*period 2*)	Variance of X+Y (*total period*)
Random factors	$\dfrac{\alpha k}{\varrho-1}$	$\dfrac{\alpha m}{\varrho-1}$	$\dfrac{\alpha(k+m)}{\varrho-1}$
Proneness (*endogenous factors*)	$\dfrac{k^2\alpha(\alpha+\varrho-1)}{(\varrho-1)^2(\varrho-2)}$	$\dfrac{m^2\alpha(\alpha+\varrho-1)}{(\varrho-1)^2(\varrho-2)}$	$\dfrac{(k+m)^2\alpha(\alpha+\varrho-1)}{(\varrho-1)^2(\varrho-2)}$
Liability (*exogenous factors*)	$\dfrac{\alpha k(\alpha+1)}{(\varrho-1)(\varrho-2)}$	$\dfrac{\alpha m(\alpha+1)}{(\varrho-1)(\varrho-2)}$	$\dfrac{\alpha(k+m)(\alpha+1)}{(\varrho-1)(\varrho-2)}$
Total	$\dfrac{\alpha k(\varrho+k-1)(\varrho+\alpha-1)}{(\varrho-1)^2(\varrho-2)}$	$\dfrac{\alpha m(\varrho+m-1)(\varrho+\alpha-1)}{(\varrho-1)^2(\varrho-2)}$	$\dfrac{\alpha(k+m)(\varrho+k+m-1)(\varrho+\alpha-1)}{(\varrho-1)^2(\varrho-2)}$

By estimating the parameters of the bivariate generalized Waring distribution we can arrive at estimates of the variance components as specified by Table 3 and hence assess the contribution of each of the three demand factors in any particular situation.

Applying this analysis to Bradford and Sugrue's (1990) data yields Table 4.

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Table 4

Estimates of the components of the variance of
the generazized Waring distribution of demand
for Branford and Sugrue's (1990)
unframed poster art data

Component	Period 1	Period 2	Overall period
Random	1.517 (43.2%)	1.498 (43.5%)	3.015 (28.8%)
Proneness (*endogenous factors*)	1.782 (50.8%)	1.737 (50.5%)	7.038 (67.2%)
Liability (*exogenous factors*)	0.21 (6%)	0.208 (6%)	0.418 (4%)
Total	3.509 (100%)	3.443 (100%)	10.471 (100%)

Inspection of this table indicates that proneness, reflecting the inherent appeal of the item to the buyer, is by far the dominating "causing" factor. In both subperiods and in the total period its contribution is above 50% with a noticeable percentage of 67.2% in the total period.

Random factors have had a contribution reflected by the largest share of the remaining percentage in all the three periods leaving the effect of exogenous factors a 6% of the total variation in each of the two subperiods and a 4% in the total period.

It seems, therefore, that in this situation proneness and factors other than liability (encompassed in the term random factors) had a higher effect with proneness to be accounted for the fact that the demand was greater for some items than for others. The low contribution of liability to the total variation might reflect a relatively equal market exposure of the items.

4. THE EFFECT OF A GENERALIZED WARING DISTRIBUTED DEMAND ON THE STOCKING POLICY

The purpose of this section is to look into the question of determining the best order quantity. The problem will be approached from a profit maximization standpoint, i.e., the determination of the optimal stocking policies in periods 1 and 2 will be centered upon maximizing the total profit during the entire selling period.

Assume that an "order up to" policy is adopted and that orders are placed at the beginning of periods 1 and 2 with no restriction on the size of the order and with instantaneous replenishment lead time. Following Bradford and Sugrue (1990) we further assume that the cost of overstocking in period 1 is deferred until the end of the selling season when any remaining stock is considered to be a total loss.

Let R, v, g and B denote the retail selling price, the acquisition cost, the salvage value and the lost sale cost per poster title respectively. Suppose that at the beginning of period 1 the management decides to stock S_1 units while at the end of period 1 (beginning of period 2) it decides to bring the stocking level up to S_2 units. Then under the generalized Waring model of section 3, the total expected profit for the entire season is

$$\Pi(S_1, S_2) = R\left\{ \sum_{x=0}^{S_1} xP(X=x) + \sum_{y=0}^{S_2} yP(Y=y) \right\} + R\left\{ S_1 \sum_{x=S_1+1}^{\infty} P(X=x) + S_2 \sum_{y=S_2+1}^{\infty} P(Y=y) \right\}$$

$$-B\left\{ \sum_{x=S_1+1}^{\infty} (x-S_1)P(X=x) + \sum_{y=S_2+1}^{\infty} (y-S_2)P(Y=y) \right\} +$$

$$+g\sum_{y=0}^{S_2} (S_2-y)P(Y=y) - vS_2 - v\sum_{x=0}^{S_1} xP(X=x) \qquad (4.1)$$

where $(X,Y) \sim BGWD(\alpha; k,m; \varrho)$.

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The optimal solution to (4.1) may be found by determining the values of S_1 and S_2 that maximize $\Pi(S_1,S_2)$. These will be the values for which the partial differences $\Delta_{S_1}\Pi(S_1,S_2) = \Pi(S_1+1,S_2)$-$\Pi(S_1,S_2)$ and $\Delta_{S_2}\Pi(S_1,S_2) = \Pi(S_1,S_2+1)$-$\Pi(S_1,S_2)$ become negative for the first time.

From (4.1) we have for the first partial difference

$$\Pi(S_1+1,S_2) - \Pi(S_1,S_2) = R(S_1+1)P(X=S_1+1) +$$

$$R\left\{ \sum_{x=S_1+2}^{\infty} P(X=x) - S_1 P(X=S_1+1)\right\} +$$

$$B\left\{ \sum_{x=S_1+2}^{\infty} P(X=x) + P(X=S_1+1)\right\} - v(S_1+1)P(X=S_1+1)$$

i.e.,

$$\Delta_{S_1}\Pi(S_1,S_2) = (B+R-v(S_1+1))P(X=S_1+1)+(R+B)P(X>S_1+1) . \quad (4.2)$$

Similarly

$$\Pi(S_1,S_2+1)-\Pi(S_1,S_2) = R(S_2+1)P(Y=S_2+1) +$$

$$R\left\{ \sum_{y=S_2+2}^{\infty} P(Y=y)-S_2 P(Y=S_2+1)\right\} + B\left\{ \sum_{y=S_2+2}^{\infty} P(Y=y)+P(Y=S_2+1)\right\} +$$

$$g\left\{ \sum_{y=0}^{S_2+1} (S_2+1-y)P(Y=y) - \sum_{y=0}^{S_2} (S_2-y)P(Y=y)\right\} - v$$

i.e.,

$$\Delta_{S_2}\Pi(S_1,S_2) = (R+B-g)P(Y=S_2+1) + (R+b-g)P(Y>S_2+1) + g - v \quad (4.31)$$

Therefore, the optimal stocking policy would be: stock S_1 units at the beginning of period 1 and S_2 units at the beginning of period 2 where S_1 and S_2 are the lowest values for which

$$(R+B)P(X \geq S_1+1) - v(S_1+1)P(X=S_1+1) < 0$$

and

$$(R+B-G)P(Y \geq S_2+1) + g - v < 0$$

where $(X,Y) \sim BGWD(\alpha; k,m; \varrho)$.

Iterating over various values of S_1 and S_2 is a straightforward process which can be further simplified in certain special cases of the BGWD. As an illustration, consider Bradford and Sugrue's (1990) data. The BGWD was applied to these data using a moment estimation procedure which led to an estimate of α close to unity. So, if for simplicity we consider the joint demand distribution over the two periods to be the BGWD(1; k,m; p), the inequalities in (4.4) reduce to

$$S_1(R+B-v\varrho) < v\varrho - (R+B)(k+\varrho+1)$$

$$P(Y=S_2+1)(S_2+\varrho+m+1) < \frac{(v-g)\varrho}{R+B-g} \qquad (4.5)$$

This is a consequence of the fact that since $X \sim UGWD(1,k; \varrho)$ and $Y \sim UGWD(1,m; \varrho)$

$$P(X>S_1+1) = \frac{k+S_1+1}{\varrho} P(X=S_1+1)$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.6)$

$$P(Y>S_2+1) = \frac{m+S_2+1}{\varrho} P(Y=S_2+1)$$

(see Dimaki and Xekalaki (1992)).

Hence dividing both sides of the first inequality in (4.4) by $P(X=S_1+1)$ and of the second inequality by $P(Y=S_2+1)$ and using (4.6) we arrive at (4.5).

In concluding, it should be emphasized that the model considered in this paper provides an innovative tool to the manager of a warehouse or a firm as apart from being an alternative to the negative binomial model it allows separate estimation of the effects of non-random factors pertaining to the inherent appeal of the items and to their market exposure. In addition to these mana-

gerial insights into the nature of the stocking problem consider-
ed here, the BGWD offers ease of calculation of the optimal stock
levels.

REFERENCES

Bradford, J.W. and Sugrue, P.K. (1990). "A Bayesian Approach to
the Two-Period Style-Goods Inventory Problem with Single
Replenishment and Heterogeneous Poisson Demands", *J. Opl.
Res.Soc.*, 41(3), 211-218.

Greenwood, M. and Woods, H.M. (1919). "On the Incidence of Indu-
strial Accidents upon Individuals with Special Reference to
Multiple Accidents". *Report of the Industrial Fatigue Rese-
arch Board*, No 4, 1-28, London: His Magesty's Stationery
Office.

Dimaki, C. and Xekalaki, E. (1992). "Towards a Unification of
Certain Characterizations by Conditional Expectations",
Ann.Instit.Statist.Math., A (to appear).

Xekalaki, E. (1981). "Chance Mechanisms for the Univariate Genera-
lized Waring Distributions". *Statistical Distributions in
in Scientific Work*, Vol.4, (*Models, Structures and Chara-
cterizations*), (C. Taillie, G.P. Patil and B. Baldessari,
eds), pp. 157-171. Dordrecht, Holland: D. Reidel.

Xekalaki, E. (1983a). "The Univariate Generalized Waring Distribu-
tion in Relation to Accident Theory: Proneness, Spells or
Contagion?". *Biometrics*, 39(3), 887-895.

Xekalaki, E. (1983b). "A Property of the Yule Distribution and its Applications". *Commun.Statist.*A, 12, 1181-1189.

Xekalaki, E. (1983c). "Hazard Functions and Life Distributions in Discrete Time". *Commun.Statist.*A, 12, 2503-2509.

Xekalaki, E. (1984a). "The Bivariate Generalized Waring Distribution and Its Application to Accident Theory", *J.Roy.Statist.Soc.*,A, 147(3), 488-498.

Xekalaki, E. (1984b). "Linear Regression and the Yule Distribution". *J.Econometrics*, 24, 397-403.

Xekalaki, E. (1988). "Buying and Stocking Under Uncertainty". *Scientific Annals*, The Athens School of Economics and Business Science, 365-375.