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SOME BIVARIATE EXTENSIONS OF THE GENERALIZED WARING DISTRIBUTION

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Summary

The bivariate generalized Waring distribution results as a mixture of the double Poisson distribution. In this paper some probability models are considered that give rise to alternative bivariate forms of the generalized Waring distribution.

1. Introduction

The bivariate generalized Waring distribution with parameters a, k, m and ϱ (B.G.W.D. $(a; k, m; \varrho)$) defined by Xekalaki [7] is the distribution with probability function (p. f.) $p_{r,l}$ given by

$$(1.1)\qquad p_{rl} = \frac{\varrho_{(k+m)}}{(a+\varrho)_{(k+m)}} \frac{a_{(r+l)}k_{(r)}m_{(l)}}{(a+k+\varrho+m)_{(r+l)}} \frac{1}{r!}\frac{1}{l!}$$

$$r = 0, 1, 2, \ldots, \quad l = 0, 1, 2, \ldots$$

where $a, k, m, \varrho > 0$ and $\alpha_{(\beta)} = \Gamma(\alpha+\beta)/\Gamma(\alpha)$, $\alpha > 0$, $\beta \in \mathbf{R}$. The probability generating function (p.g.f.) of this distribution is

$$G(s, t) = \frac{\varrho_{(k+m)}}{(a+\varrho)_{(k+m)}} F_1(a;\ k, m;\ a+k+\varrho+m;\ s, t)$$

where $F_1(a; b, b'; x, y)$ is the Appell function of the first kind defined by

$$F_1(a;\ b, b';\ c;\ x, y) = \sum_{m,n} \frac{a_{(m+n)}\, b_{(m)}\, b'_{(n)}}{c_{(m+n)}} \frac{x^m}{m!} \frac{y^n}{n!}$$

$$a, b, b', c-a-b-b' > 0, \quad (x, y) \in [-1, 1]\times[-1, 1].$$

The marginal probability distributions of X and Y, the conditional distributions of $X|(Y=y)$ and $Y|(X=x)$ as well as the distribution of $X+Y$ are all of the same form. Specifically, they are univariate generalized Waring distributions (U.G. W.D.) with p.g.f.'s expressed in terms of the Gauss hypergeometric function obtained from

$$(1.2)\qquad {}_pF_q(a_1, a_2, \ldots, a_p;\ b_1, b_2, \ldots, b_q;\ z) = \sum_{r=0}^{\infty} \frac{(a_1)_{(r)}\cdots(a_p)_{(r)}}{(b_1)_{(r)}\cdots(b_q)_{(r)}} \frac{z^r}{r!}$$

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for $p=2$, $q=1$. The U.G.W.D. is a member of the family of generalized hypergeometric distributions studied by, among others, Kemp and Kemp [2] and Sarkadi [3]. (For more information concerning the structure, properties and applications of the U.G.W.D. $(a, k; \varrho)$ the interested reader is referred to Irwin [1] and Xekalaki [4], [5], [6], [8].)

Xekalaki [7] showed that the B.G.W.D. can arise as the joint distribution of accidents incurred in two consecutive time periods by a group of people in situations where not only random factors are present, but, also, factors associated with the individual's exposure to external risk as well as psychological factors predisposing the individual to accidents. In the same paper, except for providing a satisfactory fit to accident data, the B.G.W.D. was shown to enable one to separately estimate the variance components due to random, external and psychological factors so that one can have a clue as to which kind of factors influenced the particular accident situation the most.

The derivation of the B.G.W.D. in the context of the above-mentioned accident situation was based on a mixed Poisson model. The mathematical nature of the mixing processes involved suggests the possibility of obtaining some more general forms of bivariate distributions with both marginals of the U.G.W.D. type. This can be done by slightly altering the mixing mechanisms concerned.

In the sequel, we provide various mixed models which give rise to such distributions. Some properties and limiting cases of these models are also considered.

2. Some bivariate extensions of the generalized Waring distribution

We first give a description of Xekalaki's [7] model.

MODEL 1 (Xekalaki, [7]). Let X_1, X_2 be nonnegative integer valued random variables (r.v.'s) whose joint distribution is the double Poisson with p.g.f.

$$g(s, t) = e^{\lambda_1(s-1)+\lambda_2(t-1)}, \qquad \lambda_1, \lambda_2 > 0. \tag{2.1}$$

Assume that λ_1 and λ_2 are independent gamma r.v.'s with probability density functions

$$f(\lambda_1) = \frac{v^{-k}}{\Gamma(k)} e^{-(1/v)\lambda_1} \lambda_1^{k-1}, \qquad v, k > 0 \tag{2.2}$$

and

$$g(\lambda_2) = \frac{v^{-m}}{\Gamma(m)} e^{-(1/v)\lambda_2} \lambda_2^{m-1}, \qquad m > 0, \tag{2.3}$$

respectively. Then (2.1) becomes

(2.4)
$$G(s, t) = \frac{v^{-k}}{\Gamma(k)} \int_0^\infty e^{-(\lambda_1/v)(1+v(1-s))} \lambda_1^{k-1}\, d\lambda_1 \frac{v^{-m}}{\Gamma(k)} \int_0^\infty e^{-(\lambda_2/v)(1+v(1-t))} \lambda_2^{m-1}\, d\lambda_2 =$$
$$= (1+v(1-s))^{-k}(1+v(1-t))^{-m},$$

i.e. $(X_1, X_2)|v$ follows a double negative binomial distribution with parameters $k, v/(1+v)$, m and $v/(1+v)$. Assume now that v has a beta distribution of the second kind (beta II) with parameters a and ϱ and p.d.f.

$$h(v) = \frac{\Gamma(a+\varrho)}{\Gamma(\varrho)\Gamma(a)} v^{a-1}(1+v)^{-(\varrho+a)}, \qquad a, \varrho > 0.$$

Then the resulting distribution of X_1, X_2 has p.g.f.

$$G_{x_1x_2}(s, t) = \frac{\Gamma(\varrho+a)}{\Gamma(\varrho)\Gamma(a)} \int_0^\infty v^{a-1}(1+v)^{-(a+\varrho)}(1+v(1-s))^{-k}(1+v(1-t))^{-m}\, dv =$$

$$= \frac{\varrho_{(k+m)}}{(a+\varrho)_{(k+m)}} F_1(a;\ k, m;\ a+k+m+\varrho;\ s, t),$$

i.e., $(X_1, X_2) \sim$ B.G.W.D. $(a; k, m; \varrho)$.

Xekalaki [8] has shown that under certain conditions the B.G.W.D. $(a; k, m; \varrho)$ tends to the double negative binomial distribution with parameters $k, m, \dfrac{a}{a+\varrho}$ and $\dfrac{a}{a+\varrho}$, and to the double Poisson distribution with parameters $\dfrac{ak}{a+\varrho}$ and $\dfrac{am}{a+\varrho}$. Moreover, if the scale is at our choice, the B.G.W.D. can be shown to tend to the bivariate beta II distribution with parameters k, m and ϱ or to an uncorrelated bivariate gamma distribution with parameters k, m, 1 and 1.

MODEL 2. Let X_1, X_2 be non-negative discrete r.v.'s whose joint distribution is the double Poisson with parameters λp, and λq and p.g.f. given by

$$g(s, t) = e^{\lambda(p(s-1)+q(t-1))}, \qquad \lambda, p, q > 0, \quad p+q \leqq 1. \tag{2.5}$$

Assume that λ is a r.v. having a gamma distribution with p.d.f.

$$f(\lambda) = \frac{(a/b)^a}{\Gamma(a)} e^{-(a/b)\lambda} \lambda^{a-1}, \quad \lambda, a, b > 0. \tag{2.6}$$

Then (2.5) becomes

$$\begin{aligned} G(s, t) &= \frac{(a/b)^a}{\Gamma(a)} \int_0^\infty e^{-\lambda(a/b)(1+(b/a)p(1-s)+(b/a)q(1-t))} \lambda^{a-1}\, d\lambda = \\ &= \left[1+\frac{b}{a}\{p(1-s)+q(1-t)\}\right]^{-a}, \end{aligned} \tag{2.7}$$

i.e., $(X_1, X_2)|b$ follows a bivariate negative binomial distribution with parameters $a, bp/(a+bp)$ and $bq/(a+bq)$.

If we now let b have a Beta II distribution with parameters k, ϱ and p.d.f.

$$h(b) = \frac{\Gamma(\varrho+k)a^{-1}}{\Gamma(\varrho)\Gamma(k)} \left(\frac{b}{a}\right)^{k-1} \left(1+\frac{b}{a}\right)^{-(\varrho+k)} \tag{2.8}$$

$$\varrho > 0,\ k > 0,\ b > 0,\ a > 0$$

the final resulting joint distribution of X_1, X_2 has p.g.f.

$$G_{X_1,X_2}(s,t)=\frac{\Gamma(\varrho+k)a^{-1}}{\Gamma(\varrho)\Gamma(k)}\int_0^\infty\left(\frac{b}{a}\right)^{k-1}\left(1+\frac{b}{a}\right)^{-(\varrho+k)}\times$$

$$\times\left[1+\frac{b}{a}\{p(1-s)+q(1-t)\}\right]^{-a}db= \tag{2.9}$$

$$=\frac{\varrho_{(k)}}{(a+\varrho)_{(k)}}\,{}_2F_1(a,k;\ a+k+\varrho;\ 1-p-q+ps+qt).$$

The generalized hypergeometric series in (2.9) is convergent for all $(s,t)\in[-1,1]\times\times[-1,1]$.

It can be seen that

$$G_{X_1}(s)=G_{X_1,X_2}(s,1)=\frac{\varrho_{(k)}}{(a+\varrho)_{(k)}}\,{}_2F_1(a,k;\ a+k+\varrho;\ 1-p+ps) \tag{2.10}$$

with a similar expression for the p.g.f. of X_2.

$$G_{X_1|X_2}(s)=\frac{\partial^{x_2}}{\partial t^{x_2}}G_{X_1X_2}(s,0)\Big/\frac{\partial^{x_2}}{\partial t^{x_2}}G_{X_1X_2}(1,0)=$$

$$=\frac{{}_2F_1(a+x_2,k+x_2;\ a+k+\varrho+x_2;\ 1-p-q+ps)}{{}_2F_1(a+x_2,k+x_2;\ a+k+\varrho+x_2;\ 1-q)} \tag{2.11}$$

with a similar expression for the p.g.f. of $X_2|X_1$ and

$$G_{X_1+X_2}(s)=G_{X_1,X_2}(s,s)=\frac{\varrho_{(k)}}{(a+\varrho)_{(k)}}\,{}_2F_1(a,k;\ a+k+\varrho;\ 1+(p+q)(s-1)), \tag{2.12}$$

i.e., the marginals and the convolution have all the same form.

We note that (2.10) is in fact a U.G.W.D. generalized by a binomial distribution with index 1 and probability of success p and can arise in situations where sampling is made with inclusion probability equal to p. Hence, (2.10) defines a more general distribution which includes the U.G.W.D. as a special case ($p=1$). It is interesting to see that the factorial moments of the distribution generated by (2.10) are given by

$$\mu_{(r)}(x)=p^r\frac{a_{(r)}k_{(r)}}{(\varrho-1)\ldots(\varrho-r)}. \tag{2.13}$$

It is also of interest to remark that in the case $p+q=1$, (2.9) and (2.12) reduce to

$$G_{X_1,X_2}(s,t)=\frac{\varrho_{(k)}}{(a+\varrho)_{(k)}}\,{}_2F_1(a,k;\ a+k+\varrho;\ ps+qt) \tag{2.14}$$

and

$$G_{X_1+X_2}(s)=\frac{\varrho_{(k)}}{(a+\varrho)_{(k)}}\,{}_2F_1(a,k;\ a+k+\varrho;\ s)\sim\text{U.G.W.D. }(a,k;\ \varrho). \tag{2.15}$$

Moreover, (8.1.7) becomes

$$G_{X_1|X_2}(s)=\frac{{}_2F_1(a+x_2,k+x_2;\ a+k+\varrho+x_2;\ ps)}{{}_2F_1(a+x_2,k+x_2;\ a+k+\varrho+x_2;\ p)}$$

which, provided that $\varrho > x_2$, is a weighted U.G.W.D. $(a+x_2, k+x_2; \varrho-x_2)$ and can arise in cases where the sampling chance (weight) is proportional to p^{x_1}, $p \leqq 1$.

In the more general case when $p+q \leqq 1$ it can be shown, using an argument similar to that used by Xekalaki [8] that

(i) *for* $\varrho \to \infty$, $a \to \infty$, $\dfrac{a}{a+\varrho} < 1$, $0 < k < \infty$, *the distribution defined by* (2.9) *tends to the bivariate negative binomial distribution with parameters* k, $\dfrac{ap}{a+\varrho}$ *and* $\dfrac{aq}{a+\varrho}$.

(ii) *if we let* $a \to \infty$, $k \to \infty$, $\varrho \to \infty$ *while* $\dfrac{a}{a+\varrho} \to 0$ *and* $\dfrac{ka}{a+\varrho} < \infty$, *we obtain the double Poisson distribution with parameters* $\dfrac{akp}{a+\varrho}$ *and* $\dfrac{akq}{a+\varrho}$.

MODEL 3. Consider X_1 and X_2 to be two non-negative discrete r.v.'s whose joint distribution is the double Poisson with parameters λ_1, λ_2 and p.g.f. given by

$$g(s, t) = e^{\lambda_1(s-1)+\lambda_2(t-1)}, \qquad \lambda_1, \lambda_2 > 0. \tag{2.16}$$

Assume that (λ_1, λ_2) is a random vector having a bivariate gamma distribution with p.d.f.

$$f(\lambda_1, \lambda_2) = \frac{b^{-m} e^{-(1/b)\lambda_2}}{\Gamma(k)\Gamma(m-k)} \lambda_1^{k-1}(\lambda_2-\lambda_1)^{m-k-1} \tag{2.17}$$

$$b > 0,\ m > k > 0,\ \lambda_2 \geqq \lambda_1 > 0.$$

Then, from (2.16) we have

$$\begin{aligned} G(s, t) &= \frac{b^{-m}}{\Gamma(k)\Gamma(m-k)} \int_0^\infty \int_0^{\lambda_2} e^{-(1/b)\lambda_2(1+b(1-t)+\lambda_1(1-s))} \times \\ &\quad \times \lambda_1^{k-1}(\lambda_2-\lambda_1)^{m-k-1}\, d\lambda_1\, d\lambda_2 = \\ &= \frac{1}{\Gamma(k)\Gamma(m-k)} \int_0^\infty e^{-(\lambda_2/b)(1+b(1-t))} \left(\frac{\lambda_2}{b}\right)^{m-k-1} \int_0^{\lambda_2/b} e^{\lambda_1(1-s)} \times \\ &\times \left(\frac{\lambda_1}{b}\right)^{k-1} \left(1-\frac{\lambda_1}{\lambda_2}\right)^{m-k-1} d\frac{\lambda_1}{b}\, d\frac{\lambda_2}{b} = \frac{1}{\Gamma(k)\Gamma(m-k)} \int_0^\infty e^{-(\lambda_2/b)(1+b(1-t))} \left(\frac{\lambda_2}{b}\right)^{m-1} \times \\ &\quad \times \int_0^1 e^{-\lambda_1\lambda_2(1-s)} \lambda_1^{k-1}(1-\lambda_1)^{m-k-1}\, d\lambda_1\, d\frac{\lambda_2}{b} = \\ &= \frac{1}{\Gamma(m)} \int_0^\infty {}_1F_1(k;\ m;\ \lambda_2(s-1)) e^{-(\lambda_2/b)(1+b(1-t))} \left(\frac{\lambda_2}{b}\right)^{m-1} d\frac{\lambda_2}{b} = \\ &= [1+b(1-t)]^{-m}{}_1F_0\left(k;\ ;\ \frac{b(s-1)}{1+b(1-t)}\right) = \\ &= [1+b(1-t)]^{-m+k}[1+b(1-t)+b(1-s)]^{-k} \end{aligned} \tag{2.18}$$

where ${}_1F_1(a; b; z)$ and ${}_1F_0(a; ; z)$ are obtained from (1.2) for $p=q=1$ and $p=1$, $q=0$, respectively.

If we now consider $b \sim$ Beta II $(a; \varrho)$, i.e.

$$b \sim \frac{\Gamma(a+\varrho)}{\Gamma(a)\Gamma(\varrho)} b^{a-1}(1+b)^{-(a+\varrho)}, \qquad b>0,\ a,\ \varrho>0, \tag{2.19}$$

we obtain the final joint distribution of X_1, X_2 as

$$\begin{aligned} G_{X_1X_2}(s,t) &= \frac{\Gamma(a+\varrho)}{\Gamma(a)\Gamma(\varrho)} \int_0^\infty b^{a-1}(1+b)^{-(a+\varrho)}[1+b(1-t)]^{-m+k} \times \\ &\times [1+b(1-t)+b(1-s)]^{-k}\, db = \frac{\varrho_{(m)}}{(a+\varrho)_{(m)}} F_1(a;\ m-k, k;\ a+\varrho+m;\ t, t+s-1). \end{aligned} \tag{2.20}$$

Note that the F_1 series in (2.20) is convergent for $(s, t) \in [-1, 1] \times [-1, 1]$.

We have for the marginal distributions of X_1 and X_2

$$\begin{aligned} G_{X_1}(s) &= \frac{\varrho_{(m)}}{(a+\varrho)_{(m)}} F_1(a;\ m-k, k;\ a+\varrho+m;\ 1, s) = \\ &= \frac{\varrho_{(k)}}{(a+\varrho)_{(k)}} {}_2F_1(a, k;\ a+\varrho+k;\ s) \sim \text{U.G.W.D. } (a, k;\ \varrho) \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} G_{X_2}(t) &= \frac{\varrho_{(m)}}{(a+\varrho)_{(m)}} F_1(a;\ m-k, k;\ a+\varrho+m;\ t, t) = \\ &= \frac{\varrho_{(m)}}{(a+\varrho)_{(m)}} {}_2F_1(a, m;\ a+\varrho+m;\ t) \sim \text{U.G.W.D. } (a, m;\ \varrho), \end{aligned} \tag{2.22}$$

respectively.

Thus, both marginal distributions of (2.20) are U.G.W.D.'s. This was expected due to the fact that the distributions of λ_1 and λ_2 are, from (2.17), gamma (k, b^{-1}) and gamma (m, b^{-1}), respectively.

The conditional distributions of $X_2|X_1$, $X_1|X_2$ and the distribution of X_1+X_2, however, are not of a U.G.W.D. form. Nor are they of a more general form containing the U.G.W.D. as a particular case.

It is interesting to see that for $a \to \infty$, $\varrho \to \infty$ while $\dfrac{a}{a+\varrho} \to q < 1$, $0 < k, m < \infty$ the distribution in (2.20) tends to a distribution with p.g.f.

$$\{1+q(1-t)/p\}^{-m+k}\{1+q[(1-t)+(1-s)]/p\}^{-k}, \qquad p = 1-q.$$

Moreover, for $a\to\infty, k\to\infty, m\to\infty, \varrho\to\infty$ while $\frac{a}{a+\varrho}\to 0, \frac{ak}{a+\varrho}<\infty$ and $\frac{am}{a+\varrho}<\infty$ it reduces to the double Poisson distribution with parameters $\frac{ak}{a+\varrho}$ and $\frac{am}{a+\varrho}$.

MODEL 4. Let X_1, X_2 be non-negative discrete r.v.'s and let their joint distribution be the double Poisson with parameters λ_1, λ_2 and p.d.f. given by (2.16). Assume that

$$\lambda_1 \sim \frac{\Gamma(\varrho+k)b^{-1}}{\Gamma(\varrho)\Gamma(k)}\left(\frac{\lambda_1}{b}\right)^{k-1}\left(1+\frac{\lambda_1}{b}\right)^{-(\varrho+k)}, \qquad \lambda_1>0, \quad \varrho, k, b>0 \tag{2.23}$$

$$\lambda_2 \sim \frac{\Gamma(\varrho+m)b^{-1}}{\Gamma(\varrho)\Gamma(m)}\left(\frac{\lambda_2}{b}\right)^{m-1}\left(1+\frac{\lambda_2}{b}\right)^{-(\varrho+m)}, \qquad \lambda_2>0, \quad \varrho, m, b>0. \tag{2.24}$$

Then (2.16) becomes

$$\begin{aligned} G(s,t) &= \frac{\Gamma(\varrho+k)\Gamma(\varrho+m)}{\{\Gamma(\varrho)\}^2\Gamma(k)\Gamma(m)}\int_0^\infty e^{\lambda_1(s-1)}\left(\frac{\lambda_1}{b}\right)^{k-1}\left(1+\frac{\lambda_1}{b}\right)^{-(\varrho+k)} d\frac{\lambda_1}{b}\times \\ &\times\int_0^\infty e^{\lambda_2(t-2)}\left(\frac{\lambda_2}{b}\right)^{m-1}\left(1+\frac{\lambda_2}{b}\right)^{-(\varrho+m)} d\frac{\lambda_2}{b} = \\ &= \frac{\Gamma(\varrho+k)\Gamma(\varrho+m)}{\{\Gamma(\varrho)\}^2}\psi(k;\ 1-\varrho;\ (1-s)b)\psi(m;\ 1-\varrho;\ (1-t)b) \end{aligned} \tag{2.25}$$

where ψ is the confluent hypergeometric function of the second kind defined by

$$\psi(a;\ c;\ z) = \frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}(1+t)^{c-a-1}e^{-zt}\,dt, \qquad a>0, \quad c-a>0.$$

Letting b be a r.v. with p.d.f.

$$f(b) = \frac{1}{\Gamma(a)}e^{-b}b^{a-1}, \qquad a>0,\ b>0 \tag{2.26}$$

we obtain the p.g.f. of the final resulting distribution of (X_1, X_2) as

$$G_{X_1,X_2}(s,t) = \frac{\Gamma(\varrho+k)\Gamma(\varrho+m)}{\{\Gamma(\varrho)\}^2\Gamma(a)}\int_0^\infty e^{-b}b^{a-1}\psi(k;\ 1-\varrho;\ b(1-s))\times$$
$$\times\psi(m;\ 1-\varrho;\ b(1-t))\,db.$$

It can be shown that

$$\psi(a;\ c;\ z) = \frac{\Gamma(1-c)}{\Gamma(1+a-c)}\,{}_1F_1(a;\ c;\ z).$$

Then,

$$G_{X_1,X_2}(s,t)=\frac{1}{\Gamma(a)}\int_0^\infty e^{-b}b^{a-1}{}_1F_1(k;\ 1-\varrho;\ b(1-s))\times$$

(2.27)
$$\times{}_1F_1(m;\ 1-\varrho;\ b(1-t))\,db=$$

$$=\frac{1}{\Gamma(a)}\sum_{r,l}\frac{k_{(r)}m_{(l)}}{(1-\varrho)_{(r)}(1-\varrho)_{(l)}}\frac{(1-s)^r}{r!}\frac{(1-t)^l}{l!}\int_0^\infty e^{-b}b^{a+r+l-1}\,db=$$

$$=\sum_{r,l}\frac{a_{(r+l)}k_{(r)}m_{(l)}}{(1-\varrho)_{(r)}(1-\varrho)_{(l)}}\frac{(1-s)^r}{r!}\frac{(1-t)^l}{l!}=F_2(a;\ k,m;\ 1-\varrho,1-\varrho;\ 1-s,1-t)$$

where F_2 is as defined by

$$F_2(a;\ b,b';\ c,c';\ x,y)=\sum_{m,n}\frac{a_{(m+n)}b_{(m)}b'_{(n)}}{c_{(m)}c'_{(n)}}\frac{x^m}{m!}\frac{y^n}{n!}.$$

The region of convergence for the p.g.f. given by (2.27) is $|1-s|+|1-t|<1$. Clearly, (2.16), (2.23), (2.24) and (2.26) imply that

(2.28) $\quad X_1\sim\text{Poisson}(\lambda_1)\wedge\text{Beta II}(k;\ \varrho)\wedge\text{gamma}(a;\ 1)\sim\text{U.G.W.D.}\ (a,k;\ \varrho),$

(2.29) $\quad X_2\sim\text{Poisson}(\lambda_2)\wedge\text{Beta II}(m;\ \varrho)\wedge\text{gamma}(a;\ 1)\sim\text{U.G.W.D.}\ (a,m;\ \varrho).$

Again, the convolution X_1+X_2 and the conditional distributions of $X_2|X_1$, $X_1|X_2$ are not U.G.W.D.'s.

We note that the double negative binomial and the double Poisson can be obtained as the limit of (2.27) for suitable limiting values of the parameters. In particular, for $a\to\infty$, $\varrho\to\infty$ while $\frac{a}{a+\varrho}<1$, $0<k<\infty$ and $0<m<\infty$, the double negative binomial $\left(k,m;\frac{a}{a+\varrho},\frac{a}{a+\varrho}\right)$ distribution arises. Also, for $a\to\infty$, $k\to\infty$, $m\to\infty$, $\varrho\to\infty$ while $\frac{a}{a+\varrho}\to 0$, $\frac{ak}{a+\varrho}<\infty$ and $\frac{am}{a+\varrho}<\infty$ we obtain the double Poisson $\left(\frac{ak}{a+\varrho},\frac{am}{a+\varrho}\right)$ distribution.

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