The Bivariate Yule Distribution and Some of Its Properties

Evdokia Xekalaki

University of Crete, Heraklio[1]

Summary. In this paper, a bivariate extension of the Yule distribution is defined and some of its structural properties are examined. It is shown in particular, that it can be obtained in the context of a bivariate STER model and as the only distribution with tail probabilities satisfying certain conditions.

AMS 1980 *subject classifications:* Primary 62E10; secondary 60E05.

Key words: Bivariate Yule distribution; bivariate STER model; tail probability.

1. Introduction

The Yule distribution with parameter $\varrho > 0$ (Yule (ϱ)) is the distribution whose probability function (p.f.) p_x ($\equiv \mathsf{P}(X = x)$) is given by

$$p_x = \frac{\varrho x!}{(\varrho + 1)_{(x+1)}} \qquad x = 0, 1, 2, \ldots \tag{1.1}$$

where $a_{(b)} = \Gamma(a+b)/\Gamma(a)$, $a > 0$, $b \in \mathbb{R}$. This distribution was obtained by Yule (1924) in connection with biological data and has been found to provide a useful model in various other fields. For example, Mitzutani (1953) used it to study the frequency of words comprising the vocabulary of women's magazines. Simon (1955, 1960) and Haight (1966) suggested it as a model for interpreting word association data. Further, Simon (1955) pointed out that income distributions can be expressed in terms of the Yule distribution and hence considered it in connection with economic data. For economic applications, it was also considered by Kendall (1961). In the same paper, Kendall suggested it as a model for describing bibliographic data. Xekalaki (1984c) showed that the Yule distribution can arise in the context of an econometric model. Finally, Xekalaki (1983a) obtained the Yule distribution in connection with an income under-reporting model as well as a demand distribution in inventory control problems.

Some of the applications obtained in the latter paper were based on the following mathematical model.

Let X, Y be two nonnegative, integer-valued random variables (r.v.'s) with p.f.'s $\mathsf{P}(X = x) \equiv p_x$ and $\mathsf{P}(Y = y) \equiv q_y$ respectively. Assume that

$$q_y = \frac{1}{1 - p_0} \sum_{x=y+1}^{\infty} \frac{p_x}{x}, \quad y = 0, 1, 2, \ldots . \tag{1.2}$$

[1] Now at the Athens School of Economics and Business Science.

21*

Then

$$q_r = p_r, \quad r = 0, 1, 2, \ldots$$

if and only if X has the YULE distribution with parameter ϱ and p.f. given by (1.1).

The distribution defined by (1.2) is known in the literature as the STER distribution (as its probabilities are Sums Successively Truncated from the Expectation of the Reciprocal of the r.v. X (BISSINGER (1965))). It arises often in inventory decision and income underreporting problems (see e.g. XEKALAKI (1983a)). In the context of these problems, when the distribution of demand or income is of a YULE form, some interesting results stem from (1.2) as shown by XEKALAKI (1983a, 1984c). These can be regarded as the discrete analogues of results that hold for the PARETO distribution which is a continuous approximation of the YULE distribution (KENDALL (1961)).

In the area of income (or demand) analysis it is often useful to study the distribution of income (or demand) in two consecutive time periods. This leads to the question of examining the possibility of extending the concepts of the STER and YULE distributions to the bivariate case and studying their interrelationship.

So, in this paper a bivariate YULE distribution as well as a bivariate analogue of the STER model are defined and a uniqueness property of the former is shown to stem from the latter (section 2). Further, the bivariate Yule distribution is shown to be determined uniquely by two conditions on its tail probabilities (section 3).

2. The bivariate STER model and the bivariate Yule distribution

A bivariate extension of the YULE distribution will be constructed from the bivariate generalized Waring distribution (BGWD($a; k, m; \varrho$)) defined by XEKALAKI (1984a) to be the distribution with p.f. $P(X = r, Y = l) = p_{r,l}$ given by

$$p_{r,l} = \frac{\varrho_{(k+m)}}{(a+\varrho)_{(k+m)}} \frac{a_{(r+l)} k_{(r)} m_{(l)}}{(a+k+m+\varrho)_{(r+l)}} \frac{1}{r!} \frac{1}{l!} \tag{2.1}$$

$$a, k, m, \varrho > 0; \quad r = 0, 1, 2, \ldots; \quad l = 0, 1, 2, \ldots$$

Definition 2.1. A random vector (X, Y) will be said to have the bivariate YULE distribution with parameter ϱ if its p.f. is given by (2.1) for $a = k = m = 1$, i.e. if

$$p_{r,l} = \frac{\varrho_{(2)} (r+l)!}{(\varrho+1)_{(r+l+2)}}, \quad r, l = 0, 1, 2, \ldots \tag{2.2}$$

$$\varrho > 0.$$

This distribution can be thought of as a natural two-dimensional extension of the YULE distribution as given by (1.1) as both of the marginal distributions are univariate YULE distributions with parameter ϱ. It is also worth noting that the probabilities $p_{r,l}$ are symmetrical in r and l ($p_{r,l} = p_{l,r}$, $r, l = 0, 1, 2, \ldots$) and constant on the line $r+l = n$, n constant ($p_{r,n-r} = p_{r',n-r'}$, $r \neq r'$, $r, r' = 0, 1, \ldots, n$). Furthermore, as it follows from a result given by XEKALAKI (1984b) it can be

considered to be the discrete analogue of a one-parameter bivariate extension of the Pareto distribution with probability density function

$$f(x, y) = (a + \varrho + 1)^\varrho \, \varrho_{(2)} \, (x + y + \varrho + 2)^{-(\varrho + 2)}, \quad x, y, \varrho > 0 \, .$$

Definition 2.2. Let $p_{r,l}$, $r, l = 0, 1, 2, \ldots,$ be the p.f. of a discrete random vector (X_1, X_2), i.e., $p_{r,l} = P(X_1 = r, X_2 = l)$, with $c^{-1} = \sum_{r \geq 1} \sum_{l \geq 1} \frac{rl}{r+l} \, P_{r,l} > 0$. Then the distribution defined by

$$P(Y_1 = r, Y_2 = l) \equiv q_{r,l} = c \sum_{x = r+1}^{\infty} \sum_{y = l+1}^{\infty} \frac{p_{x,y}}{x + y} \qquad r, l = 0, 1, 2, \ldots \tag{2.3}$$

will be called the bivariate STER distribution.

Another distribution that can also be constructed from (2.1) and that is going to be used in what follows is the bivariate Waring distribution.

Definition 2.3. A random vector (X, Y) will be said to have the bivariate Waring distribution with parameters a and ϱ (BWD$(a; \varrho)$) if its p.f. is given by (2.1) for $k = m = 1$, i.e. if

$$p_{r,l} = \frac{\varrho_{(2)}}{(a + \varrho)_{(2)}} \cdot \frac{a_{(r+l)}}{(a + \varrho + 2)_{(r+l)}}, \quad r, l = 0, 1, 2, \ldots \qquad a, \varrho > 0 \, . \tag{2.4}$$

Obviously, BWD $(1; \varrho) \sim$ BYD(ϱ).
The following theorem can now be shown.

Theorem 2.1. *Let (X_1, X_2), (Y_1, Y_2) be random vectors with nonnegative integer-valued components. Assume that the p.f.'s of (X_1, X_2) and (Y_1, Y_2) satisfy (2.3). Then the distribution of (Y_1, Y_2) is the BWD$(2; \varrho)$, $\varrho > 0$ defined by (2.4) for $a = 2$ if and only if the distribution of (X_1, X_2) is the BYD $(\varrho + 1)$, $\varrho > 0$.*

Proof.

Necessity. Assume that (2.3) is true and let $(X_1, X_2) \sim$ BYD $(\varrho + 1)$ i.e.

$$p_{x,y} = \frac{(\varrho + 1)_{(2)} \, (x + y)!}{(\varrho + 2)_{(x+y+2)}}, \quad x, y = 0, 1, 2, \ldots \tag{2.5}$$

We will show that $(Y_1, Y_2) \sim$ BWD$(2; \varrho)$.
Substituting for $p_{x,y}$ from (2.5), equation (2.3) becomes

$$\begin{aligned}
q_{r,l} &= c \, (\varrho + 1)_{(2)} \sum_{x = r+l}^{\infty} \sum_{y = l+1}^{\infty} \frac{(x + y - 1)!}{(\varrho + 2)_{(x+y+2)}} \\
&= c \, (\varrho + 1)_{(2)} \sum_{x,y} \frac{(x + y + r + l + 1)!}{(\varrho + 2)_{(x+y+r+l+4)}} \\
&= c \, (\varrho + 1)_{(2)} \frac{(r + l + 1)!}{(\varrho + 2)_{(r+l+4)}} \sum_{x,y} \frac{(r + l + 2)_{(x+y)}}{(\varrho + r + l + 6)_{(x+y)}} \\
&= \frac{c \, (\varrho + 1)_{(2)} \, 2_{(r+l)}}{(\varrho + 2)_{(r+l+4)}} \frac{(\varrho + r + l + 4)_{(2)}}{(\varrho + 2)_{(2)}} = \frac{c \, (\varrho + 1)_{(2)}}{(\varrho + 2)_{(2)}} = \frac{2_{(r+l)}}{(\varrho + 2)_{(r+l+2)}}
\end{aligned} \tag{2.6}$$

$r, l = 0, 1, 2, \ldots, \quad i = 1, 2 \, .$

Here $\sum_{x,y}$ represents the double summation $\sum_{x=0}^{\infty} \sum_{y=0}^{\infty}$. Summing both sides of (2.6) over r, l we get

$$c = \varrho \, (\varrho + 3) \, .$$

Then

$$q_{r,l} = \frac{\varrho_{(2)} \, 2_{(r+l)}}{(\varrho + 2)_{(r+l+2)}} \, , \quad r, l = 0, 1, 2, \ldots \tag{2.7}$$

But this is the p.f. of the $BWD(2; \varrho)$.

Sufficiency

Let (2.3) be true and let $(Y_1, Y_2) \sim BWD(2; \varrho)$. We will show that $(X_1, X_2) \sim$ $\sim BYD \, (\varrho + 1)$.

From (2.3) we obtain for $r, l = 0, 1, 2, \ldots$

$$q_{r,l} - q_{r+1,l} = c \sum_{x=l+1}^{\infty} \frac{p_{r+1,x}}{x + r + 1} \, , \tag{2.8}$$

$$q_{r,l} - q_{r,l+1} = c \sum_{x=r+1}^{\infty} \frac{p_{x,l+1}}{x + l + 1} \, , \tag{2.9}$$

$$q_{r,l} - q_{r+1,l+1} = c \left\{ \sum_{x=l+1}^{\infty} \frac{p_{r+1,x}}{x + r + 1} + \sum_{x=r+1}^{\infty} \frac{p_{x,l+1}}{x + l + 1} - \frac{p_{r+1,l+1}}{r + l + 2} \right\} \, . \tag{2.10}$$

Hence, from (2.8), (2.9) and (2.10), it follows that

$$q_{r+1,l+1} + q_{r,l} - q_{r+1,l} - q_{r,l+1} = c \, \frac{p_{r+1,l+1}}{r + l + 2} \, , \quad r, l = 0, 1, 2, \ldots \tag{2.11}$$

Substituting for $q_{r,l}$ from (2.7), equation (2.11) becomes

$$p_{r,l} = c^{-1} \, \varrho_{(2)} \, \frac{(r+l)!}{(\varrho + 4)_{(r+l)}} \, .$$

Summing over r, l we obtain $c = \varrho \, (\varrho + 3)$. Hence

$$p_{r,l} = \frac{(\varrho + 1)_{(2)} \, (r+l)!}{(\varrho + 2)_{(r+l+2)}} \, , \quad r, l = 0, 1, 2, \ldots$$

But this is the p.f. of the $BYD \, (\varrho + 1)$. Hence the theorem is established.

3. A uniqueness result based on tail probabilities

We now provide the following uniqueness theorem.

Theorem 3.1. *Let* (X_1, X_2) *be a random vector of nonnegative integer-valued components with p.f.* $p_{r,l}, r, l = 0, 1, 2, \ldots$ *Then*

$$\mathsf{P} \, (X_1 > r, \, X_2 = l) = \mathsf{P} \, (X_1 = r, \, X_2 > l) = a \, (r + l + 1) \, p_{r,l}$$
$$0 < a < 1, \quad r, l = 0, 1, 2, \ldots$$

if and only if $(X_1, X_2) \sim BYD \left(\dfrac{1}{a} - 1 \right)$.

The proof of this theorem can be obtained as a special case of a more general result given by the following theorem

Theorem 3.2. *Let* (X_1, X_2) *be a random vector with nonnegative, integer-valued components and let* $p_{r,l} = P(X_1 = r, X_2 = l)$, $r, l = 0, 1, 2, \ldots$ *Then*

$$P(X_1 > r, X_2 = l) = P(X_1 = r, X_2 > l) = (ar + al + b) p_{r,l} \qquad (3.1)$$

$$0 < a < 1, \quad b > 0, \quad r, l = 0, 1, 2, \ldots$$

if and only if $(X_1, X_2) \sim BWD\left(\dfrac{b}{a}; \dfrac{1}{a} - 1\right)$

Proof.

Necessity

Assume that $(X_1, X_2) \sim \mathrm{BWD}(\alpha; \varrho)$. We will show that equations (3.1) are satisfied.

Substituting for $p_{r,l}$ in $P(X_1 > r, X_2 = l) = \sum\limits_{x=r+1}^{\infty} p_{r,l}$ we have

$$P(X_1 > r, X_2 = l) = \frac{\varrho_{(2)}}{(\alpha + \varrho)_{(2)}} \sum_{x=r+1}^{\infty} \frac{\alpha_{(x+l)}}{(\alpha + \varrho + 2)_{(x+l)}}$$

$$= \frac{\varrho_{(2)}}{(\alpha + \varrho)_{(2)}} \frac{\alpha_{(r+l+1)}}{(\alpha + \varrho + 2)_{(r+l+1)}} \sum_{x=0}^{\infty} \frac{(\alpha + r + l + 1)_{(x)}}{(\alpha + \varrho + r + l + 3)_{(x)}}$$

$$= \frac{\varrho_{(2)}}{(\alpha + \varrho)_{(2)}} \frac{\alpha_{(r+l+1)}}{(\alpha + \varrho + 2)_{(r+l+1)}} \frac{\alpha + \varrho + r + l + 2}{\varrho + 1}$$

$$= \frac{\alpha + r + l}{\varrho + 1} p_{r,l}$$

$$r, l = 0, 1, 2, \ldots$$

In a similar way we prove that

$$P(X_1 = r, X_2 > l) = \frac{\alpha + r + l}{\varrho + 1} p_{r,l}, \quad r, l = 0, 1, 2, \ldots$$

Hence the two tail probabilities in (3.1) are equal and of the form $(ar + al + b) p_{r,l}$ with $a = \dfrac{1}{\varrho + 1}$, $b = \dfrac{\alpha}{\varrho + 1}$. Obviously $0 < a < 1$, $b > 0$. Therefore, necessity has been established.

Sufficiency

Assume that equations (3.1) are true. We will show that $(X_1, X_2) \sim \mathrm{BWD}$ $\left(\dfrac{b}{a}; \dfrac{1}{a} - 1\right)$.

One may observe that

$$P(X_1 > r, X_2 = l) - P(X_1 > r + 1, X_2 = l) = p_{r+1,l} \qquad (3.2)$$

$$P(X_1 = r, X_2 > l) - P(X_1 = r, X_2 > l + 1) = p_{r,l+1}, \quad r, l = 0, 1, 2, \ldots$$

Then, combining (3.2) with (3.1) we obtain the equations

$$p_{r+l,l} - \frac{r+l+\dfrac{b}{a}}{r+l+\dfrac{b+a+1}{a}}\, p_{r,l} = 0$$

$$p_{r,l+1} - \frac{r+l+\dfrac{b}{a}}{r+l+\dfrac{b+a+1}{a}}\, p_{r,l} = 0\,, \quad r,\, l = 0,\, 1,\, 2,\, \ldots$$

But these are uniquely satisfied by the p.f. of the $\mathrm{BWD}\left(\dfrac{b}{a};\ \dfrac{1}{a}-1\right)$ under the condition $\sum\limits_{r,l} p_{r,l} = 1$. This completes the proof of the theorem.

Obviously, the result of theorem 3.1 can be obtained from theorem 3.2 for $b = a$.

Note: The univariate analogue of the result in theorem 3.2 has been shown by XEKALAKI (1983b).

References

BISSINGER, B. H. (1965). A type resisting distribution generated from considerations of an inventory decision model. In *Classical and Contagious Discrete Distributions*, G. P. PATIL, ed. Pergamon Press and Statistical Publishing Society, Calcutta, pp. 15—17.

HAIGHT, F. A. (1966). Some statistical problems in connection with word association data. *J. Math. Psychol.* **3**, 217—233.

KENDALL, M. G. (1961). Natural law in the social sciences. *J. Roy. Statist. Soc. Ser.* **A, 124**, 1—16.

MITZUTANI, S. (1953). Vocabulary in women's magazines. *Kokken-Hokoku,* **4**, National Research Institute, Tokyo. (In Japanese).

SIMON, H. A. (1955). On a class of skew distribution functions. *Biometrika,* **42**, 425—440.

SIMON, H. A. (1960). Some further notes on a class of skew distribution functions. *Information and Control,* **3**, 80—88.

XEKALAKI, E. (1983a). A property of the Yule distribution and its applications. *Comm. Statist. Ser. A (Theory and Methods)* **12** (10), 1181—1189.

XEKALAKI, E. (1983b). "Hazard functions and life distributions in discrete time." *Comm. Statist., Ser. A (Theory and Methods).* **12** (21), 2503—2509.

XEKALAKI, E. (1983c). The bivariate Yule distribution and some of its properties. *Technical Report no* 92, Department of Statistics and Actuarial Science, University of Iowa, Iowa City.

XEKALAKI, E. (1984a). The bivariate generalized Waring distribution and its application to accident theory. *J. Roy. Statist. Soc.* **A 147**, 488—490.

XEKALAKI, E. (1984b). Models leading to the bivariate generalized Waring distribution. *Utilitas Mathematicae* **25**, 263—290.

XEKALAKI, E. (1984 c). Linear regression and the Yule distribution. *J. Econ.* **24**, 397–403.
YULE, G. W. (1924). "A mathematical theory of evolution based on the conclusions of
Dr. J. W. Willis, F. R. S." *Philos. Trans. Roy. Soc. London, Ser.* **B, 213**, 21–87.

EVDOKIA XEKALAKI
Department of Statistics and Informatics
Athens School of Economics and Business Science
76 Patision Street
10434 Athens, Greece