

## Letter to the Editor

### Comments on the paper of Shan et al.: The multivariate Waring distribution

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In a recent paper in *Scientometrics*, SHAN et al. (2004) claim that they present a multivariate extension of the Waring distribution. However, the parametric form considered by the authors as defining a multivariate extension of the Waring distribution is not new and neither are the results provided therein.

The paper is thus merely an application to scientific productivity data (the only novelty in the paper) of the shifted version of a special case of a distribution that has been known in the literature since 1984. In particular:

- (a) the two-dimensional case was defined by XEKALAKI (1984a), as the bivariate generalized Waring distribution with positive real parameters  $\alpha, k, m$  and  $\rho$  ( $BGWD(\alpha; k, m; \rho)$ ) and probability generating function

$$G(s, t) = \frac{\rho(\alpha)}{(\rho + k + m)(\alpha)} F_1(\alpha; k, m; \alpha + k + m + \rho; s, t) \quad (1)$$

and

- (b) in the  $n$ -dimensional case it was defined by XEKALAKI (1986a), as the multivariate generalized Waring distribution with positive real parameters  $\alpha, k_1, \dots, k_n$  and  $\rho$  ( $MGWD(\alpha; k_1, k_2, \dots, k_n; \rho)$ ) and probability generating function

$$G(s_1, s_2, \dots, s_n) = \frac{\rho(\alpha)}{(\rho + \sum_{i=1}^n k_i)(\alpha)} F_D(\alpha; k_1, \dots, k_n; \alpha + \sum_{i=1}^n k_i + \rho; s_1, \dots, s_n). \quad (2)$$

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Here,  $F_1$  and  $F_D$  are Appell's and Lauricella's generalized hypergeometric functions, respectively, defined by

$$F_1(\alpha; \beta, \gamma; \delta; z, u) = \sum_{r=0}^{\infty} \sum_{l=0}^{\infty} \frac{\alpha_{(r+l)} \beta_{(r)} \gamma_{(l)} z^r u^l}{\delta_{(r+l)} r! l!}$$

and

$$F_D(\alpha; \beta_1, \dots, \beta_n; \delta; z_1, \dots, z_n) = \sum_{r_1=0}^{\infty} \dots \sum_{r_n=0}^{\infty} \frac{\alpha_{(\sum_{i=1}^n r_i)} \prod_{i=1}^n \beta_{(r_i)}}{\delta_{(\sum_{i=1}^n r_i)} \prod_{i=1}^n r_i!}$$

with  $\alpha_{(\beta)}$  denoting the ascending factorial defined as the ratio

$$\Gamma(\alpha + \beta) / \Gamma(\alpha), \quad \alpha > 0, \quad \beta \in \mathbb{R}.$$

(The interested reader can find a complete list of related publications of the present author on this distribution and its special cases in the references below).

From the above, and the other results contained in the related papers, one can easily see that what SHAN et al. (2004) define as the multivariate Waring distribution is not new. It is precisely, in the above notation, the MGWD  $(1; b_1, b_2, \dots, b_n; a)$  of XEKALAKI (1986a), shifted one unit in the direction of each  $x_i$ ,  $i=1, 2, \dots, n$ . This can be readily seen to hold by noting that shifting the  $MGWD(\alpha; k_1, k_2, \dots, k_n; \rho)$  one unit in the direction of each of the  $x_i$ 's yields a distribution with probability generating function

$$g(s_1, \dots, s_n) = \frac{\rho_{(\alpha)}}{(\rho + \sum_{i=1}^n k_i)_{(\alpha)}} \left\{ \prod_{i=1}^n s_i \right\} F_D(\alpha; k_1, \dots, k_n; \alpha + \sum_{i=1}^n k_i + \rho; s_1, \dots, s_n)$$

and probability function

$$P(X_1 = x_1, \dots, X_n = x_n) = \frac{\rho_{(\alpha)}}{(\rho + \sum_{i=1}^n k_i)_{(\alpha)}} \frac{\alpha_{(\sum_{i=1}^n x_i - n)}}{(\alpha + \sum_{i=1}^n k_i + \rho)_{(\sum_{i=1}^n x_i - n)}} \prod_{i=1}^n \frac{(k_i)_{(x_i-1)}}{(x_i-1)!}$$

$$x_i = 1, 2, \dots; \quad i = 1, 2, \dots, n. \quad (3)$$

For  $\rho = \alpha$ ,  $k_i = b_i$  and  $\alpha = 1$ , this is exactly the probability function in relationship (2) of the paper of SHAN et al. (2004), through which they define what they

term the multivariate Waring distribution. For a proof of this, observe that the probability function in relationship (2) of the paper can be written as

$$\begin{aligned}
 P(X_1 = x_1, \dots, X_n = x_n; \alpha, b_1, \dots, b_n) &= \alpha \frac{\Gamma(\sum_{i=1}^n x_i - n + 1) \Gamma(\sum_{i=1}^n b_i + \alpha)}{\Gamma(\sum_{i=1}^n x_i + \sum_{i=1}^n b_i - n + \alpha + 1)} \prod_{i=1}^n \frac{\Gamma(x_i + b_i - 1)}{\Gamma(x_i) \Gamma(b_i)} \\
 &= \alpha \frac{1}{(\alpha + \sum_{i=1}^n b_i)} \frac{\prod_{i=1}^n (b_i)_{(x_i-1)}}{(\sum_{i=1}^n x_i - n)} \\
 &= \frac{\alpha}{\alpha + \sum_{i=1}^n b_i} \frac{1}{(\alpha + \sum_{i=1}^n b_i + 1)} \frac{\prod_{i=1}^n (b_i)_{(x_i-1)}}{(\sum_{i=1}^n x_i - n)},
 \end{aligned}$$

which indeed coincides with (3) given above, for  $\alpha = 1$ ,  $b_i = k_i$ .

All the results SHAN et al. (2004) prove (properties, theorems), as well as all the results that they say they intend to present in a subsequent paper, have already been derived, or follow directly, as immediate consequences of results obtained in relevant papers already published (see, e.g. XEKALAKI (1984a, b, 1985c, 1986a)).

Although SHAN et al. (2004) thank a referee for pointing out "a book on multivariate distributions" (presumably JOHNSON et al. (1997) cited in the paper), it seems that they did not read carefully the appropriate section. They erroneously refer to the distribution in relations (7) and (8) of their paper as the generalized Waring distribution (their relations (7) and (8) are shifted versions of it). They also erroneously cite Irwin's work on the Waring distribution (Remark 1, Relation (8)). In fact, the distribution in (8) is not what Irwin called a generalized Waring distribution and was not defined in 1963. It is a shifted version of IRWIN's (1975) generalized Waring distribution leading to his (simple) Waring distribution (IRWIN, 1963) as a special case.

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