BUYING AND STOCKING UNDER UNCERTAINTY

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ABSTRACT

An invetory decision model is considered whereby the demand for the item, the stock in hand and the lead time period are considered to be random variables. The interrelationships of these three item characteristics are then studied in the framework of a scheme for deciding when to place an order for additional material. The effect of a Yule demand distribution is studied and the implications of such an assumption on the distribution of the mean lead time are examined.

Keywords and Phrases: Yule distribution; Pareto Distribution; Inventory model; reorder point system.

1. INTRODUCTION

Inventory managers are usually faced with the question as to how large a quantity of the demanded material to order and most important as to when to place the order so that the warehouse does not run out stock. Their buing and stocking policy very much depends on the fluctuations of certain characteristics of the item to be ordered such as demand, stock level, lead time periods etc. Very often the time to replenish the stock is specified by the stock level: An order is placed when invetory reaches a specified position. This scheme is known as the **reorder point system**. Prichard and Eagle (1965) established a decision rule by means of a function associated with the fraction of the lead time for which the item is out of stock.

Let X be a non-negative integer valued random variable representing the demand for an item in units ordered and let λ be a fixed constant representing the lead time period. Assume that the fraction of λ out of stock is represented by the fluctuations of a random variable T. Then according to Prichard and Eagle's decisions rule an order for a quantity is placed when stock reachs a level y for which E(T), the expected fraction of lead time out of stock, does not exceed a given length λ_o.

By inspecting figure 1 one can easily deduce the similarity of the triangles $\overset{\Delta}{\lambda o x}$ and $\overset{\Delta}{R\, X\, y}$ and hence conclude that

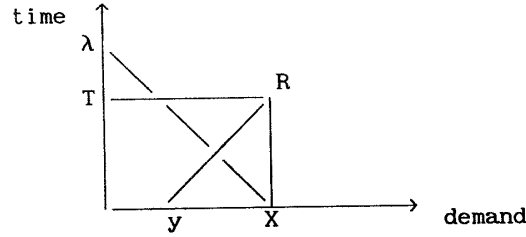$$\frac{X-y}{T} = \frac{X}{\lambda} \qquad\qquad (1.1)$$

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figure 1.

which implies that

$$E(T) = \lambda \sum_{x=y+1}^{\infty} \frac{x-y}{x} P(X=x) \qquad (1.2)$$

Hence, an inventory manager who wishes to ensure that the item will not be out of stock longer than a specified length of time λ_o in each lead time will have to choose the reorder point y so that

$$\lambda \sum_{x=y+1}^{\infty} \frac{x-y}{x} P(X=x) \le \lambda_o \qquad (1.3)$$

or, equivalently so that

$$P(X>y) - y \sum_{x=y+1}^{\infty} \frac{P(X=x)}{x} \le \lambda_o / \lambda \qquad (1.4)$$

Note that in this set up only the demand for an item has been considered random. All the other item characteristics i.e. the lead time and the stock level have been assumed fixed. Indeed this is the case with most inventory problems of this nature.

In this paper a more realistic view is taken. The on hand inventory is assumed to be a random variable instead of continually reviewed constant. Further the assumption is made that the lead time is of a random length. The effects of these

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assumptions on the interrelationship of the item characteristics is then examined in section 2. The subsequent two sections study such inventory situations on the assumption of a Yule demand distribution (section 2) and the reorder point decision rule is looked upon in terms of the distribution of demand (section 3).

2. THE RANDOM FACTOR

Consider an inventory model in which the demand for an item during a lead time is a non-negative interger valued random variable X with probability function $P(X=r) = p_r$, $r=0,1,2,\ldots$. Let Y be another non-negative, integer-valued random variable representing the amount (in item units) of the inventory on hand during the same lead time which is assumed to be a random variable itself denoted by L and distributed independently of X and Y. Further, let q_r, $r=0,1,2,\ldots$ represent the probability function of Y, i.e.

$P(Y=r) = q_r$, $r=0,1,2,\ldots$ and denote by $F_L(l)$, $l>0$ the distribution function of L. If now T represents the fraction of L during which the item will be out of stock, an argument similar to that used to derive (1.1) leads to

$$\frac{X}{L} = \frac{X-Y}{T} \tag{2.1}$$

This implies that

$$E(T|Y=y)=E(L)\left\{ P(X>y) - y\sum_{x=y+1}^{\infty} \frac{P(X=x)}{x} \right\} \tag{2.2}$$

$$y=0,1,2,\ldots$$

Obviously (2.2) is meaningful only in cases of shortage when

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Y<X. (If Y≥X then T=0 as a logical consequence). In such cases

since P(Y<X)=1 we have that

$$q_r \equiv P(Y=r) = P(Y=r \mid Y<X)$$

Therefore

$$q_r = \frac{\displaystyle\sum_{x=r+1}^{\infty} P(Y=r \mid X=x)\, p_x}{\displaystyle\sum_{r=0}^{\infty} \sum_{x=r+1}^{\infty} P(Y=r \mid X=x)\, p_x}$$

But

$$\sum_{r=0}^{\infty} \sum_{x=r+1}^{\infty} P(Y=r \mid X=x)\, p_x = \sum_{x=1}^{\infty} p_x \sum_{r=0}^{x-1} P(Y=r \mid X=x)$$

$$= \sum_{x=1}^{\infty} p_x$$

$$1-p_0$$

Therefore

$$q_r = \frac{1}{1-p_0} \sum_{x=r+1}^{\infty} P(Y=r \mid X=x)\, p_x \qquad\qquad (2.3)$$

$$r=0,1,2,\ldots$$

It would therefore be interesting to specify the form of the conditional distribution of Y given that (X=x) so as one would be able to express the relationship between X and Y during a lead time L more explicitly. The theorem that follows provides a result which is of central importance in this direction.

Theorem 2.1. Let L be a positive random variable with distribution function $F_L(l)$, $l>0$. Let T be another positive

random variable whose conditional distribution given that $(L=l)$ is given by

$$F_{T|(L=l)}(t) = \frac{t}{l}, \quad t \in (0,l) \tag{2.4}$$

Then the r.v. $Z = \dfrac{T}{L}$ is uniformly distributed on the interval $(0,1)$.

Proof:

Let $F_Z(z)$ denote the d.f. of Z. Then

$$F_Z(z) = P(Z \leq z) = P\left[\frac{T}{L} \leq z\right]$$

$$= P(T \leq zL) = \int_0^\infty P(T \leq zl \,|\, L=l) \, d\, F_L(l)$$

$$= \int_0^\infty F_{T|(L=l)}(zl) \, d\, F_L(l).$$

Then using (2.4) we obtain

$$F_Z(z) = z \int_0^\infty d\, F_L(l) = z.$$

But this is the distribution function of the uniform distribution on an interval of unit length, i.e. on an interval of the form $(c, c+1)$. By the definition of Z it follows that $c=0$. Hence the theorem has been established.

Put in the context of the invetory model considered the result of this theorem implies that if the fraction T of a lead time L during which the item will be out of stock for a given length l of L is uniformly distributed in $(0,l)$ then T/L the proportion of time out stock, is uniformly distributed in $(0,1)$. This, taking into account the fact that a uniform distribution for $T|(L=l)$ is a natural assumption, implies in turn that, wherever

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X>0, (X-Y)/X is a uniform random variable defined on (0,1). As a
result Y/X is also uniformly distributed in (0,1). Since Y is an
integer valued random variable this implies that

$$Y = [RX] \quad \text{whenever } X>0 \qquad (2.5)$$

where R is a uniformly distributed in (0,1) independently of X.
Here $[\alpha]$ denotes the integral part of α. Hence

$$
\begin{aligned}
P(Y=r|X=x) &= P([RX]=r|X=x) \\
&= P(r \leq RX < r+1 \,|\, X=x) \\
&= P\left[\frac{r}{x} \leq R < \frac{r+1}{x} \,\bigg|\, X=x \right] \\
&= P\left[\frac{r}{x} \leq R < \frac{r+1}{x} \right] \\
&= \frac{1}{x} \; .
\end{aligned}
$$

i.e. under (2.5)

$$P(Y=r)|X=x) = \frac{1}{x} \, , \qquad r=0,1,2,\ldots,x-1$$

Hence (2.3) reduces to

$$q_r = \frac{1}{1-p_o} \sum_{x=r+1}^{\infty} \frac{P_x}{x} \qquad r=0,1,\ldots \qquad (2.6)$$

Therefore the second term in the right hand side of (2.2)
represents the fluctuations of the stock in hand. In the light of
this result (2.2) leads to a decision rule which amounts to
selecting the reorder point to be the value y_o of Y for which

$$P(X>y_o)-y_o(1-p_o) \, P(Y=y_o) \leq c \qquad (2.7)$$

where $c = \lambda_o/E(L)$, and λ_o is an administratively set constant.

The algorithm for the determination of y_o will obviously
depend on the form of the distribution of X and Y. Xekalaki
(1983, 1984) obtained some results connecting the distributions of
X and Y.

Theorem 2.2. (Xekalaki, 1983). Let X, Y be non-negative integer valued r.v's such that $P(Y=r|X=x) = \frac{1}{x}$, $r=0,1,\ldots,x-1$. Then X and Y are indentically distributed if and only if X has a Yule distribution with probability function given by

$$P(X=x) = \frac{\rho x!}{(\rho+1)(\rho+2)\ldots(\rho+x+1)}, \quad \begin{array}{l} x=0,1,\ldots \\ \rho>0 \end{array} \qquad (2.8)$$

Theorem 2.3. (Xekalaki, 1984). Let X be a non-negative, integer-valued random variable. Then X is Yule distributed with probability function given by (2.8) if and only if

$$P(X>r) = \frac{1}{\rho} (r+1) P(X=r), \qquad r=0,1,2,\ldots \qquad (2.9)$$

Consider now an inventory situation where the Yule distribution with probability function as given by (2.7) may be appropriate for modeling the demand fluctuations and shortage can be regarded to be effected through (2.5). Then theorems 2.2 and 2.3 imply that the decision rule will select the reorder point y_o so that

$$\frac{(y_o+\rho+1)}{\rho(\rho+1)} P_{y_o} \leq c$$

or equivalently so that

$$\frac{\Gamma(y_o+\rho+1)}{\Gamma(y_o+1)} \geq c_o \qquad (2.10)$$

where $c_o = \Gamma(\rho+1)/(c(\rho+1))$.

3. DECIDING IN TERMS OF THE LEAD TIME DISTRIBUTION

In this section the problem of determining when to order will be looked at in terms of the probability distribution of the lead time. Xekalaki (1983) showed that the fluctuations of demand can be desrcibed by the Yule distribution with probability function as given by (2.8). The derivation was based upon the following hypothesis.

Let Z, the number of orders arriving at a warehouse, be Poisson distributed with parameter ϑ characteristic of the buyer's behaviour and let $X_1, X_2, \ldots$ the numbers of item units ordered by the various customers, be independent and identically distributed indepently of Z according to a logarithmic distribution with probability function

$$P(X_1 = r) = \frac{1}{\vartheta} \frac{\left(1 - e^{-\vartheta}\right)^r}{r}, \qquad r = 1, 2, \ldots .$$

Then $X = X_1 + X_2 + \ldots + X_Z$, the demand for the item in terms of the total number of item units ordered follows for a given buyer (fixed ϑ) a distribution with probability generating function.

$$G_{X|\vartheta}(s) = \exp\left\{ \vartheta \left[\frac{\ln\left[1 - \left(1 - e^{-\vartheta}\right)s\right]}{-\vartheta} - 1 \right] \right\}$$

$$= \left[e^{\vartheta} - (e^{\vartheta} - 1)s \right]^{-1} .$$

In other words for a given buyer whose behaviour is reflected by the parameter ϑ the distribution of demand is the geometric

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distribution with parameter $e^{-\vartheta}$. If now differences in the buying behaviour from buyer to buyer are effected through an exponential distribution for ϑ with parameter $\rho > 0$ the resulting distribution of X has probability generating function.

$$G_X(s) = \rho \int_0^\infty G_\vartheta(s) \ e^{-\rho\vartheta} \ d\vartheta$$

$$= \rho \sum_{r=0}^\infty s^r \int_0^1 \vartheta^\rho \ (1-\vartheta)^r d\vartheta$$

$$= \rho \sum_{r=0}^\infty \frac{r!}{(\rho+1)_{(r+1)}} s^r$$

i.e. the distribution of demand is the Yule with parameter ρ.

In the sequel, an alternative model giving rise to a Yule demand distributon is suggested. Assume that during a lead time of given length t orders for an item occur according to a homogeneous Poisson process $\{N(t), \ t>0\}$ with parameter $\lambda=1$. Then for the given length of time the distribution of orders has probability generating function

$$G_{N(t)}(s) = e^{t(s-1)} \tag{3.1}$$

Assume that t follows a distribution that is a scale mixture of the exponential distribution i.e. a distribution defined by

$$dF_L(t) = \left\{ \int_0^\infty \frac{1}{\alpha} e^{-t/\alpha} \ dF(\alpha) \right\} dt, \quad \alpha>0 \tag{3.2}$$

where $F(\alpha)$, $\alpha>0$ is a proper distribution function. Then the distribution of the demand X for the item will coincide with the distribution of N(L). Therefore, the probability generating function $G_X(s)$ of X will be given by

$$G_X(s) = G_{N(L)}(s) = E_L\left[G_{N(t)}(s)\right]$$

$$= \int_0^\infty G_{N(t)}(s) \; d \; F_L(t)$$

$$= \int_0^\infty \int_0^\infty G_{N(t)}(s) \; \frac{1}{\alpha} \; e^{-\frac{1}{\alpha} t} \; d \; F(\alpha) \; dt$$

i.e.

$$G_X(s) = \int_0^\infty \int_0^\infty \frac{1}{\alpha} e^{-t[1+\alpha(1-s)]/\alpha} \; dF(\alpha) \; dt \qquad (3.3)$$

If now α follows a Pareto distribution with parameter ρ (Pearson type VI) i.e. if

$$dF(\alpha) = \rho \; (1+\alpha)^{-(\rho+1)} \; d\alpha \qquad (3.4)$$

we obtain form (3.3)

$$G_X(s) = \rho \int_0^\infty \int_0^\infty \frac{1}{\alpha} e^{-t[1+\alpha(1-s)]/\alpha} (1+\alpha)^{-(\rho+1)} \; d\alpha \; dt$$

$$= \rho \int_0^\infty \left[\frac{1}{\alpha} \int_0^\infty e^{-t[1+\alpha(1-s)]/\alpha} \; dt \right] (1+\alpha)^{-(\rho+1)} d\alpha$$

$$= \rho \int_0^\infty [1+\alpha(1-s)]^{-1} (1+\alpha)^{-(\rho+1)} \; d\alpha$$

$$= \rho \sum_{r=0}^\infty s^r \int_0^\infty \alpha^r (1+\alpha)^{-(\rho+r+2)} \; d\alpha$$

$$= \rho \sum_{r=0}^\infty s^r \; \frac{\Gamma(r+1)\Gamma(\rho+1)}{\Gamma(\rho+r+2)}$$

$$= \rho \sum_{r=0}^\infty \frac{r!}{(\rho+1)(\rho+2)\ldots(\rho+r+1)} \; s^r$$

But this is the probability generating function of the Yule distribution as defined by (2.8). Hence if the distribution of lead time L is a Pareto mixture of the exponential distribution the distribution of demand X is the Yule distribution. The converse is also true i.e. if X is Yule distributed then the distribution of L is a Pareto scale mixture of the exponential distribution. This follows from a more general result shown by

Xekalaki and Panaretos (1987). In fact, this result goes even further as it leads to a one-to-one correspondence between the mixing distribution F(α) in (3.2) and the distribution of the demand X whenever X can be regarded as the image N(L) of the lead time L through a homogeneous Poisson process {N(t), t>0} with parameter λ=1 as indicated by the theorem that follows.

Theorem 3.1. (Xekalaki and Panaretos, 1987). Let {N(t), t>0} be a homogeneous Poisson process with parameter λ=1. Let Z_1, Z_2 be two independent non-negative random variables that are distributed independently of {N(t), t>0} with distribution functions satisfying

$$h_{Z_i}(z) = \int_0^\infty \frac{\alpha_i^\beta}{\Gamma(\beta)} z^{\beta-1} e^{\alpha_i z} dF_i(\alpha_i) \qquad (3.5)$$

$$z, \beta > 0, \quad i=1,2$$

Then $F_1 = F_2$ if and only if $N(Z_1)$ and $N(Z_2)$ are identically distributed.

The implication of this result in the context of the model just considered will become obvious if one observes that the distribution F(α), α>0 in (3.2) represents the distribution of the mean of an exponential lead time. Indeed we have from (3.2) that conditional on α the mean lead time is given by

$$E(L|\alpha=a) = \int_0^\infty t \frac{1}{a} e^{-t/a} dt = a$$

Hence the result of theorem 3.2, brought within the framework of the model considered in this section, leads to the following conclusion.

The distribution of demand X is the Yule distribution with parameter ρ if and only if the distribution of the lead time L is exponential with a mean $E(L|a)$ that has a Pareto distribution with the same parameter ρ.

Theorem 3.3. (Xekalaki and Panaretos, 1987). Let X be a random variable having the Yule distribution with paremeter ρ as defined by (2.8). Let U be another random variable distributed according to the Pareto distribution with parameter ρ as defined by (3.4) and consider $U_1, U_2, ..$ to be a sequence of mutually indepent Pareto (1) random variables independent of U. Then

$$P(X \geq k) = P(U > U_1 + U_2 + ... + U_k) \qquad (3.6)$$

$$k = 1, 2, ...$$

Theorem 3.4. (Xekalaki and Panaretos, 1987). Let $U, U_1, U_2, ...$ be independent positive random variables such that $U_1, U_2, ...$ have the Pareto distribution with parameter 1. Then U has the Pareto distribution with parameter $\rho > 0$ if and only if either of the following conditions is satisfied

$$(i) \quad P\left[U > U_1 + ... + U_k + U_{k+1} \mid U > U_1 + ... + U_k\right] = \frac{k+1}{k+\rho+1}, \qquad (3.7)$$

$$k = 0, 1, 2,$$

$$(ii) \quad \sum_{r=k+1}^{\infty} \frac{P(U_1 + ... + U_r < U < U_1 + ... + U_{r+1})}{r} = \frac{1}{\rho+1} \quad P(U_1 + ... + U_k < U < U_1 + ... + U_{k+1})$$

$$(3.8)$$

From the definition of the random variable X and the derivation of its distribution it is obvious that the random variable $E(L|\alpha)$ plays the role of the r.v. U in (3.6). So, if we

let

$$U = E(L|\alpha)$$

and define

$$U_i = E(L_i|\alpha), \quad i=1,2,\ldots$$

where L_i is a continous random variable representing the lead time for the i-th ordered unit (if a unit is ordered at the time the previous ordered unit is delivered) then (3.6) states that the events {at least k item units are ordered} and {the mean lead time exceeds the aggregate of the mean lead times for the k units if ordered individually-each at the time of delivery of the previous item} are equiprobable. Then the decision rule for determining the stock position at which an order should be placed can be expressed in terms of the distribution of the mean lead time.

From theorems 2.2, 3.3 and 3.4 if follows that if X,Y and U represent the demand for an item, the stock on hand and the mean lead time respectively then

$$P(X=r) = P(Y=r) = P\left[U_1+\ldots+U_r \le U < U_1+\ldots+U_{r+1}\right]$$

Therefore the inequality in (2.7) becomes

$$P\left[U > U_1+\ldots+U_{y_o+1}\right] - y_o \frac{1}{\rho+1} P\left[U_1+\ldots+U_{y_o} \le U < U_1+\ldots+U_{y_o+1}\right] \le c$$

or, equivalently

$$\frac{\rho+y_o+1}{\rho+1} P\left[U > U_1+\ldots+U_{y_o+1}\right] - \frac{y_o}{\rho+1} P\left[U > U_1+\ldots+U_{y_o}\right] \le c$$

i.e.

$$P\left[U > U_1+\ldots+U_{y_o}\right] \left\{(\rho+y_o+1) P\left[U > U_1+\ldots+U_{y_o+1} \mid U > U_1+\ldots+U_{y_o}\right]-y_o\right\} \le c(\rho+1)$$

By condition (3.7) of 3.4 this inequality implies that

$$P\left[U > U_1 + \ldots + U_{y_o}\right] \leq c(\rho+1) \qquad (3.9)$$

The equivalence of (3.9) to (2.10) can be easily checked using theorems 2.3 and 3.3..

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