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# A METHOD FOR OBTAINING THE PROBABILITY DISTRIBUTION OF $m$ COMPONENTS CONDITIONAL ON $l$ COMPONENTS OF A RANDOM VECTOR

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## 1. THE RESULT

Let  $\mathbf{X} = (X_1, X_2, \dots, X_s)$  be a random vector of nonnegative and integer valued components and let  $p_x$  and  $G_{\mathbf{X}}(\mathbf{u}) \equiv G_{X_1, X_2, \dots, X_s}(u_1, u_2, \dots, u_s)$  denote its probability function (p.f.) and its probability generating function (p.g.f.), respectively, i.e.,  $p_x = P(X_1 = x_1, X_2 = x_2, \dots, X_s = x_s)$  and  $G_{\mathbf{X}}(\mathbf{u}) = \sum_{x_1, \dots, x_s} p_x \prod_{i=1}^{x_i} u_i^{x_i}$ . Here  $\sum_{x_1, \dots, x_s}$  denotes the multiple summation  $\sum_{k_1}^{\infty} \sum_{k_2}^{\infty} \dots \sum_{k_s}^{\infty}$  where  $k_i \in I^+ \cup \{+\infty\}$ ,  $i = 1, 2, \dots, s$ . Obtaining the p.f. of the conditional distribution of a random vector consisting of  $m$ ,  $m < s$  components of  $\mathbf{X}$  given some of the remaining  $s - m$  components from  $p_x$  can sometimes be tedious, especially when  $p_x$  is of a complicated form. However, utilizing  $G_{\mathbf{X}}(\mathbf{u})$  the task of obtaining the conditional p.g.f. of the particular vector can become easier. Subrahmaniam (1966) showed that, for  $s = 2$ , the p.g.f. of  $X_1$  conditional on  $X_2$  can be obtained by

$$G_{X_1|X_2}(u) = \frac{\partial^{x_2}}{\partial u^{x_2}} G_{X_1 X_2}(u, 0) \div \frac{\partial^{x_2}}{\partial u^{x_2}} G_{X_1 X_2}(1, 0).$$

Steyn and Roux (1970), based on a restatement of the theorem of compound probability distributions, gave the following formula

$$G_x(\mathbf{u}) = \sum_{x_{m+1}, \dots, x_s} p_{x_{m+1}, \dots, x_s} u_{m+1}^{x_{m+1}} \dots u_s^{x_s} G_{X_1 \dots X_m | X_{m+1} \dots X_s}(u_1, \dots, u_m)$$

from which the p.g.f. of the conditional distribution follows if the joint p.g.f.  $G_x(\mathbf{u})$  and the marginal p.f.  $p_{x_{m+1}, \dots, x_s}$  are known.

We now prove the following theorem which provides a multivariate version of Subrahmaniam's result.

**THEOREM.** Let  $X_1, \dots, X_s$  be discrete r.v.'s with joint p.g.f.  $G_x(\mathbf{u})$ . Then,

$$(1.1) \quad \begin{aligned} & G_{X_1 \dots X_m | X_{m+1} \dots X_s}(u_1, \dots, u_m) = \\ & \frac{\partial^x}{\partial u_{m+1}^{x_{m+1}} \dots \partial u_s^{x_s}} G_x(u_1, \dots, u_m, 0, \dots, 0) \div \\ & \div \frac{\partial^x}{\partial u_{m+1}^{x_{m+1}} \dots \partial u_s^{x_s}} G_x(1, \dots, 1, \underset{m}{0}, \underset{s-m}{0}, \dots, 0) \end{aligned}$$

where  $x = x_{m+1} + x_{m+2} + \dots + x_s$ .

*Proof.* We have

$$G_{x_1, \dots, x_m; u_{m+1}, \dots, u_s}(u_1, \dots, u_m) = \sum_{x_1, \dots, x_m} p_{x_1, \dots, x_m; x_{m+1}, \dots, x_s} u_1^{x_1} u_2^{x_2} \dots u_m^{x_m}$$

(1.2)

$$\sum_{x_1, \dots, x_m} \frac{p_{x_1, \dots, x_s}}{p_{x_{m+1}, \dots, x_s}} u_1^{x_1} u_2^{x_2} \dots u_m^{x_m} = \frac{1}{p_{x_{m+1}, \dots, x_s}} \sum_{x_1, \dots, x_m} p_{x_1, \dots, x_s} u_1^{x_1} u_2^{x_2} \dots u_m^{x_m}.$$

We will show that

$$\sum_{x_1, \dots, x_m} p_{x_1, \dots, x_s} u_1^{x_1} \dots u_m^{x_m} = \frac{1}{\prod_{i \geq m+1} (x_i)!} \frac{\partial^x}{\partial u_{m+1}^{x_{m+1}} \dots \partial u_s^{x_s}} G_x(u_1, \dots, u_m, 0, \dots, 0),$$

(1.3)

where  $x = x_{m+1} + \dots + x_s$ . We have

$$\begin{aligned} & \frac{\partial^x}{\partial u_s^{x_s} \dots \partial u_{m+1}^{x_{m+1}}} G_x(u_1, \dots, u_m, 0, \dots, 0) = \\ &= \frac{\partial^{x_s}}{\partial u_s^{x_s}} \left[ \frac{\partial^{x_{s-1}}}{\partial u_{s-1}^{x_{s-1}}} \left[ \dots \left[ \frac{\partial^{x_{m+1}}}{\partial u_{m+1}^{x_{m+1}}} G_x(u_1, \dots, u_s) \right] \dots \right] \right]_{u_{m+1} = \dots = u_s = 0}. \end{aligned}$$

But

$$\begin{aligned} &= \frac{\partial^{x_{m+1}}}{\partial u_{m+1}^{x_{m+1}}} G_x(u_1, \dots, u_m, 0, u_{m+2}, \dots, u_s) = \\ &= \sum_{x_1, \dots, x_m, x_{m+2}, \dots, x_s} p_{x_1, \dots, x_s} u_1^{x_1} \dots u_m^{x_m} (x_{m+1})! u_{m+1}^{x_{m+1}} \dots u_s^{x_s}. \end{aligned}$$

So

$$\begin{aligned} & \frac{\partial^{x_{m+2} + x_{m+1}}}{\partial u_{m+2}^{x_{m+2}} \dots \partial u_{m+1}^{x_{m+1}}} G_x(u_1, \dots, u_m, 0, 0, u_{m+3}, \dots, u_s) = \\ &= \sum_{x_1, \dots, x_m, x_{m+3}, \dots, x_s} p_{x_1, \dots, x_s} u_1^{x_1} \dots u_m^{x_m} (x_{m+1})! (x_{m+2})! u_{m+3}^{x_{m+3}} \dots u_s^{x_s}. \end{aligned}$$

By induction we obtain (1.3). We notice that

$$\begin{aligned} & \frac{\partial^x}{\partial u_{m+1}^{x_{m+1}} \dots \partial u_s^{x_s}} G_x(1, \dots, 1, 0, \dots, 0) = \\ &= \prod_{i \geq m+1} x_i! \sum_{x_1, \dots, x_m} p_{x_1, \dots, x_s} = \prod_{i \geq m+1} x_i! p_{x_{m+1}, \dots, x_s}. \end{aligned}$$

So that

$$(1.4) \quad p_{x_{m+1}, \dots, x_s} = \frac{1}{\prod_{i \geq m+1} x_i!} \frac{\partial^x}{\partial u_{m+1}^{x_{m+1}} \dots \partial u_s^{x_s}} G_x(1, \dots, 1, 0, \dots, 0).$$

Using (1.2), (1.3) and (1.4) we obtain the result.

*Note.* Obvious modifications to the proof of the theorem lead to a more general formula concerning the p.g.f. of  $(X_{i_1}, X_{i_2}, \dots, X_{i_m})$  conditional on  $(X_{i_{m+1}}, X_{i_{m+2}}, \dots, X_{i_l})$ , where  $(i_1, i_2, \dots, i_m)$  and  $(i_{m+1}, i_{m+2}, \dots, i_l)$  are permutations of the subscripts  $(1, 2, \dots, s)$  taking  $m$  and  $l$  at a time respectively ( $m, l \geq 1; m + l \leq s$ ).

## 2. SOME EXAMPLES

As an illustration of the result shown in the previous section we consider obtaining the p.g.f. of the conditional distribution of  $(X_1, X_2, \dots, X_m) | (X_{m+1}, X_{m+2}, \dots, X_s)$ ,  $m < s$  in two cases.

*Example 1.* Let  $(X_1, X_2, \dots, X_s)$  have an  $s$ -variate binomial distribution with p.g.f.  $G_X(\mathbf{u}) = (p_0 + p_1 u_1 + p_2 u_2 + \dots + p_s u_s)^n$ ,  $n \in I^+$ ,  $p_i \geq 0$ ,  $i = 0, 1, 2, \dots, s$ ;  $\sum_{i=1}^s p_i = 1 - p_0$ . Then

$$\begin{aligned} G_{X_1, \dots, X_m | X_{m+1}, \dots, X_s}(u_1, \dots, u_m) &= \\ &= n^{(x)} \prod_{i=m+1}^s p_i^{x_i} (p_0 + p_1 u_1 + \dots + (p_m u_m)^{n-x} \div n^{(x)} \prod_{i=m+1}^s p_i^{x_i} \left(1 - \sum_{i=m+1}^s p_i\right)^{n-x}) = \\ &= (p'_0 + p'_1 u_1 + \dots + p'_m u_m)^{n-x}, \end{aligned}$$

where  $x$  is defined as in the theorem,  $n^{(x)} = \prod_{i=1}^s (n - i + 1)$ , and  $p'_i = p_i / \left(1 - \sum_{i=m+1}^s p_i\right)$ ,  $i = 0, 1, 2, \dots, s$ .

*Example 2.* Let  $(X_1, X_2, \dots, X_s)$  have an  $s$ -variate negative binomial distribution with p.g.f.  $G_X(\mathbf{u}) = (1 + \theta_1(1 - u_1) + \dots + \theta_s(1 - u_s))^{-k}$ ,  $k > 0$ ,  $\theta_i > 0$ ,  $i = 1, 2, \dots, s$ . Then,

$$\begin{aligned} G_{X_1, \dots, X_m | X_{m+1}, \dots, X_s}(u_1, \dots, u_m) &= \\ &= k_{(x)} \prod_{i=m+1}^s \theta_i^{x_i} (1 + \theta_1(1 - u_1) + \dots + \theta_m(1 - u_m))^{-k-x} \div k_{(x)} \prod_{i=m+1}^s \theta_i^{x_i} = \\ &= (1 + \theta_1(1 - u_1) + \dots + \theta_m(1 - u_m))^{-k-x}, \end{aligned}$$

where  $x$  is defined as before.

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