Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE @ DIRECT•
Journal of Statistical Planning and Inference 131 (2005) 231–252

journal of statistical planning and inference

www.elsevier.com/locate/jspi

Identifying the Pareto and Yule distributions by properties of their reliability measures

Evdokia Xekalaki*, Caterina Dimaki

Department of Statistics, Athens University of Economics and Business, 76 Patission St., 104 34 Athens, Greece

Received 23 September 1999; received in revised form 1 February 2004; accepted 27 February 2004

Abstract

This paper provides characterizations of a family of distributions in the context of reliability theory placing emphasis on a particular family member, the Yule distribution. In particular, it is shown that the distribution of a non-negative, integer-valued random variable X with $E(X) < +\infty$ is uniquely identified to belong to the class of distributions consisting of the geometric, the Waring and the negative hypergeometric distributions if and only if anyone of the following conditions is satisfied:

a. The mean residual life is a linear function of time.
b. The vitality function is a linear function of time.
c. The product of the hazard rate by the mean residual life is constant.

Characterizations of the Yule distribution based on reliability measures of its size-biased version are also provided. Continuous analogues of the results are considered. These include characterizations of the exponential and the beta of the first and second kind distributions.

Keywords: Reliability; Hazard rate; Mean residual life; Vitality function; Characterization; Yule distribution; Geometric distribution; Negative hypergeometric distribution; Waring distribution; Exponential distribution; Beta distribution; Pareto distribution; Uniform distribution; Size biased distribution

1. Introduction

Reliability theory can be regarded as the theory of prediction, estimation or optimization of the probability of survival, the mean life or, more generally, the life distribution

* Corresponding author. Fax: +30-1-8238798.
E-mail address: exek@aueb.gr (E. Xekalaki).

0378-3758/$ - see front matter
doi:10.1016/j.jspi.2004.02.005

of a component. A plethora of books are available presenting reliability theory at various levels of mathematical sophistication. Some of them are aimed primarily at engineers, presenting statistical techniques at an introductory level and emphasizing their practical applicability. At the other end of the spectrum, lie mathematical texts which give detailed coverage of the probabilistic models. In between, one finds a number of texts on survival data analysis, some orientated more towards engineering applications and others towards medical data. Some of the books that can be included in the last two categories are: Mann et al. (1974), Kalbfleisch and Prentice (1980), Lawless (1982), Cox and Oakes (1984), Crowder et al. (1991). Finally, the book by Kleinbaum (1995) provides an easy to follow introduction to the main concepts and techniques of the subject.

Recently, studying the properties of life distributions has become important, especially due to the applicability of reliability theory to areas other than those related to engineering that gave the theory its first impetus, e.g. economics, environmental studies, etc. (See, for example, Epstein and Sobel (1954), Reinhardt (1968), Bryson and Siddiqui (1969), Harris (1970), Basu (1971), Brindley and Thompson (1972), Laurent (1975), Marshall (1975), Block (1977), Friday and Patil (1977), Shimi and Tsokos (1977), Muth (1977), Morrison (1978), Esary and Marshall (1979), Block and Savits (1981), Gupta (1979), Thompson (1981), Roy and Mukherjee (1986), Ecstein and Wolpin (1990), Narendranathan and Stewart (1993)).

Among the best known life distributions are the exponential, the Weibull, the gamma and the lognormal distributions. Xekalaki (1983) highlighted the potential use of Yule distributions for describing life distributions in discrete time models, as well as the use of the Pareto distribution, suggested as a potential continuous time model.

In this paper, we consider characterizations of a family of models in discrete time consisting of the geometric, the Waring and the negative hypergeometric distributions by properties of certain reliability measures with emphasis on a particular family member, the Yule distribution. Continuous analogues of the results are also provided. These include characterizations of the exponential, Pearson's Type VI (beta of the second kind) and its special case, the Pareto, as well as characterization of the Pearson's Type I (beta of the first kind) distributions. In particular, following some introductory general results in Section 2, we show that the linearity of the form of the mean residual life of a random variable or the constancy of the product of its hazard function and mean residual life is a necessary and sufficient condition that characterizes the family of geometric, Waring and negative hypergeometric distributions in the discrete case (Section 3) and the family of exponential and beta distributions of the first and second kind in the continuous case (Section 5). Corollaries to the main results lead to characterizations of interesting special cases of the members of the above families such as the Yule and discrete uniform distributions in the discrete case and the Pareto and uniform distributions in the continuous case. In Section 4, we provide characterizations of Waring and Yule distributions on the basis of the form of their size biased versions or of certain reliability measures of the latter. Finally, continuous analogues of these characterizations referring to the Pareto distribution are provided in Section 5.

2. A general result

Let X be a nonnegative, integer valued random variable representing the failure time of a component and let its probability function be denoted by $P_X(t)$, $t=0,1,2,\ldots$ The failure model consists of the specification of the functional form of the probability function $P_X(t)$ and the values of its parameters. Associated to the probability function $P_X(t)$ is the probability that the component has not failed at time t known as the *reliability function*, $\bar{F}_X(t) = 1 - F_X(t)$. Here, $F_X(t) = P(X \leqslant t)$ is the cumulative probability of failure up to time t. In the sequel, we denote by $h_X(t)$ the *hazard rate* function also known in the literature as the *failure rate* function. As is well known, $h_X(t)$ reflects the instantaneous potential for an event to occur given survival up to time t and it is defined as follows:

$$h_X(t) = \frac{P(X=t)}{P(X \geqslant t)}, \quad t=0,1,2,\ldots . \tag{1}$$

Further, we refer to the function

$$\mu_X(t) = E[X - t \,|\, X > t] = \frac{1}{\bar{F}_X(t)} \sum_{x=t}^{+\infty} \bar{F}_X(x), \quad t=0,1,\ldots \tag{2}$$

as the *mean residual life* of the component at time t. Obviously, $\mu_X(t)$ expresses the expected additional life time given that a component has survived until time t and is well defined when $E(X) < +\infty$.

In Kupta and Loo (1989) a new measure of the ageing process has been introduced. It measures the "vitality" of a time period in terms of the increase in average lifespan which results from surviving that time period. This measure, termed as the *vitality function*, is denoted by $v_X(t)$ and defined as $v_X(t) = E[X \,|\, X > t]$. Obviously, $v_X(t) = \mu_X(t) + t$. Finally, $\sigma_X^2(t) = Var[X - t \,|\, X > t]$ is the *residual life variance function*.

The concepts defined above relate to the reliability of the functioning of a system and play a key role in optimizing its use since the form of any one of them uniquely determines the life distribution of the system as highlighted by the theorem that follows. (For more details see Barlow and Proscham (1965)).

Theorem 1. *Let X be a nonnegative, integer valued random variable with reliability function $\bar{F}_X(t)$. The distribution of X is uniquely determined by the form of anyone of the following functions*:

i. *the survival or reliability function*
ii. *the hazard rate function $h_X(t)$*
iii. *the mean residual life function $\mu_X(t)$, provided that $E(X) < +\infty$*

Proof. The proof of statement (i) is well known and straightforward. Statement (ii) follows from the fact that (1) is equivalent to

$$P(X = r+1) - \frac{[1 - h_X(r)]h_X(r+1)}{h_X(r)} P(X = r) = 0, \tag{3}$$

whose unique solution is

$$P(X=r)=P(X=0)\prod_{i=0}^{r-1}\frac{[1-h_X(i)]h_X(i+1)}{h_X(i)}, \quad r=0,1,\ldots . \tag{4}$$

The proof of part (iii) follows as a special case of the results obtained independently by Glänzel et al. (1984) and Dimaki and Xekalaki (1996) on characterizing discrete distributions by the form of the conditional expectation $E[g(X)\,|\,X>t]$ for $g(X)=X-t$, $t=0,1,2,\ldots$.

By means of the next theorem, a direct relation between the hazard rate and the mean residual life is established. □

Theorem 2. *Let X be a nonnegative integer valued random variable with $E(X)<+\infty$ and reliability function $\bar{F}_X(t)$. Then, its hazard rate $h_X(t)$ can be directly expressed in terms of its mean residual life $\mu_X(t)$ by the relation*

$$h_X(t+1)=\frac{\mu_X(t+1)-\mu_X(t)+1}{\mu_X(t+1)}, \quad t=0,1,\ldots . \tag{5}$$

Proof. Observe that (2) yields

$$\mu_X(t)\bar{F}_X(t)=\sum_{x=t}^{+\infty}\bar{F}_X(x), \quad t=1,2,\ldots . \tag{6}$$

Specializing this for $t=r$ and $r+1$ and subtracting the resulting equations, we obtain

$$\mu_X(r+1)\bar{F}_X(r+1)-\mu_X(r)\bar{F}_X(r)=-\bar{F}_X(r), \quad r=0,1,\ldots \tag{7}$$

leading to

$$\mu_X(t+1)P(X>t+1)-\mu_X(t)P(X>t)=-P(X>t).$$

Therefore,

$$\frac{\mu_X(t+1)-\mu_X(t)+1}{\mu_X(t+1)}=\frac{P(X=t+1)}{P(X\geqslant t+1)}.$$

Hence, the theorem has been established since, by relation (1), $h_X(t+1)=P(X=t+1)/P(X\geqslant t+1)$. □

Corollary 1. *Let X be a nonnegative integer valued random variable with $E(X)<+\infty$ and reliability function $\bar{F}_X(t)$. Then, its hazard rate $h_X(t)$ can be expressed in terms of its vitality function by the relation*

$$h_X(t+1)=\frac{v_X(t+1)-v_X(t)}{v_X(t+1)-(t+1)}, \quad t=0,1,\ldots . \tag{8}$$

3. Identifiability of the geometric, Waring and negative hypergeometric distributions

In this section, some characteristic results already known in the literature as well as some new ones mainly concerning the Yule distribution are obtained as consequences of Theorem 1. Before stating the main results, we introduce some notation and terminology:

Definition 1. A nonnegative, integer valued random variable X is said to have the univariate generalized Waring distribution ($UGWD$) with parameters α, k and ρ if its probability function is given by,

$$p_x = P(X = x) = \frac{\rho_{(k)}}{(\alpha+\rho)_{(k)}} \frac{\alpha_{(x)} k_{(x)}}{(\alpha+k+\rho)_{(x)}} \frac{1}{x!},$$

$$x = 0, 1, 2, \ldots; \ \alpha > 0, \ k > 0, \ \rho > 0, \tag{9}$$

where

$$\beta_{(\gamma)} = \frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)}, \quad \beta > 0, \ \gamma \in R.$$

Definition 2. A nonnegative integer valued random variable X is said to have the Waring distribution with parameters α, and ρ if its probability function is given by,

$$p_x = P(X = x) = \frac{\rho \alpha_{(x)}}{(\alpha+\rho)_{(x+1)}}, \quad x = 0, 1, 2, \ldots; \ \alpha > 0, \ \rho > 0. \tag{10}$$

Definition 3. A nonnegative integer valued random variable X is said to have the Yule distribution with parameter ρ if its probability function is given by,

$$p_x = P(X = x) = \frac{\rho x!}{(\rho+1)_{(x+1)}}, \quad x = 0, 1, \ldots; \ \rho > 0. \tag{11}$$

Obviously, $X \sim \mathrm{Yule}(\rho)$ iff $X \sim \mathrm{Waring}(1, \rho)$ and $X \sim \mathrm{Waring}(\alpha, \rho)$ iff $X \sim UGWD(\alpha, 1; \rho)$. (For more details about these distributions, see Irwin (1975) and Johnson et al. (1993, Sections 6.10.3 and 6.10.4)).

Definition 4. A nonnegative integer valued random variable X is said to have the geometric distribution with parameter q if its probability function is given by

$$p_x = P(X = x) = (1-q)q^x, \quad x = 0, 1, 2, \ldots . \tag{12}$$

Note that the geometric distribution is a limiting case of the Yule (ρ) distribution as ρ increases.

Definition 5. A nonnegative integer valued random variable X is said to have the negative hypergeometric distribution with parameters k, h and l if its probability function is given by

$$p_x = P(X = x) = \binom{-k}{x}\binom{-h}{l-x} \Big/ \binom{-k-h}{l}$$

$$= \binom{l}{x} \frac{k_{(x)} h_{(l-x)}}{(k+h)_{(l)}}, \quad x = 0, 1, \ldots l; k, h, l > 0. \tag{13}$$

By the next theorem, a class of distributions that consists of the geometric, the negative hypergeometric and the Waring distributions (thus containing the Yule distribution as a member) is shown to result when the mean residual life is a linear function of time.

Theorem 3. *Let X be a non negative, integer valued random variable defined on $\{0,1,\ldots,l\}, l \in \{1,2,\ldots\} \cup \{+\infty\}$ with $E(X) < +\infty$. Then, its mean residual life is a linear function of time t given by*

$$\mu_X(t) = a + bt, \quad a,b \in R, \ a > 1; \ t = 0,1,\ldots,l-1 \tag{14}$$

if and only if one of the following statements holds:

(i) *X has a geometric distribution with parameter $q = (a-1)/a$*
(ii) *X has a Waring distribution with parameters $\alpha = (a-1)/b - 1$ and $\rho = 1/b + 1$.*
(iii) *X has a negative hypergeometric distribution on $\{0,1,\ldots,l\}$ with parameters, $k = 1, h, l > 0$*

(*These cases correspond to $b = 0$, $b > 0$ and $b < 0$, respectively*).

Proof. In the case where X has a distribution with infinite support, relation (14) implies for $t \to +\infty$ that $b \geqslant 0$. By Theorem 1, the distribution of X is uniquely determined by its mean residual life $\mu_X(t)$. It follows therefore from (12), that relationship (14) is uniquely satisfied, for $b=0$, by the geometric distribution with parameter $q=(a-1)/a$. In the case $b > 0$, we obtain by the definition of the mean residual life

$$\mu_X(t) = \frac{1}{\bar{F}_X(t)} \sum_{x=t}^{+\infty} \bar{F}_X(x) = \frac{1}{P(X > t)} \sum_{x=0}^{+\infty} P(X > x + t). \tag{15}$$

Also, using relation (10) with $\rho > 1$, we can show that

$$P(X > t) = \frac{1}{\rho}(\alpha + \rho + t + 1)P(X = t + 1), \quad t = 0,1,2,\ldots$$

(see Xekalaki, 1983). Therefore,

$$\begin{aligned}
\mu_X(t) &= \frac{1}{(\alpha + \rho + t + 1)P(X = t + 1)} \sum_{x=0}^{+\infty} (\alpha + \rho + x + t + 1)P(X = x + t + 1) \\
&= \frac{1}{\alpha + \rho + t + 1} \sum_{x=0}^{+\infty} (\alpha + \rho + x + t + 1) \frac{(\alpha + t + 1)_{(x)}}{(\alpha + \rho + t + 2)_{(x)}} \\
&= \frac{1}{\rho}\left(\alpha + \rho + t + 1 + \frac{\alpha + t + 1}{\rho - 1}\right). \\
&= \frac{\alpha + \rho + t}{\rho - 1}
\end{aligned}$$

Hence, when $b > 0$, Theorem 1 implies that condition (14) is uniquely satisfied by the Waring distribution with parameters α and ρ, for $a = (\alpha + \rho)/(\rho - 1)$ and $b = 1/(\rho - 1)$.

If X has a finite support ($l < +\infty$), it follows from (14), for $t = l-1$, that $a+b(l-1)=1$. Thus, since $a > 1$, it follows that $b=(1-a)/(l-1) < 0$. Also, by the definition of the mean residual life, we have from (14), for $t=1$, that $a+b=E(X)/(1-p_0) > 0$. Hence, using (13) with $m=1$, we obtain

$$
\begin{aligned}
P(X > t) &= \sum_{x=t+1}^{l} \binom{-1}{x}\binom{-h}{l-x} \Big/ \binom{-1-h}{l} \\
&= \sum_{x=t+1}^{l} \frac{l!}{(l-x)!}\,\frac{h_{(l-x)}}{(h+1)_{(l)}} \\
&= \frac{(l-t)_{(t+1)}}{(l-t+h)_{(t+1)}} \sum_{x=0}^{l-t-1} \frac{(l-t-1)!}{(h-x-t-1)!}\,\frac{h_{(l-x-t-1)}}{(h+1)_{(h-t-1)}} \\
&= \frac{l^{(t+1)}}{(l+h)^{(t+1)}} \sum_{x=0}^{h-t-1} \binom{-1}{x}\binom{-h}{l-t-x-1} \Big/ \binom{-1-h}{l-t-1} \\
&= \frac{l+n-t-1}{h}\,\frac{l!}{(l-t-1)!}\,\frac{h_{(l-t-1)}}{(h+1)_{(l)}} \\
&= \frac{h+l-t-1}{h} \binom{-1}{t+1}\binom{-h}{l-t-1} \Big/ \binom{-1-h}{l} \\
&= \frac{l+h-t-1}{h} P(X = t+1).
\end{aligned}
$$

(Here $\beta^{(r)}$ denotes the descending factorial $\beta(\beta-1)\cdots(\beta-r+1), \beta \geqslant r$ ($\beta^{(0)}=1$)). Therefore,

$$
\begin{aligned}
\mu_X(t) &= \frac{1}{(l+h-t-1)P(X=t+1)} \sum_{x=0}^{l-t} (l+h-t-x-1)P(X=t+x+1) \\
&= \frac{(l-t-1)!}{(l+h-t-1)h_{(l-t-1)}} \sum_{x=0}^{l-t} \frac{(l+h-t-x-1)h_{(l-t-x-1)}}{(l-t-x-1)!} \\
&= \frac{(l-t-1)!}{h_{(l-t)}} \sum_{x=0}^{l-t} \frac{h_{(l-t-x)}}{(l-t-x)!}(l-t-x) \\
&= \frac{1}{l-t}\,\frac{h+l-t}{h}\left(l-t-\frac{l-t}{h+l-t}\right) \\
&= \frac{h+l-t-1}{h}.
\end{aligned}
$$

Hence, when $b < 0$, Theorem 1 implies that condition (13) uniquely leads to the negative hypergeometric distribution on $\{0, 1, \ldots, l\}$, $l = 1 + (1 - a)/b$ with parameters $k = 1$, h and l, for $a = 1 + (l - 1)/h$ and $b = -1/h$. This completes the proof of the theorem. □

Note that, using (5) and Xekalaki's (1983) main result, an alternative proof of the theorem can be obtained.

The following characterizations of the Yule distribution can be obtained as direct consequences of the above theorem.

Corollary 2. *Let X be defined as in Theorem 3. Then, X follows the Yule (ρ) distribution, with $\rho = 1/b + 1$ if and only if its mean residual life is a linear function of time t given by*

$$\mu_X(t) = 1 + b(t + 2), \quad t = 0, 1, 2, \ldots .$$

Corollary 3. *Let X be defined as in Theorem 3. Then, X follows the Yule (ρ) distribution with $\rho = 1/b + 1$ if and only if its vitality is a linear function of time t given by,*

$$v_X(t) = 1 + (b + 1)(t + 1), \quad t = 0, 1, 2, \ldots .$$

Note that when $h = 1$, the negative hypergeometric distribution reduces to the discrete uniform distribution on $\{0, 1, \ldots, l\}$. Hence, the following characterization of the uniform distribution can be obtained.

Corollary 4. *Let X be defined as in Theorem 3. Then, X follows the discrete uniform distribution on $\{0, 1, \ldots, a\}$ if and only if its mean residual life is a linear function of t given by*

$$\mu_X(t) = a - t, \quad a \in R, \ a > 1; \ t = 0, 1, 2, \ldots, \alpha - 1$$

or, equivalently, if and only if its vitality function is constant, and, in particular,

$$v_X(t) = a, \quad t = 0, 1, 2, \ldots, \alpha - 1.$$

Next theorem shows that the distribution of a nonnegative, integer valued random variable X is uniquely determined as geometric, Waring or negative hypergeometric according as the product of its hazard rate and its mean residual life is a constant equal to, exceeding or exceeded by unity. Before proceeding with the main result, we prove the following lemma.

Lemma 1. *Let X be defined on $\{0, 1, 2, \ldots, l\}$, $l \in \{1, 2, \ldots\} \cup \{+\infty\}$ and such that $E(X) < +\infty$, $p_0 = P(X = 0) < 1$ and*

$$h_X(t) = \frac{1}{a + bt}, \quad a, b \in R, \ a > 0, \ t = 0, 1, 2, \ldots, l. \tag{16}$$

Then (i) $a > 1$, (ii) $b < 1$, (iii) $E(X) = (a - 1)/(1 - b)$, (iv) X has a distribution with finite support if and only if $b < 0$ and in this case $b = (1 - a)/l$.

Proof. For a proof of (i) and (iv), see Xekalaki (1983). To prove (ii), observe that from (16) it follows that $P(X \geqslant t) = (a + bt)P(X = t)$, which, upon summation over t, yields $E(X) + 1 = a + bE(X)$ or, equivalently, $a - 1 = (1 - b)E(X)$. Since $a > 1$ and $E(X) > 0$ by the definition of X, it follows that $b < 1$. Then, (iii) follows immediately. Hence the lemma has been established. □

Theorem 4. *Let X be a nonnegative, integer valued random variable defined on $\{0, 1, \ldots, l\}$, $l \in \{1, 2, \ldots\} \cup \{+\infty\}$ with $P(X = 0) < 1$ and $E(X) = m < +\infty$. Then,*

$$h_X(t)\mu_X(t) = c, \quad t = 0, 1, 2, \ldots, l - 1; \quad c \in R \ldots \tag{17}$$

if and only if one of the following statements holds:

(i) *X has a geometric distribution with parameter $q = m/(m + 1)$, for $c = 1$*
(ii) *X has a Waring distribution with parameters $\alpha = m/(c - 1)$ and $\rho = c/(c - 1)$, for $c > 1$*
(iii) *X has a negative hypergeometric distribution on $\{0, 1, \ldots, m/(1 - c)\}$ with parameters $k = 1$, $h = c/(1 - c)$ and $l = m/(1 - c)$, for $c < 1$.*

Proof. The necessity part is straightforward. For sufficiency, observe that

$$h_X(t + 1) = \frac{\mu_X(t + 1) - \mu_X(t) + 1}{\mu_X(t + 1)}$$

or, equivalently,

$$\mu_X(t + 1) - \mu_X(t) + 1 = \mu_X(t + 1)h_X(t + 1). \tag{18}$$

Combining (17) and (18) we obtain, $\mu_X(t + 1) - \mu_X(t) = c - 1$, $t = 0, 1, 2, \ldots, l - 1$. The unique solution of this difference equation is given by

$$\mu_X(t) = c' + (c - 1)t, \quad t = 0, 1, 2, \ldots, l - 1, \tag{19}$$

where, as implied by (17) for $t = 0$ and the definition of the mean residual life and hazard function, $c' = m + c$. In view of (17), the above relationship is equivalent to

$$h_X(t) = \frac{1}{(c + m)/c + (c - 1)t/c}, \quad t = 0, 1, 2, \ldots, l - 1. \tag{20}$$

From Lemma 1, when X has a distribution with an infinite support, it follows that $c \geqslant 1$. Then, if $c = 1$, relation (20) reduces to the well known characteristic property of the geometric distribution with parameter $q = m/(m + 1)$. If on the other hand, $c > 1$, (20) leads to the Waring distribution with parameters $\alpha = m/(c - 1)$ and $\rho = c/(c - 1)$ (see Xekalaki, 1983).

In the case where X has a finite support ($l < +\infty$), it follows, from Lemma 1, that $c < 1$. It follows, moreover, that $l = m/(1 - c)$, i.e., $m/(1 - c)$ is a positive integer. Then, since $h_X(l) = 1$, it follows from (20) that $h_X(t) = \frac{1}{(c+m)/c+(c-1)t/c}$, $t = 0, 1, 2, \ldots, m/(1 - c)$. This implies that X has a negative hypergeometric distribution on $\{0, 1, \ldots, m/(1 - c)\}$ with parameters $k = 1$, $h = c/(1 - c)$ and $l = m/(1 - c)$ (see Xekalaki, 1983). □

Corollary 5. *Let X be defined as in Theorem* 4. *Then,*

$$h_X(t)\mu_X(t) = m + 1, \quad t = 0, 1, 2, \ldots$$

if and only if X has a Yule distribution with parameter $\rho = (m+1)/m$.

4. Characterizations of Waring and Yule distributions based on reliability measures of their size-biased versions

In studies in reliability, biometry and survival analysis, weighted distributions are frequently appropriate for certain natural sampling schemes. It would, therefore, be useful to develop relationships between the survival function, $\bar{F}_X(t)$, the failure rate, $h_X(t)$, and the mean residual life, $\mu_X(t)$, of the original distribution and the corresponding measures of its weighted version. Gupta and Keating (1986) obtained relations for the reliability measures of the size-biased version of a distribution and proved some interesting characterization results concerning the continuous case. Jain et al. (1989) working with continuous time models, generalized, among other results, those obtained by Gupta and Keating for a general form of the weight function.

In this section, we characterize the Yule distribution by properties of the reliability measures of its weighted version. Before proceeding to the main results, we introduce some notation and terminology.

Let us consider a nonnegative random variable X with probability function $f_X(x)$, $f_X(0) < 1$, and let $w(X)$ be a positive weight function. Assume that $E[w(X)]$ exists. Denote a new probability function by $f_X^w(x) = w(x)f_X(x)/E[w(X)]$ and let X^w be the random variable whose probability function is $f_X^w(x)$. Then, $f_X(x)$ is referred to as the *original probability function* and $f_X^w(x)$ as the *weighted probability function*. In the context of statistical analysis and in the case where $w(x) = x^\alpha$, $\alpha > 0$, $f_X^w(x)$ is known as the *size-biased version of $f_X(x)$ of order α* and the random variable X^w associated with it is denoted by $X^{*\alpha}$. In the special case where $\alpha = 1$, the random variable X^w is denoted by $X^* \equiv X^{*1}$ and $f_X^w(x)$, simply called *size-biased version of $f_X(x)$*, is given by $f_X^w(x) = f_X^*(x) = xf_X(x)/E[X]$. Finally, in the case where $w(x) = x^{(j)} = x(x-1)\cdots(x-j+1)$, the random variable X^w is denoted by $X^{*(j)}$ and its probability function $f_X^{*(j)}(x)$, referred to as the *size-biased version of $f_X(x)$ of factorial order j*, is given by $f_X^{*(j)}(x) = x^{(j)}f_X(x)/E[X^{(j)}]$, $x = j, j+1, \ldots$ provided that the factorial moment of order j exists, i.e. $E[X^{(j)}] < +\infty$.

In the sequel, we characterize a variant of the Waring distribution by the form of its size-biased version of factorial order j. More precisely, it will be shown that the distribution of a random variable X is uniquely determined as a variant of the generalized Waring distribution if its size-biased form of factorial order j follows the shifted j units to the right univariate generalized Waring distribution.

Theorem 5. *Let X and $X^{*(j)}$ be nonnegative, integer valued random variables defined as above. Assume that $E[X^{(j)}] \equiv \mu^{(j)}$ with $0 < \mu^{(j)} < +\infty$ and let the probability functions of X and $X^{*(j)}$ be p_x and $p_x^{*(j)}$ respectively, satisfying the condition, $p_x^{*(j)} = x^{(j)}p_x/E[X^{(j)}]$, $x = j, j+1, \ldots$. Then, the random variable $X^{*(j)}$ follows a shifted j*

units to the right generalized Waring $(\alpha+j, k+j; \rho-j)$ distribution, $\alpha, k>0$, $\rho>j$ if and only if X is distributed according to a variant of the generalized Waring $(\alpha, k; \rho)$ distribution, with probability function

$$p_x = \begin{cases} d_x, & x=0,1,2,\dots,j-1, \\ c_j \dfrac{\rho_{(k)}}{(\alpha+\rho)_{(k)}} \dfrac{\alpha_{(x)}k_{(x)}}{(\alpha+k+\rho)_{(x)}} \dfrac{1}{x!}, & x=j, j+1,\dots, \end{cases} \tag{21}$$

where c_j and d_x arbitrary constants such that $\sum_{x=0}^{+\infty} p_x = 1$.

Proof (Necessity). Let X be a random variable distributed according to a variant of the univariate generalized Waring $(\alpha, k; \rho)$ distribution, with probability function given by (21). Obviously, $\mu^{(j)}=E[X^{(j)}]=\alpha_{(j)}k_{(j)}/((\rho-1)(\rho-2)\cdots(\rho-j))$, $j=0,1,2,\dots,[\rho]$, where $[\rho]$ denotes the integer part of ρ. Consequently,

$$\begin{aligned} p_x^{*(j)} &= \frac{(\rho-j)_{(j)}\rho_{(k)}(\alpha+j)_{(x-j)}(k+j)_{(x-j)}}{(\alpha+\rho)_{(k)}(\alpha+k+\rho)_{(j)}(\alpha+k+\rho+j)_{(x-j)}} \frac{1}{(x-j)!} \\ &= \frac{(\rho-j)_{(k+j)}(\alpha+j)_{(x-j)}(k+j)_{(x-j)}}{(\alpha+\rho)_{(k+j)}(\alpha+k+\rho+j)_{(x-j)}} \frac{1}{(x-j)!}, \quad x=j, j+1,\dots . \end{aligned} \tag{22}$$

Therefore, the random variable $X^{*(j)}$ follows a shifted j units to the right univariate Waring $(\alpha+j, k+j; \rho-j)$ distribution, $\alpha, k>0$, $\rho>j$.

Sufficiency: Let $X^{*(j)}$ be a random variable distributed according to a shifted j units to the right univariate Waring $(\alpha+j, k+j; \rho-j)$ distribution, $\alpha, k>0$, $\rho>j$. Then,

$$p_x^{*(j)} = \frac{(\rho-j)_{(k+j)}(\alpha+j)_{(x-j)}(k+j)_{(x-j)}}{(\alpha+\rho)_{(k+j)}(\alpha+k+\rho+j)_{(x-j)}} \frac{1}{(x-j)!}, \quad x=j,\ j+1,\dots .$$

By the definition of $p_x^{*(j)}$, we have $p_x^{*(j)}=x^{(j)}p_x/E[X^{(j)}]$, $x=j, j+1,\dots$. Therefore,

$$\frac{x^{(j)}p_x}{E[X^{(j)}]} = \frac{(\rho-j)_{(k+j)}(\alpha+j)_{(x-j)}(k+j)_{(x-j)}}{(\alpha+\rho)_{(k+j)}(\alpha+k+\rho+j)_{(x-j)}} \frac{1}{(x-j)!}$$

or, equivalently,

$$\begin{aligned} p_x &= \frac{\mu^{(j)}}{x!/(x-j)!} \frac{(\rho-j)_{(k+j)}}{(\alpha+\rho)_{(k+j)}} \frac{(\alpha+j)_{(x-j)}(k+j)_{(x-j)}}{(\alpha+k+\rho+j)_{(x-j)}} \frac{1}{(x-j)!} \\ &= \mu^{(j)} \frac{(\rho-j)_{(j)}\rho_{(k)}(\alpha+j)_{(x-j)}(k+j)_{(x-j)}}{(\alpha+\rho)_{(k)}(\alpha+k+\rho)_{(j)}(\alpha+k+\rho+j)_{(x-j)}} \frac{1}{x!}, \quad x=j,\ j+1,\dots \end{aligned}$$

leading to

$$p_x = \mu^{(j)} \frac{(\rho-j)_{(j)}\rho_{(k)}\alpha_{(x)}k_{(x)}}{(\alpha+\rho)_{(k)}\alpha_{(j)}k_{(j)}(\alpha+k+\rho)_{(x)}} \frac{1}{x!}, \quad x=j,\ j+1,\dots .$$

Therefore, the random variable X follows a variant of the univariate generalized Waring $(\alpha, k; \rho)$ distribution, with probability function as given by (21) with

$$c_j = \mu^{(j)} \frac{(\rho-j)_{(j)}}{\alpha_{(j)}k_{(j)}}. \qquad \square$$

As shown by the next theorem, the Yule distribution is characterized by the form of the hazard rate function of its size biased version. In particular, it is shown that the hazard rate of the size biased version of X is directly related to the corresponding measure of the original random variable. Before providing the main result, we prove the following lemmas. The first lemma shows that the univariate generalized Waring distribution is form invariant under size biased sampling of order 1 with displacement of the origin. The second lemma expresses the hazard rate function of a size biased distribution in terms of the hazard rate function of its parent distribution.

Lemma 2. *Let X, X^* be random variables taking values in $\{1,2,3,\ldots,\}$ with $E(X) < +\infty$ and probability functions $p_r = P(X=r)$, $p_r^* = P(X^*=r)$, $r=1,2,\ldots$ respectively. Assume that $p_r^* = rp_r/E(X)$, $r=1,2,\ldots$. Then X^* is distributed according to the shifted generalized Waring $(\alpha+1, k+1; \rho-1)$ distribution if and only if the distribution of X is the zero truncated univariate generalized Waring $(\alpha, k; \rho)$ distribution, $\alpha, k > 0$, $\rho > 1$.*

Proof. By Theorem 5 it follows that X^* *is* distributed according to the shifted generalized Waring $(\alpha+1, k+1; \rho-1)$ distribution if and only if the distribution of X is a variant of the generalized Waring distribution with probability function given by (21) for $d_0 = 0$, $j = 1$. Therefore,

$$p_x = c_1 \frac{\rho_{(k)}}{(\alpha+\rho)_{(k)}} \frac{\alpha_{(x)} k_{(x)}}{(\alpha+k+\rho)_{(x)}} \frac{1}{x!}, \quad x = 1,2,\ldots .$$

Consequently,

$$c_1 \left[\sum_{x=0}^{+\infty} \frac{\rho_{(k)}}{(\alpha+\rho)_{(k)}} \frac{\alpha_{(x)} k_{(x)}}{(\alpha+k+\rho)_{(x)}} \frac{1}{x!} - \frac{\rho_{(k)}}{(\alpha+\rho)_{(k)}} \right] = 1$$

$$c_1 = \frac{(\alpha+\rho)_{(k)}}{(\alpha+\rho)_{(k)} - \rho_{(k)}}$$

leading to the zero truncated univariate generalized Waring $(\alpha, k; \rho)$ distribution. □

Lemma 3. *For any nonnegative, integer-valued random variable X with probability function $p_x \neq 0$, $x \geqslant 1$ and $E(X) < +\infty$, the following relationship holds:*

$$\frac{h_X(t)}{h_{X^*}(t)} = \frac{v_X(t-1)}{t}, \quad t = 1,2,\ldots , \tag{23}$$

where $h_X(t)$, $v_X(t)$ are the hazard rate function and the vitality function of the random variable X, respectively, while $h_{X^*}(t)$ is the hazard rate function of its size biased version X^*.

Proof.

$$\frac{h_X(t)}{h_{X^*}(t)} = \frac{p_t}{P(X \geqslant t)} \Bigg/ \frac{tp_t/E(X)}{\sum_{x \geqslant t} xp_x/E(X)} = \frac{\sum_{x \geqslant t} xp_x}{tP(X \geqslant t)}$$

$$= \frac{E(X \mid X \geqslant t)}{t} = \frac{E(X \mid X > t-1)}{t} = \frac{v_X(t-1)}{t}. \quad \square$$

Theorem 6. *Let X, X^* be random variables taking values in $\{1,2,3,\ldots\}$. Assume that $E(X) < +\infty$ and let the probability functions of X, X^* satisfy the condition $p_r^* = rp_r/E(X)$, $r = 1,2,\ldots$, where $p_r = P(X = r)$, $p_r^* = P(X^* = r)$, $r = 1,2,\ldots$. Then, X follows the zero truncated Yule (ρ) distribution, $\rho > 1$ if and only if*

$$h_{X^*}(t) = c_1 \frac{1}{\rho+1+t} - c_2 \frac{1}{1+\rho t}, \tag{24}$$

where $c_1 = (\rho-1)_{(2)}/(1-\rho_{(2)})$ and $c_2 = (\rho-1)_{(3)}/(1-\rho_{(2)})$.

Proof (Necessity). Let X have the zero-truncated Yule (ρ) distribution with $\rho > 1$. By analogy to the result of Xekalaki (1983) for the untruncated case, it follows that

$$h_X(t) = \frac{\rho}{\rho+t+1}, \quad t = 1,2,\ldots .$$

Also, from Corollary 2,

$$\mu_X(t) = \frac{\rho+t+1}{\rho-1}, \quad t = 1,2,\ldots .$$

These relationships, combined with (23) of Lemma 3 and the fact that $v_X(t) = \mu_X(t) + t$, lead to (24).

Sufficiency: Assume that (24) holds. The shifted generalized Waring $(2,2;\rho-1)$ distribution satisfies (24) as just shown. Since the distribution of a random variable is uniquely determined by its hazard rate, it follows that X^* is shifted generalized Waring $(2,2;\rho-1)$ distributed. Then, by Lemma 2, X has the zero truncated Yule (ρ) distribution, $\rho > 1$. □

Consider a random variable and its size biased version. The theorem that follows proves that the distribution of the original random variable is characterized as Yule if the ratio of the hazard rate of the original random variable to that of its corresponding size biased version is constant.

Theorem 7. *Let X, X^* be random variables taking values in $\{1,2,3,\ldots\}$. Assume that $E(X) < +\infty$ and let the probability functions of X, X^* satisfy the condition $p_r^* = rp_r/E(X)$, $r = 1,2,\ldots$, where $p_r = P(X = r)$, $p_r^* = P(X^* = r)$, $r = 1,2,\ldots$. Then, X follows the shifted Yule $(\rho+1)$ distribution, $\rho > 0$ if and only if*

$$\frac{h_X(t)}{h_{X^*}(t)} = \frac{\rho+1}{\rho}. \tag{25}$$

Proof (Necessity). Let X be a random variable distributed according to a shifted univariate Yule $(\rho+1)$ distribution with probability function

$$p_x = \frac{(\rho+1)(x-1)!}{(\rho+2)_{(x)}}, \quad x = 1,2,\ldots .$$

Then, $E(X) = (\rho+1)/\rho$, $h_X(t) = (\rho+1)/(\rho+t+1)$, $t = 1,2,\ldots$ and, since $p_r^* = rp_r/E(X)$, $r = 1,2,\ldots$, it follows that $h_{X^*}(t) = \rho/(\rho+t+1)$, which leads to (25).

Sufficiency: Assume that (25) holds. Using Lemma 3, we obtain

$$v_X(t-1) = t\,\frac{\rho+1}{\rho}.$$

This leads to $\mu_X(t) = (\rho + 1 + t)/\rho$, which, by analogy to the result of Corollary 2 for the Yule distribution, implies that X follows the shifted Yule $(\rho + 1)$ distribution. □

Corollary 6. *Let X, X^* be random variables taking values in $\{1,2,3,\ldots\}$. Assume that $E(X) < +\infty$ and let the probability functions of X, X^* satisfy the condition $p_r^* = rp_r/E(X)$, $r = 1,2,\ldots$, where $p_r = P(X = r)$, $p_r^* = P(X^* = r)$, $r = 1,2,\ldots$. Then, X follows the shifted Yule $(\rho+1)$ distribution, $\rho > 1$ if anyone of the following relations holds*:

1. $h_X(t) - h_{X^*}(t) = 1/(t + \rho + 1)$.
2. $\mu_X(t)/\mu_{X^*}(t) = (\rho - 1)/\rho$.
3. $\mu_{X^*}(t) - \mu_X(t) = (1/(\rho - 1)_{(2)})(t + \rho + 1)$.
4. $h_X(t)/h_{X^*}(t) - \mu_X(t)/\mu_{X^*}(t) = 2/\rho$.
5. $v_{X^*}(t) - v_X(t) = (1/(\rho - 1)_{(2)})(t + \rho + 1)$.
6. $v_X(t)/v_{X^*}(t) = (\rho - 1)(\rho + 1)(t + 1)/[\rho(\rho t + \rho + 1)]$.

Before proceeding with the proof of the next theorem, we shall state and prove the following lemma which is a characterization of the shifted 1 unit to the right Yule $(\rho + 1)$ distribution, $\rho > 0$ based on its tail probability.

Lemma 4. *Let X and Z be nonnegative, integer valued random variables. Assume that $E(X) < +\infty$ and that the probability functions of X and Z satisfy the relationship*

$$q_r^+ = \frac{\sum_{j=r+1}^{+\infty} p_j}{E(X)}, \qquad r = 0,1,2\ldots\,, \tag{26}$$

where $p_r = P(X = r)$ and $q_r^+ = P(Z = r)$. Then, the zero truncated distribution of Z is the shifted $UGWD(1,2;\rho), \rho > 0$, if and only if the distribution of X is the shifted Yule$(\rho + 1), \rho > 0$.

Proof (Necessity). Let X be a random variable distributed according to a shifted univariate *Yule*$(\rho + 1)$ distribution with probability function

$$p_x = \frac{(\rho+1)(x-1)!}{(\rho+2)_{(x)}}, \qquad x = 1,2,\ldots \quad \text{and} \quad E(X) = \frac{\rho+1}{\rho}.$$

The zero truncated distribution of the random variable Z has probability function given by

$$P(Z = r \mid Z > 0) = \frac{q_r^+}{1 - q_0^+} = \frac{\sum_{j=r+1}^{+\infty} p_j}{E(X) - 1}.$$

It is also known, from Xekalaki (1984), that $\sum_{j=r+1}^{+\infty} p_j = P(X > r) = rP(X = r)/(\rho+1)$. Therefore,

$$\frac{q_r^+}{1-q_0^+} = \frac{[r/(\rho+1)][(\rho+1)(r-1)!]/(\rho+2)_{(r)}}{(\rho+1)/\rho-1} = \frac{(\rho)_{(2)}2_{(r-1)}1_{(r-1)}}{(\rho+1)_{(2)}(\rho+3)_{(r-1)}}\frac{1}{(r-1)!}.$$

So, the distribution of the random variable $Z\,|\,(Z>0)$ is the shifted $UGWD(1,2;\rho)$, $\rho > 0$.

Sufficiency. Let $Z\,|\,Z>0$ be a random variable having the shifted $UGWD(1,2;\rho)$, $\rho>0$. Then,

$$P(Z=r\,|\,Z>0) = \frac{q_r^+}{1-q_0^+} = \frac{(\rho)_{(2)}2_{(r-1)}1_{(r-1)}}{(\rho+1)_{(2)}(\rho+3)_{(r-1)}}\frac{1}{(r-1)!}.$$

But,

$$\frac{q_r^+}{1-q_0^+} = \frac{\sum_{j=r+1}^{+\infty} p_j}{E(X)-1}, \qquad r=1,2,\ldots .$$

Therefore,

$$\frac{\sum_{j=r+1}^{+\infty} p_j}{E(X)-1} = \frac{(\rho)_{(2)}2_{(r-1)}1_{(r-1)}}{(\rho+1)_{(2)}(\rho+3)_{(r-1)}}\frac{1}{(r-1)!}$$

or, equivalently,

$$\sum_{j=r+1}^{+\infty} p_j = (E(X)-1)\frac{(\rho)_{(2)}2_{(r-1)}}{(\rho+1)_{(2)}(\rho+3)_{(r-1)}}. \tag{27}$$

Specializing this relation for $r+1$ and subtracting it from (23), it follows that,

$$p_{r+1} = (E(X)-1)\frac{(\rho)_{(2)}2_{(r-1)}}{(\rho+1)_{(2)}(\rho+3)_{(r-1)}} - (E(X)-1)\frac{(\rho)_{(2)}2_{(r)}}{(\rho+1)_{(2)}(\rho+3)_{(r)}}.$$

Hence,

$$p_r = (E(X)-1)\frac{\rho(\rho+1)}{(\rho+1)(\rho+2)}\left[\frac{2_{(r-2)}}{(\rho+3)_{(r-2)}} - \frac{2_{(r-1)}}{(\rho+3)_{(r-1)}}\right],$$

where $r=1,2,\ldots$. Since $\sum_{j=1}^{+\infty} p_r = 1$, we have

$$1 = (E(X)-1)\frac{\rho(\rho+1)}{(\rho+1)(\rho+2)}\sum_{r=1}^{+\infty}\left[\frac{1_{(r-1)}}{(\rho+2)_{(r-2)}/(\rho+2)} - \frac{2_{(r-1)}}{(\rho+3)_{(r-1)}}\right]$$

$$= (E(X)-1)\frac{\rho(\rho+1)}{(\rho+1)(\rho+2)}\left[(\rho+2)\sum_{r=1}^{+\infty}\frac{1_{(r-1)}}{(\rho+2)_{(r-2)}} - \sum_{r=1}^{+\infty}\frac{2_{(r-1)}}{(\rho+3)_{(r-1)}}\right]$$

$$= (E(X)-1)\frac{\rho(\rho+1)}{(\rho+1)(\rho+2)}\left[(\rho+2)\frac{(\rho+1)}{\rho} - \frac{(\rho+2)}{\rho}\right].$$

Then, $(E(X)-1)\rho = 1$ and

$$p_r = \frac{1}{\rho}\left[\frac{\rho_{(2)}2_{(r-2)}}{(\rho+1)_{(2)}(\rho+3)_{(r-2)}} - \frac{\rho_{(2)}2_{(r-1)}}{(\rho+1)_{(2)}(\rho+3)_{(r-1)}}\right]$$

$$= \frac{(r-1)!}{(\rho+2)_{(r-1)}} - \frac{r!}{(\rho+2)_{(r)}}$$

$$= \frac{(\rho+1)(r-1)!}{(\rho+2)_{(r)}}, \quad r = 1, 2, \ldots .$$

Hence, the distribution of X is the shifted 1 unit to the right Yule$(\rho+1), \rho > 0$. □

The tail probability function of X as given by relationship (26) is the probability function of the weighted distribution of X with weight function $w(X) = 1/h_X(X)$. Therefore, the following characterization of the shifted Yule $(\rho+1)$ distribution, $\rho > 0$ can be proved.

Theorem 8. *Let X, X^w be random variables taking values in $\{1, 2, 3, \ldots\}$ with $E(X) < +\infty$ and probability functions $p_r = P(X = r)$, $p_r^w = P(X^w = r)$, $r = 1, 2, \ldots$ respectively. Assume that*

$$p_r^w = \frac{w(r)p_r}{E[w(X)]}, \quad r = 1, 2, \ldots \quad \textit{with } w(X) = \frac{1}{h_X(X)}.$$

Then, X is distributed according to the shifted Yule $(\rho+1)$ distribution, $\rho > 0$ if and only if $h_{X^w}(t) = \rho/(\rho+t+1)$.

Proof. Let X be a random variable distributed according to the shifted Yule $(\rho+1)$, $\rho > 0$ distribution.

Then,

$$p_r^w = \frac{w(r)p_r}{E[w(X)]} = \frac{P(X \geqslant r)}{E[1/h_X(X)]}, \quad r = 1, 2, \ldots .$$

But,

$$E\left[\frac{1}{h_X(X)}\right] = \sum_{x=0}^{+\infty}\frac{1}{h_X(x)}P(X = x) = \sum_{x=0}^{+\infty}P(X \geqslant x) = E(X).$$

Therefore,

$$p_r^w = \frac{P(X \geqslant r)}{E(X)} = \frac{\sum_{j=r+1}^{+\infty} p_r}{E(X)}.$$

Consequently, by Lemma 4, $X^w\,|\,(X^w > 0)$ follows the shifted $UGWD(1,2;\rho)$ distribution if and only if X follows a shifted $Yule(\rho+1)$ distribution. Also, $X^w\,|\,(X^w > 0)$ follows the shifted $UGWD(1,2;\rho)$ distribution if and only if $X^w\,|\,(X^w > 0)$ follows a 0− truncated Yule (ρ) distribution leading to $X^w \sim \text{Yule}(\rho)$, which, as shown by Xekalaki (1983), is equivalent to $h_{X^w}(t) = \rho/(\rho+t+1)$. □

Corollary 7. *Let X, X^* be random variables taking values in $\{1,2,3,\ldots\}$. Assume that $E(X) < +\infty$ and let the probability functions of X, X^* satisfy the relationship $p_r^* = rp_r/E(X)$, $r = 1,2,\ldots$, where $p_r = P(X=r)$, $p_r^* = P(X^*=r)$, $r=1,2,\ldots$. Let Z be a non-negative integer valued random variable with probability function q_r^+ satisfying the following relationship $q_r^+ = \sum_{j=r+1}^{+\infty} p_j/E(X)$, $r = 0,1,2\ldots$. Then, the random variables X^* and $Z\,|\,(Z>0)$ follow the same distribution if and only if $h_X(t) = (\rho+1)/(\rho+t+1)$.*

5. Continuous analogues-characterizations of exponential, beta and Pareto distributions

In this section, we provide continuous analogues for most of the results obtained in earlier sections.

Let X be a continuous random variable on $[0,+\infty)$ representing the failure time of a component and let its probability density function be denoted by $f_X(t)$. The probability that the component has not failed at time t is known as the *reliability function*, $\bar{F}_X(t) = 1 - F_X(t)$.

In the sequel, we denote by $h_X(t)$ the *hazard rate* function, which reflects the instantaneous potential for an event to occur given survival up to time t and it is defined as follows:

$$h_X(t) = \frac{f_X(t)}{P(X \geqslant t)}, \quad t \geqslant 0 \quad \text{or } h_X(t) = \frac{f_X(t)}{\bar{F}_X(t)}, t \geqslant 0.$$

Further, we refer to the function

$$\mu_X(t) = E[X - t\,|\,X > t] = \frac{1}{\bar{F}_X(t)} \int_t^{+\infty} \bar{F}_X(x)\,dx, \quad t \geqslant 0$$

as the *mean residual life* of the component at time t. This is well defined whenever $E(X) < +\infty$.

Also, $v_X(t)$ is termed as the *vitality function*, and it is defined as $v_X(t) = E[X\,|\,X > t]$. Obviously, $v_X(t) = \mu_X(t) + t$. Finally, $\sigma_X^2(t) = \text{Var}[X - t\,|\,X > t]$ is the *residual life variance function*.

It is known that the distribution of a nonnegative random variable is uniquely determined by the form of its hazard rate or that of its mean residual life. Also, the hazard rate is directly related to the mean residual life, i.e. $h_X(t) = (1 + (d/dt)\mu_X(t))/\mu_X(t)$; $t \geqslant 0$ and to the vitality function, i.e. $h_X(t) = (d/dt)v_X(t)/(v_X(t) - t)$; $t \geqslant 0$.

In the sequel, we demonstrate that the results obtained in Sections 3 and 4 bear complete analogies to results concerning lifelength distributions in continuous time obtained under similar hypotheses. The geometric distribution and its continuous analogue, the exponential distribution, arise, for example, on the assumption of a constant mean residual life. On the other hand, the Pareto distribution is a continuous approximation of the Yule distribution and, as shown by Xekalaki and Panaretos (1988), there exists a duality association between Yule and Pareto distributions comparable to the duality association between the geometric and the exponential distributions. Moreover, just like the geometric distribution is a limiting case of the Yule distribution, the exponential distribution is a limiting case of the Pareto distribution. It would, therefore, be interesting to examine whether continuous versions of the specific results already proved for the Yule distribution lead to characterizations concerning the Pareto distribution with probability density function (pdf) given by

$$f_X(x) = \alpha\theta^{\alpha}x^{-(\alpha+1)}, \quad x \geqslant \theta;\ \alpha, \theta > 0. \tag{29}$$

In fact, the following theorems can be shown whose results indicate that this is the case. Some of the obtained results hold for more general forms of distributions. It is quite interesting, in particular, that as shown below (Theorem 9), a mean residual life of form (14) with $t \in [0, +\infty)$ leads to the beta distribution of the second kind (Pearson type VI) when $b > 0$, which has been shown by Irwin (1975) to be the continuous analogue of the Waring distribution. Moreover, when $b < 0$, the continuous life distribution that arises is the beta of the first kind (Pearson type I) to which the negative hypergeometric was shown by Xekalaki (1983) to provide a discrete approximation.

Definition 6. A random variable taking values in $[0, 1]$ is said to have the beta distribution of the first kind (Pearson type I) with parameters p and q if its probability density function is given by

$$f(x) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}x^{p-1}(1-x)^{q-1}, \quad 0 \leqslant x \leqslant 1, \ p, q > 0.$$

Definition 7. A nonnegative random variable X is said to have the beta distribution of the second kind (Pearson type VI) with parameters p and q if its probability density function is given by

$$f(x) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}x^{p-1}(1+x)^{-(p+q)}, \quad x > 0, \ p, q > 0.$$

Theorem 9. *Let X be a continuous random variable defined on $(0, l)$, $l \in R^{+} \cup \{+\infty\}$ with $E(X) < +\infty$. Then, its mean residual life is given by*

$$\mu_X(t) = \alpha + bt, \quad \alpha, b \in R, \ \alpha > 1, \ t \in (0, l) \tag{30}$$

if and only if the distribution of X belongs to the Pearson family of continuous distributions with a density of the form

$$f(x) = \frac{b+1}{\alpha}\left(1 + \frac{b}{\alpha}t\right)^{-(2+1/b)}, \quad t \in (0, l). \tag{31}$$

Proof. It can be observed that, by the definition of the mean residual life, (30) holds if and only if

$$(\alpha + bt)P(X > t) = \int_t^{+\infty} P(x > x)\,\mathrm{d}x,$$

which, upon differentiation with respect to t, is equivalent to

$$(b+1)P(X > t) - (\alpha + bt)f(t) = 0.$$

Differentiating again with respect to t, the latter relationship is equivalent to

$$\frac{f'(t)}{f(t)} = -\frac{2b+1}{\alpha + bt},$$

whose unique solution is (31). Hence, the theorem has been established. □

Now, observe that, as in the discrete case, (14) implies that $b \geqslant 0$ if X has an infinite support ($l = +\infty$) and $b < 0$ if X has a finite support ($l < +\infty$). Then, when $b > 0$, relationship (31) represents the probability density function of the beta distribution of the second kind (Pearson type VI) with parameters $p = 1$ and $q = 1 + 1/b$ and scale parameter b/α. This is precisely the Pareto distribution as defined by (29) for $\alpha = 1 + 1/b$ and $\theta = b/\alpha$. When $b \to 0$, (31) reduces to $f(t) = (1/\alpha)\mathrm{e}^{-t/\alpha}$, $t > 0$, leading to the well known result for the exponential lifelength distribution. Finally, if $b < 0$ whence $l < +\infty$, (31) represents the probability density function of the beta distribution of the first kind (Pearson type I) with parameters $p = 1$ and $q = -1/b$ and scale parameter $-b/\alpha$.

Corollary 8. *Let X be defined as in Theorem 9. Then, X follows the uniform distribution on $(0, \alpha)$ if and only if its mean residual life is a linear function of time given by*

$$\mu_X(t) = \alpha - t, \quad \alpha \in R,\ a > 1,\ t \in (0, \alpha)$$

or, equivalently, if and only if its vitality function is constant and, in particular,

$$v_X(t) = \alpha,\ t \in (0, \alpha).$$

Note that results analogous to those of Lemma 1 in the discrete case, hold in the continuous case too as summarized by the following lemma.

Lemma 5. *Let X be defined on $(0, l)$, $l \in R^+ \cup \{+\infty\}$ and such that $p_0 = P(X = 0) < 1$, $E(X) < +\infty$ and $h_X(t) = 1/(\alpha + bt)$, $\alpha, b \in R$, $\alpha > 0$, $t \in (0, l)$. Then,* (i) *$\alpha > 1$,* (ii) *$b < 1$,* (iii) *$E(X) = \alpha/(1-b)$ and* (iv) *X has a distribution with finite support ($l < +\infty$) if and only if $b < 0$ and, in this case, $b = -\alpha/l$.*

Then, by analogy to the results of Theorem 4, the following results can be shown.

Theorem 10. *Let X be a nonnegative random variable defined as in Lemma 5. Assume that $E(X) = m < +\infty$. Then,*

$$h_X(t)\mu_X(t) = c, \quad t \in (0, l),\ c \in R$$

if and only if one of the following statements holds:

(i) *X has an exponential distribution with parameter $1/(1+m)$, for $c=1$*
(ii) *X has a beta distribution of the second kind with parameters $p=1$, $q=c/(c-1)$ and scale parameter $(c-1)/(c+m)$, for $c>1$*
(iii) *X has a beta distribution of the first kind with parameters $p=1$, $q=c/(1-c)$ and scale parameter $(1-c)/(c+m)$, for $c<1$.*

Summarizing results that are already known in the literature as well as the ones shown in this section, the following hold for the Pareto distribution.

Corollary 9. *Let X be a continuous random variable defined on $[\theta,+\infty)$, $\theta>0$. Then, X follows the Pareto (θ,α) distribution, $\theta,\alpha>0$ with pdf given by relation (28) if and only if:*

(i) *the hazard rate is inversely proportional to time*
and, provided that $E(X)<+\infty$,
if and only if anyone of the following conditions is satisfied
(ii) *the mean residual life is a linear function of time*
(iii) *the product of the hazard rate by the mean residual life is constant.*

Proof. For a proof of statement (i), see Xekalaki (1983). Statement (ii) follows immediately from Theorem 9, while statement (iii) is a direct consequence of Theorem 10. (Note that a multivariate version of (ii) can be found in Jupp and Mardia, 1982).

Before proceeding with the remaining continuous analogues, let us note the following:

A random variable X^w is said to have the weighted distribution corresponding to a positive random variable X with a weight function $w(x)$ if its probability density function is given by $f_X^w(x)=w(x)f_X(x)/E[w(X)]$, provided that $E[w(X)]$ exists. Here, we restrict lifetimes to density functions $f_X(x)\neq 0$, $x>0$.

Theorem 11. *Let X, X^* be random variables defined on $[1,+\infty)$. Assume that $E(X)<+\infty$ and let the probability density functions of X, X^* satisfy the condition $f_X^*(x)=xf_X(x)/E(X)$, $x\geqslant 1$. Then X follows the Pareto $(1,\alpha)$ distribution, $\alpha>1$ if and only if*

$$\frac{h_X(t)}{h_{X^*}(t)}=\frac{\alpha}{\alpha-1}. \tag{32}$$

Theorem 12. *Let X, X^* be random variables defined on $[1,+\infty)$. Assume that $E(X)<+\infty$ and let the probability density functions of X, X^* satisfy the condition $f_X^*(x)=xf_X(x)/E(X)$, $x\geqslant 1$. Then, X follows the Pareto $(1,\alpha)$ distribution, $\alpha>2$ if and only if anyone of the following relations holds:*

1. $h_X(t)-h_{X^*}(t)=1/t$
2. $\mu_X(t)/\mu_{X^*}(t)=(\alpha-2)/(\alpha-1)$

3. $\mu_{X^*}(t) - \mu_X(t) = t/((\alpha-1)(\alpha-2))$
4. $h_X(t)/h_{X^*}(t) - \mu_X(t)/\mu_{X^*}(t) = 2/(\alpha-1)$
5. $v_{X^*}(t) - v_X(t) = t/((\alpha-1)(\alpha-2))$
6. $v_X(t)/v_{X^*}(t) = \alpha/(\alpha-1)$.

Theorem 13. *Let X, X^w be random variables defined on $[\theta, +\infty); \theta > 0$ with $E(X) < +\infty$ and probability density functions $f_X(x)$ and $f_X^w(x)$, respectively. Assume that, $f_X^w(x) = w(x)f_X(x)/E[w(X)]$, $x \geqslant \theta$ with $w(X) = 1/h_X(X)$. Then, X is distributed according to the Pareto (θ,α) distribution, $\theta, \alpha > 0$ if and only if $h_{X^w}(t) = (\alpha-1)/t$; $t \geqslant \theta$, $\theta, \alpha > 0$.*

Acknowledgements

We would like to thank the referees for constructive comments.

References

Barlow, R.E., Proscham, F., 1965. Mathematical Theory of Reliability. Wiley, New York.

Basu, A.P., 1971. Bivariate failure rate. J. Amer. Statist. Assoc. 66, 103–104.

Block, H.W., 1977. Multivariate reliability classes. In: Krishnaiah, P.R. (Ed.), Applications in Statistics. North Holland, Amsterdam, pp. 79–88.

Block, H.W., Savits, T.H., 1981. Multivariate classes of life distributions in reliability theory. Math. Oper. Res. 6 (3), 453–461.

Brindley, E.C., Thompson, W.A., 1972. Dependence and ageing aspects of multivariate survival. J. Amer. Statist. Assoc. 67, 822–830.

Bryson, M.C., Siddiqui, M.M., 1969. Some criteria for ageing. J. Amer. Statist. Assoc. 64, 1472–1483.

Cox, D.R., Oakes, D., 1984. Analysis of Survival Data. Chapman and Hall, London.

Crowder, M.J., Kimber, A.C., Smith, R.L., Sweeting, T.J., 1991. Statistical Analysis of Reliability Data. Chapman and Hall, London.

Dimaki, C., Xekalaki, E., 1996. Towards a unification of certain characterizations by conditional expectations. Ann. Inst. Statist. Math. 48, 157–168.

Ecstein, Z., Wolpin, K.I., 1990. Estimating a market equilibrium search model from panel data on individuals. Econometrica 58 (4), 783–808.

Epstein, B., Sobel, M., 1954. Some theorems relevant to life testing from an exponential distribution. Ann. Math. Statist. 25, 373–381.

Esary, J.D., Marshall, A.W., 1979. Multivariate distributions with increasing hazard rate average. Ann. Probab. 7 (2), 359–370.

Friday, D.S., Patil, G.P., 1977. A bivariate exponential model with applications to reliability and computer generation of random variables. In: Tsokos, C.P., Shimi, I.N. (Eds.), The Theory and Applications of Reliability, Vol. 1. Academic Press, New York, pp. 527–549.

Glänzel, W., Telcs, A., Schubert, A., 1984. A characterization by truncated moments and its application to Pearson-type distributions. Z. Wahrsch. Verw. Gebiete 66 (2), 173–183 (See also, Glanzel, W., Telcs, A., Schubert, A. Correction to: "A characterization by truncated moments and its application to Pearson-type distributions" [Z. Wahrsch. Verw. Gebiete, 66 (1984), (2), 173–183; MR 86b:62022]; Probab. Theory Related Fields 74 (1987), (2), 317.62E10).

Gupta, R.C., 1979. Waiting time paradox and size biased sampling. Comm. Statist.—Theory and Methods 8, 601–607.

Gupta, R.C., Keating, J.P., 1986. Relations for reliability variance and their applications in reliability. J. Statist. Plann. Inference 16, 329–335.

Harris, R., 1970. A multivariate definition for increasing hazard rate distribution functions. Ann. Math. Statist. 41 (2), 713–717.

Irwin, J.O., 1975. The generalized Waring distribution. J. Roy. Statist. Soc. A (Part I) 138, 18–31; (Part II) 204–227; (Part III) 374–384.

Jain, K., Singh, H., Bagai, I., 1989. Relations for reliability measures of weighted distributions. Comm. Statist.—Theory and Methods 18 (2), 4393–4412.

Johnson, N.L., Kotz, S., Kemp, A.W., 1993. Univariate Discrete Distributions. Wiley, New York.

Jupp, P.E., Mardia, K.V., 1982. A characterization of the multivariate Pareto distribution. Ann. Statist. 10, 1021–1024.

Kalbfleisch, J., Prentice, R.L., 1980. The Statistical Analysis of Failure Time Data. Wiley, New York.

Kleinbaum, D.G., 1995. Survival Analysis. Springer, New York.

Kupta, J., Loo, S., 1989. The hazard and vitality measures of ageing. J. Appl. Probab. 26, 532–542.

Laurent, A.G., 1975. Failure and mortality from wear and ageing. In: Patil, G.P., Kotz, S., Ord, J.K. (Eds.), The Teissier Model. Statistical Distributions in Scientific Work, Vol. 2. Reidel, Dordrecht and Boston, pp. 301–320.

Lawless, J.F., 1982. Statistical Models and Methods for Lifetime Data. Wiley, New York.

Mann, N.R., Schafer, R.E., Singpurwalla, N.D., 1974. Methods for Statistical Analysis of Reliability and Lifetime Data. Wiley, New York.

Marshall, A.W., 1975. Multivariate distributions with monotone hazard rate. In: Barlow, R., Fussel, J., Singpurwalla, N. (Eds.), Reliability and Fault Tree Analysis. SIAM, Philadelphia, pp. 259–284.

Morrison, D.G., 1978. On linearly increasing mean residual lifetimes. J. Appl. Probab. 15, 617–620.

Muth, E.G., 1977. Reliability models with positive memory derived from the mean residual life function. In: Tsokos, C.P., Shimi, I.N. (Eds.), The Theory and Applications of Reliability, Vol. II. Academic Press, New York, pp. 401–435.

Narendranathan, W., Stewart, M.B., 1993. Modelling the probability of leaving unemployment: competing risks models with flexible base-line hazards. Appl. Statist. 42 (1), 63–83.

Reinhardt, H.E., 1968. Characterizing the exponential distribution. Biometrics 24, 437–438.

Roy, D., Mukherjee, S.P., 1986. A note on characterizations of the Weibull distribution. Sankhya, Series A 48 (2), 250–253.

Thompson, W.A., 1981. On the foundations of reliability. Technometrics 23 (1), 1–13.

Xekalaki, E., 1983. Hazard functions and life distributions in discrete time. Comm. Statist.—Theory Methods 12, 2503–2509.

Xekalaki, E., 1984. Linear regression and the Yule distribution. J. Econometrics 24, 397–403.

Xekalaki, E., Panaretos, J., 1988. On the association of the Pareto and the Yule distributions. Teor. Veroyatnast. i Primenen. 33 (1), 206–210 (also in: Theory of Probability and its Applications).