ON THE INDEPENDENCE OF THE STANDARDIZED ONE-STEP-AHEAD PREDICTION ERRORS IN ARCH MODELS

Stavros Degiannakis• and Evdokia Xekalaki*

***Abstract*-- In statistical modeling contexts, the use of one-step-ahead prediction errors for testing hypotheses on the forecasting ability of an assumed model has been widely considered (see, e.g. Xekalaki et al. (2003, in Stochastic Musings, J.Panaretos (ed.), Laurence Erlbaum), Degiannakis and Xekalaki (2005, Journal of Applied Stochastic Models in Business and Industry). Quite often, the testing procedure requires independence in a sequence of recursive standardized prediction errors, which cannot always be readily deduced particularly in the case of econometric modeling. In this paper, the results of a series of Monte Carlo simulations reveal that independence can be assumed to hold. They are also indicative of a chi-square distribution for the sum of squared standardized one-step-ahead prediction errors. Some theoretical justification of these findings can be traced in Degiannakis and Xekalaki's results (2005, Journal of Applied Stochastic Models in Business and Industry).**

***Index terms*-- ARCH models, Standardized Prediction Error Criterion, Model selection, Monte Carlo Simulation, Predictability, One-step-ahead Prediction Errors.**

I. INTRODUCTION

Degiannakis and Xekalaki (2005), based on the standardized prediction error criterion (SPEC) proposed a model selection algorithm for ARCH models. The results in Degiannakis and Xekalaki (2001b) show that the SPEC model selection procedure has a satisfactory performance in selecting the model that generates "better" volatility predictions. Moreover, the SPEC algorithm was applied on a simulated options market (Xekalaki and Degiannakis 2005) and on trading S&P500 options on a daily base (Degiannakis and Xekalaki 2001a). In both studies, an agent based on the SPEC model selection algorithm achieves the highest performance. The general conclusion is that the prediction performance improves if one switches models over time, as *jumping* reflects learning in the marketplace. The switching from one model to another governed by the SPEC model selection rule (at each of a sequence of points in time, use the model with the lowest sum of squared standardized one-step-ahead prediction errors) achieves the highest predictive ability performance. However, model selection procedures based on standardized one-step-ahead prediction errors often require independence in a sequence of recursive standardized prediction errors, which cannot always be readily deduced particularly in the case of econometric modeling. In this paper, on the basis of the results of a series of Monte Carlo simulations, it is conjectured that independence holds and the sum of squared standardized one-step-ahead prediction errors is chi-square distributed.

II. THE ARCH PROCESS

An ARCH process, ε_t, is presented as:

$$\begin{gathered} \varepsilon_t = z_t \sigma_t \\ z_t \overset{i.i.d.}{\sim} N(0,1) \\ \sigma_t^2 = g(\sigma_{t-1}, \sigma_{t-2}, ..., \varepsilon_{t-1}, \varepsilon_{t-2}, ...) \end{gathered} \tag{1}$$

where z_t is a sequence of independently and identically distributed random variables with autocorrelation, $Cor(z_t, z_{t+\tau})$, approximately $N(0, T^{-1})$ distributed, σ_t is a time-varying, positive measurable function of the information set at time $t-1$ and $g(.)$ could be a linear or nonlinear functional form that has been presented in the ARCH literature.

Since very few financial time series have a constant conditional mean of zero, an ARCH model can be presented in a κ^{th} order autoregressive form by letting ε_t be the innovation process in a linear regression:

$$\begin{gathered} y_t = \sum_{i=1}^{\kappa} (c_i y_{t-i}) + \varepsilon_t \\ \varepsilon_t \mid I_{t-1} \equiv z_t \sigma_t \sim N(0, \sigma_t^2) \\ z_t \overset{i.i.d.}{\sim} N(0,1) \\ \sigma_t^2 = g(\sigma_{t-1}, \sigma_{t-2}, ..., \varepsilon_{t-1}, \varepsilon_{t-2}, ...) \end{gathered} \tag{2}$$

The disturbances, ε_t, are normally distributed with time varying conditional variance $\sigma_t^2 = E_{t-1}(\varepsilon_t^2)$. The most commonly used conditional variance function is the GARCH(1,1) model: $\sigma_t^2 = a_0 + a_1 \varepsilon_{t-1}^2 + b_1 \sigma_{t-1}^2$.

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A wide range of proposed ARCH models is covered in surveys such as Bera and Higgins (1993), Bollerslev et al. (1992), Bollerslev et al. (1994), Degiannakis and Xekalaki (2004), Gourieroux (1997) and Hamilton (1994).

III. Monte Carlo Study: Simulation of the AR(1)GARCH(1,1) Process

In the sequel, a Monte Carlo simulation is used to provide evidence for the assumption of independently and identically distributed standardized one-step-ahead prediction errors. The procedure consists of three stages:

1. Generate data from the AR(1)GARCH(1,1) process

- Generate a series of 32000 values from the standard normal distribution, i.e. $z_t \overset{i.i.d.}{\sim} N(0,1)$.
- Generate an equal number of values $\{\varepsilon_t\}_{t=1}^{32000}$ of the innovation ARCH process, by multiplying the collection $\{z_t\}_{t=1}^{32000}$ by a specific conditional variance form, or $\varepsilon_t = z_t \sqrt{\sigma_t^2}$, for $\sigma_t^2 = 0.0001 + 0.12\varepsilon_{t-1}^2 + 0.8\sigma_{t-1}^2$.
- Generate a first order autoregressive processes, $y_t = 0.06 y_{t-1} + \varepsilon_t$, for the conditional mean, based on the values $\{\varepsilon_t\}_{t=1}^{32000}$ of the innovation process.

Figure 1 plots the simulated processes, Figure 2 presents the relevant histograms and descriptive statistics, while Figure 3 depicts the histograms of the chi-square distribution with T degrees of freedom. The chi-square distributed process, with T degrees of freedom, is constructed as $\sum_{t=1}^{T} z_t^2$. According to results obtained in literature (e.g. Engle and Mustafa 1992), the shocks to the variance, $E_t(\varepsilon_t^2) - E_{t-1}(\varepsilon_t^2) = \varepsilon_t^2 - \sigma_t^2 \equiv v_t$, generate a martingale difference sequence (in the sense that it cannot be predicted from its past). These shocks are neither serially independent nor identically distributed. Looking at the autocorrelations of the variables of interest, the sequences generated by z_t are serially uncorrelated, while those generated by the shocks to the variance v_t are autocorrelated, and those by the conditional variance σ_t^2 highly correlated. As z_t defines a sequence of i.i.d. random variables, the transformations of z_t , ($|z_t|^d$, $\forall d > 0$), are uncorrelated in each case. Figure 4 presents the autocorrelation of transformations of the processes defined by $z_t, v_t, \sigma_t, \varepsilon_t$. The half-length of the 95% confidence interval for the estimated sample autocorrelation equals $1.96/\sqrt{T} = 0.0113$, in the case of a process with independently and identically normally distributed components. On the other hand, the processes defined by σ_t^2 is autocorrelated at any lag, while both of those defined by v_t and ε_t are autocorrelated in half of the cases.

Ding and Ganger (1996) and Karanasos (1996) give the autocorrelation function of the squared errors for the GARCH(1,1) process and Karanasos (1999) extends the results to the GARCH(p,q) model. He and Teräsvirta (1999) derive the autocorrelation function of the squared and absolute errors for a family of first order ARCH processes.

The percentages of estimated autocorrelations that are outside the 95% confidence interval are presented in Table 1.

Table 1. Percentages of autocorrelations outside the 95% confidence interval ($\tau = 1,2,...,100$)

$d =$	0.5	1	1.5	2	2.5	3
$Cor(\lvert z_t\rvert^d, \lvert z_{t+\tau}\rvert^d)$	6%	6%	5%	4%	5%	6%
$Cor(\lvert \varepsilon_t\rvert^d, \lvert \varepsilon_{t+\tau}\rvert^d)$	36%	44%	45%	47%	47%	44%
$Cor(\lvert v_t\rvert^d, \lvert v_{t+\tau}\rvert^d)$	56%	54%	44%	30%	20%	15%
$Cor(\sigma_t^d, \sigma_{t+\tau}^d)$	92%	97%	98%	95%	93%	86%

2. Estimate the parameters of the AR(1)GARCH(1,1) model

- The AR(1)GARCH(1,1) model is applied, for the data produced from the AR(1)GARCH(1,1) process. Dropping out the first 1000 data, maximum likelihood estimates of the parameters are obtained by numerical maximization of the log-likelihood function, using a rolling sample of constant size equal to 1000[1]. At each of a sequence of points in time, the maximum likelihood parameter vector, $\hat{\theta}_t \equiv (\hat{c}_{1,t}, \hat{a}_{0,t}, \hat{a}_{1,t}, \hat{b}_{1,t})$, is being estimated in order to compute the conditional mean and variance:

$$\begin{aligned} \hat{y}_{t+1|t} &= \hat{c}_{1,t} y_t \\ \hat{\sigma}_{t+1|t}^2 &= \hat{a}_{0,t} + \hat{a}_{1,t}\varepsilon_{t|t}^2 + \hat{b}_{1,t}\sigma_{t|t}^2 \, . \end{aligned} \tag{3}$$

Thus, the model is estimated 30000 times. Note that $\varepsilon_{t|t}^2$ and $\sigma_{t|t}^2$ belong to the I_t , so are considered as observable.

3. Compute the standardized one-step-ahead prediction errors, $\hat{z}_{t+1|t} = (y_{t+1} - \hat{y}_{t+1|t})\hat{\sigma}_{t+1|t}^{-1}$

The SPEC model selection algorithm uses the sum of the squared standardized one-step-ahead prediction errors, or $\sum_{t=1}^{T} \hat{z}_{t+1|t}^2$. According to Degiannakis and Xekalaki (2005), under the assumption of constancy of parameters over time, $(\hat{\theta}_t) = (\hat{\theta}_{t+1}) = ... = (\hat{\theta}_T) = (\hat{\theta})$, the estimated standardized one-step-ahead prediction errors $\hat{z}_{t+1|t}, \hat{z}_{t+2|t+1}, ..., \hat{z}_{T+1|T}$ are asymptotically independently standard normally distributed.

- The one-step-ahead estimated processes are presented in Figure 5. Figure 6 presents the relevant histograms and the descriptive statistics. The one-step-

[1] Maximum likelihood estimates of the parameters are obtained by numerical maximization of the log-likelihood function using the Marquardt algorithm (Marquardt 1963). The quasi-maximum likelihood estimator (QMLE) is used, as according to Bollerslev and Wooldridge (1992), it is generally consistent, has a normal limiting distribution and provides asymptotic standard errors that are valid under non-normality.

ahead standardized prediction error process, conditional on the information set available at time t, $\hat{z}_{t+1|t} = \left(y_{t+1} - \hat{y}_{t+1|t}\right)\hat{\sigma}_{t+1|t}^{-1}$, is approximately normally distributed, while $\hat{z}_{t+1|t}^2$ is chi-square distributed with 1 degree of freedom. Moreover, if $\hat{z}_{t+1|t}^2$ is independently chi-square distributed, $\sum_{t=1}^{T} \hat{z}_{t+1|t}^2$ should also be chi-square distributed with T degrees of freedom, and mean and variance:

$$E\left[\sum_{t=1}^{T} \hat{z}_{t+1|t}^2\right] = T \text{ and } V\left[\sum_{t=1}^{T} \hat{z}_{t+1|t}^2\right] = 2T . \tag{4}$$

Figure 7 plots the histograms of $\sum_{t=1}^{T} \hat{z}_{t+1|t}^2$. All the histograms are almost identical to the simulated chi-square histograms. Moreover, if $\hat{z}_{t+1|t}$ is a sequence of i.i.d. variables, then the sample autocorrelation, $Cor\left(\hat{z}_{t+1|t}, \hat{z}_{t+\tau+1|t+\tau}\right)$, is approximately $N\left(0, T^{-1}\right)$ distributed and the autocorrelation of any transformation of $\hat{z}_{t+1|t}$, $Cor\left(\left|\hat{z}_{t+1|t}\right|^d, \left|\hat{z}_{t+\tau+1|t+\tau}\right|^d\right)$, $\forall d > 0$, is also $N\left(0, T^{-1}\right)$ distributed. Figure 8 presents the autocorrelation of transformations of the processes $\hat{z}_{t+1|t}, \hat{\varepsilon}_{t+1|t}, \hat{v}_{t+1|t}, \hat{\sigma}_{t+1|t}$. Since the sum of squared standardized one-step-ahead prediction errors is chi-square distributed, and the transformations of $\hat{z}_{t+1|t}$ are not autocorrelated, our findings point towards the independence of the standardized one-step-ahead innovations, $\hat{z}_{t+1|t}$.

IV. Monte Carlo Study: Simulation of the GARCH, EGARCH and TARCH Processes

In the sequel, the assumption that the standardized one-step-ahead prediction errors are independently and identically distributed (or equivalently that the sum of T one-step-ahead prediction errors is chi-square distributed) is investigated for higher order of autoregressive processes for the conditional mean and conditional variance functions of the following types:

The GARCH(p,q) model, Bollerslev (1986):

$$\sigma_t^2 = a_0 + \sum_{i=1}^{q}\left(a_i \varepsilon_{t-i}^2\right) + \sum_{i=1}^{p}\left(b_i \sigma_{t-i}^2\right). \tag{5}$$

The EGARCH(p,q) model, Nelson (1991):

$$\ln\left(\sigma_t^2\right) = a_0 + \sum_{i=1}^{q}\left(a_i \left|\frac{\varepsilon_{t-i}}{\sigma_{t-i}}\right| + \gamma_i \left(\frac{\varepsilon_{t-i}}{\sigma_{t-i}}\right)\right) + \sum_{i=1}^{p}\left(b_i \ln\left(\sigma_{t-i}^2\right)\right) \tag{6}$$

The TARCH(p,q) model, Glosten et al. (1993):

$$\sigma_t^2 = a_0 + \sum_{i=1}^{q}\left(a_i \varepsilon_{t-i}^2\right) + \gamma_1 \varepsilon_{t-1}^2 d_{t-1} + \sum_{i=1}^{p}\left(b_i \sigma_{t-i}^2\right), \tag{7}$$

where $d_t = 1$ if $\varepsilon_t < 0$, and $d_t = 0$ otherwise.

The procedure followed is comprised of the following steps:

1. Eight processes have been generated with coefficients presented in Table 2

Table 2. Coefficients of the simulated processes

Model	c_1	c_2	c_3	a_0	a_1	a_2	b_1	γ_1
AR(1) GARCH(1,1)	0.05	-	-	0.002	0.05	-	0.91	-
AR(1) EGARCH(1,1)	0.05	-	-	0.2	0.05	-	0.2	0.1
AR(1) TARCH(1,1)	0.05	-	-	0.002	0.15	-	0.7	-0.08
AR(1) GARCH(1,2)	0.05	-	-	0.002	0.05	0.08	0.8	-
AR(1) TARCH(1,2)	0.05	-	-	0.002	0.15	0.05	0.7	-0.08
AR(3) GARCH(1,1)	0.1	0.03	-0.02	0.002	0.05	-	0.91	-
AR(3) EGARCH(1,1)	0.12	0.07	-0.03	0.001	0.05	-	0.2	0.1
AR(3) TARCH(1,1)	0.1	0.03	-0.02	0.002	0.15	-	0.7	-0.08

2. Estimate the parameters of the simulated processes

- At each of a sequence of points in time, the maximum likelihood parameter vector $\hat{\theta}_t \equiv \left(\hat{c}_{1,t}, \hat{c}_{2,t}, \hat{c}_{3,t}, \hat{a}_{0,t}, \hat{a}_{1,t}, \hat{a}_{2,t}, \hat{b}_{1,t}, \hat{\gamma}_{1,t}\right)$ is being estimated. The models are estimated 30000 times and the conditional mean and variance are computed in (8)-(11):

The κ^{th} order Autoregressive process

$$\hat{y}_{t+1|t} = \sum_{i=1}^{\kappa}\left(\hat{c}_{i,t} y_{t+1-i}\right). \tag{8}$$

The GARCH(1,q) model

$$\hat{\sigma}_{t+1|t}^2 = \hat{a}_{0,t} + \sum_{i=1}^{q}\left(\hat{a}_{i,t} \varepsilon_{t-i+1|t}^2\right) + \hat{b}_{1,t} \sigma_{t|t}^2 . \tag{9}$$

The EGARCH(1,1) model

$$\hat{\sigma}_{t+1|t}^2 = \exp\left(\hat{a}_{0,t} + \hat{a}_{1,t}\left|\frac{\varepsilon_{t|t}}{\sigma_{t|t}}\right| + \hat{\gamma}_{1,t}\left(\frac{\varepsilon_{t|t}}{\sigma_{t|t}}\right) + \hat{b}_{1,t} \ln\left(\sigma_{t|t}^2\right)\right) \tag{10}$$

The TARCH(1,q) model

$$\hat{\sigma}_{t+1|t}^2 = \hat{a}_{0,t} + \sum_{i=1}^{q}\left(\hat{a}_{i,t} \varepsilon_{t-i+1|t}^2\right) + \hat{\gamma}_{1,t} \varepsilon_{t|t}^2 d_t + \hat{b}_{1,t} \sigma_{t|t}^2 , \tag{11}$$

where $d_t = 1$ if $\varepsilon_t < 0$, and $d_t = 0$ otherwise.

3. Compute the standardized one-step-ahead prediction errors $\hat{z}_{t+1|t} = \left(y_{t+1} - \hat{y}_{t+1|t}\right)\hat{\sigma}_{t+1|t}^{-1}$ and examine the following properties:

- Histogram, mean and variance of $\left\{\hat{z}_{t+1|t}^2\right\}_{t=1}^{30000}$.
- Histogram, mean and variance of $\left\{\sum_{j=t-T+1}^{t} \hat{z}_{j+1|j}^2\right\}$, for $t = T(T)30000$.[2]
- Sample autocorrelation of transformations of $\hat{z}_{t+1|t}$, $Cor\left(\left|\hat{z}_{t+1|t}\right|^d, \left|\hat{z}_{t+\tau+1|t+\tau}\right|^d\right)$, for $\tau = 1(1)100$ and $d = 0.5(0.5)3$.

[2] Here, $T = a(b)c$ denotes $T = a, a+b, a+2b, \ldots, c-b, c$.

Due to space limitations the histograms of $\{\hat{z}^2_{t+1|t}\}_{t=1}^{30000}$, $\{\sum_{j=t-T+1}^{t}\hat{z}^2_{j+1|j}\}$ for, $t = T(T)30000$, and the autocorrelation of the processes $Cor\left(\left|\hat{z}_{t+1|t}\right|^d, \left|\hat{z}_{t+\tau+1|t+\tau}\right|^d\right)$, for $\tau = 1(1)100$ and $d = 0.5(0.5)3$, for each of the eight generated processes are not presented here, but they are available in Degiannakis and Xekalaki (2003). The evidence from our findings is in support of the hypothesis of independently and identically distributed standardized one-step-ahead prediction errors in this case too.

V. Monte Carlo Study: Simulation of the GARCH(1,1) Process for Various Coefficient Values

In this section, one more set of GARCH(1,1) processes is simulated in order to investigate if changes in the coefficients affect the distribution of squared standardized one-step-ahead prediction errors. The steps followed are:

- Generate 18 series of 20000 values from the standard normal distribution $z_t \overset{i.i.d.}{\sim} N(0,1)$.
- Generate a series $\{\varepsilon_t\}_{t=1}^{20000}$ of 20000 values for each of 18 innovation GARCH(1,1) processes by multiplying the generated values of z_t by σ_t from $\sigma_t^2 = 0.002 + 0.05\varepsilon_{t-1}^2 + b_1^{(k)}\sigma_{t-1}^2$, where $b_1^{(k)} = 0.05 * k$ for $k = 1,2,...,18$.
- Estimate the parameters of the 18 GARCH(1,1) models.
- Compute $\hat{z}_{t+1|t} = (y_{t+1} - \hat{y}_{t+1|t})\hat{\sigma}^{-1}_{t+1|t}$.

The histograms of $\{\sum_{j=t-T+1}^{t}\hat{z}^2_{j+1|j}\}$, for $t = T(T)20000$ and the autocorrelation functions $Cor\left(\left|\hat{z}_{t+1|t}\right|^d, \left|\hat{z}_{t+\tau+1|t+\tau}\right|^d\right)$, $\tau = 1(1)100$ and $d = 0.5(0.5)3$, are similar to these plotted in section III.

VI. Conclusion

As seen above, our findings are in support of the hypothesis of independence of the $\hat{z}_{t+1|t}$ and of a sum of squared standardized one-step-ahead prediction errors being chi-square distributed. Moreover, changes in the types of conditional variance function, the order of the autoregressive process of the conditional mean and as well as the values of the coefficients considered do not appear to affect these findings.

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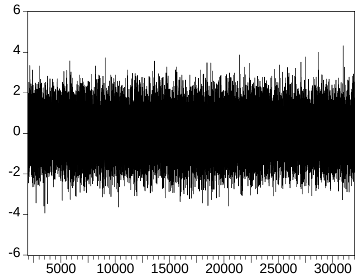
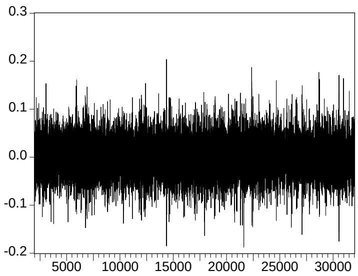
FIGURES

Figure 1. The simulated processes

The simulated z_t values ($\{z_t\}_{t=2000}^{32000}$)



The simulated ε_t values ($\{\varepsilon_t\}_{t=2000}^{32000}$)



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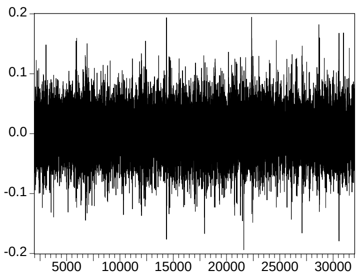


The simulated σ_t^2 values ($\{\sigma_t^2\}_{t=2000}^{32000}$)

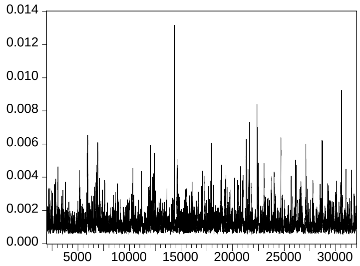
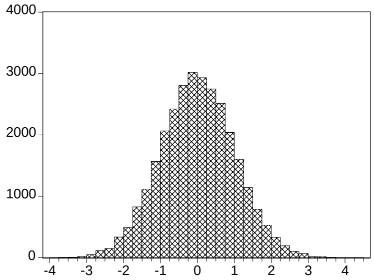


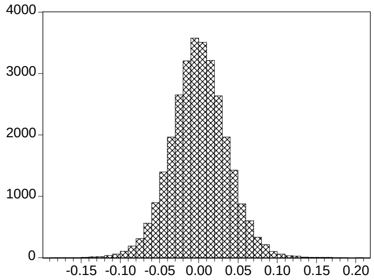
Figure 2. Histograms and descriptive statistics of the simulated processes

The simulated z_t values ($\{z_t\}_{t=2000}^{32000}$)



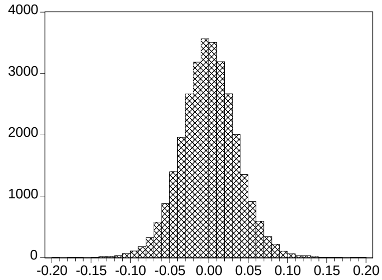
Mean	0.006139
Median	0.001398
Maximum	4.327886
Minimum	-3.948533
Std. Dev.	0.999047
Skewness	0.015896
Kurtosis	3.004320

The simulated ε_t values ($\{\varepsilon_t\}_{t=2000}^{32000}$)



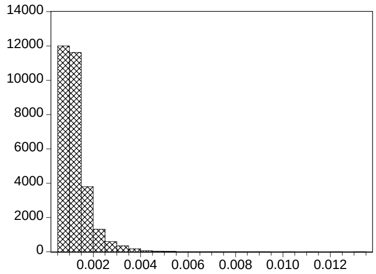
Mean	0.000218
Median	3.81E-05
Maximum	0.204209
Minimum	-0.187684
Std. Dev.	0.035500
Skewness	0.017497
Kurtosis	3.703915

The simulated y_t values ($\{y_t\}_{t=2000}^{32000}$)



Mean	0.000232
Median	6.16E-05
Maximum	0.194647
Minimum	-0.194074
Std. Dev.	0.035562
Skewness	0.023013
Kurtosis	3.712639

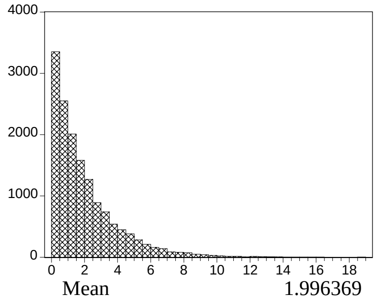
The simulated σ_t^2 values ($\{\sigma_t^2\}_{t=2000}^{32000}$)



Mean	0.001256
Median	0.001089
Maximum	0.013171
Minimum	0.000536
Std. Dev.	0.000620
Skewness	3.575072
Kurtosis	30.14328

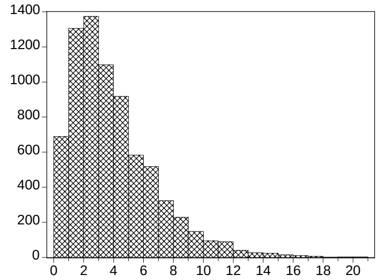
Figure 3. Histograms of simulated chi-square distributed process with T degrees of freedom

$T = 2$



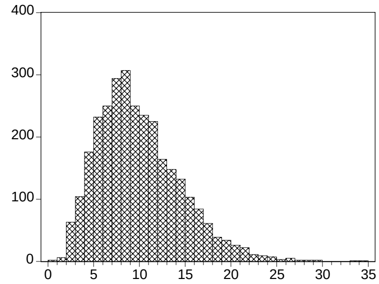
Mean	1.996369
Variance	3.997227
Observations	15000.00

$T = 4$



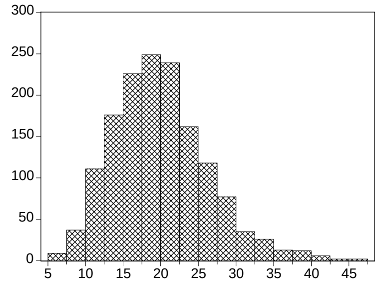
Mean	3.992739
Variance	8.121678
Observations	7500.000

$T = 10$



Mean	9.981847
Variance	20.56190
Observations	3000.000

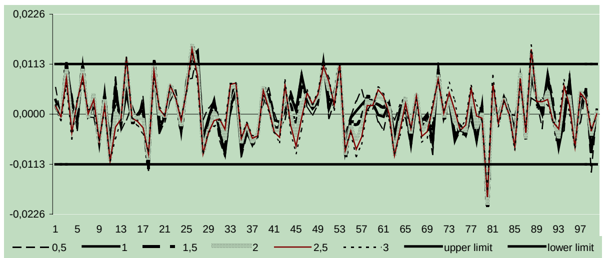
$T = 20$



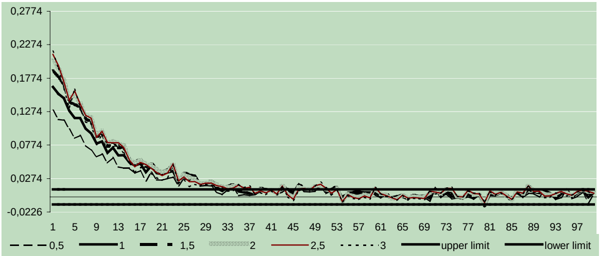
Mean	19.96369
Variance	39.83026
Observations	1500.000

Figure 4. Autocorrelation of transformations of the processes z_t, ε_t, v_t, σ_t

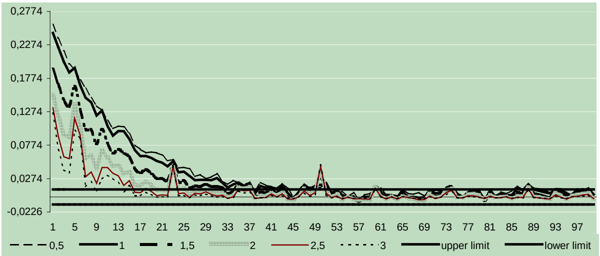
$$Cor\left(|z_t|^d, |z_{t+\tau}|^d\right),\ d = 0.5(0.5)3,\ \tau = 1(1)100$$



$$Cor\left(\left|\varepsilon_t\right|^d, \left|\varepsilon_{t+\tau}\right|^d\right),\ d = 0.5(0.5)3,\ \tau = 1(1)100$$



$$Cor\left(\left|v_t\right|^d, \left|v_{t+\tau}\right|^d\right),\ d = 0.5(0.5)3,\ \tau = 1(1)100$$



$$Cor\left(\sigma_t^d, \sigma_{t+\tau}^d\right),\ d = 0.5(0.5)3,\ \tau = 1(1)100$$

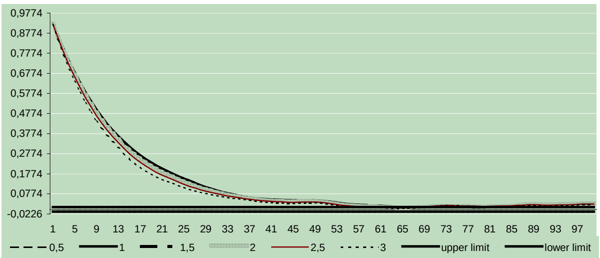
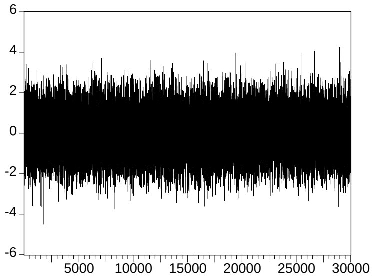
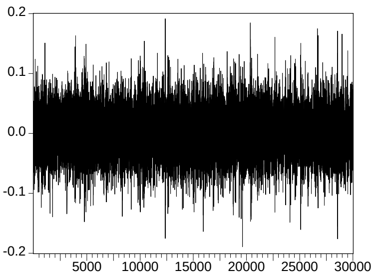


Figure 5. The one-step-ahead estimated processes

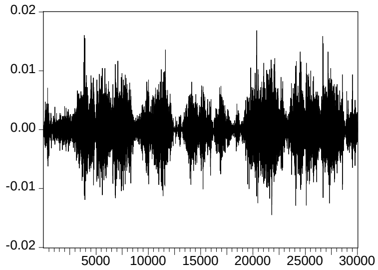
The estimated z_t values ($\left\{\hat{z}_{t+1|t}\right\}_{t=1}^{30000}$)



The estimated ε_t values ($\left\{\hat{\varepsilon}_{t+1|t}\right\}_{t=1}^{30000}$)



The estimated y_t values ($\left\{\hat{y}_{t+1|t}\right\}_{t=1}^{30000}$)



The estimated σ_t^2 values ($\left\{\hat{\sigma}_{t+1|t}^2\right\}_{t=1}^{30000}$)

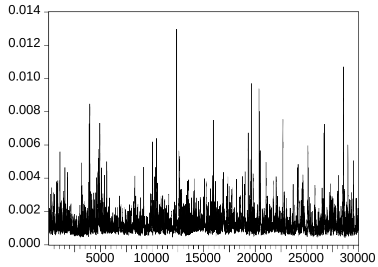
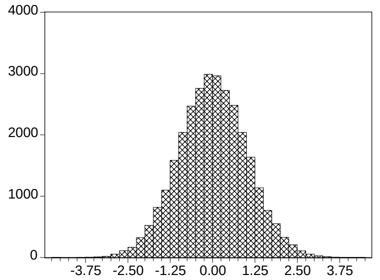


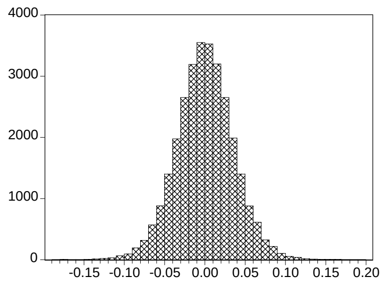
Figure 6. Histograms and descriptive statistics of the one-step-ahead estimated processes

The estimated z_t values ($\left\{\hat{z}_{t+1|t}\right\}_{t=1}^{30000}$)



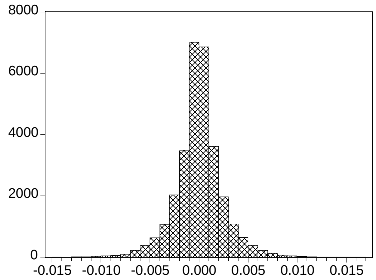
Mean	0.006218
Median	0.003156
Maximum	4.256031
Minimum	-4.508652
Std. Dev.	1.004477
Skewness	0.016695
Kurtosis	3.027268

The estimated ε_t values ($\left\{\hat{\varepsilon}_{t+1|t}\right\}_{t=1}^{30000}$)



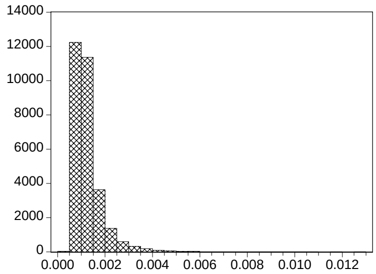
Mean	0.000214
Median	9.65E-05
Maximum	0.191628
Minimum	-0.189207
Std. Dev.	0.035514
Skewness	0.014996
Kurtosis	3.699045

The estimated y_t values ($\left\{\hat{y}_{t+1|t}\right\}_{t=1}^{30000}$)



Mean	1.73E-05
Median	4.14E-07
Maximum	0.016871
Minimum	-0.014472
Std. Dev.	0.002390
Skewness	0.086885
Kurtosis	5.895584

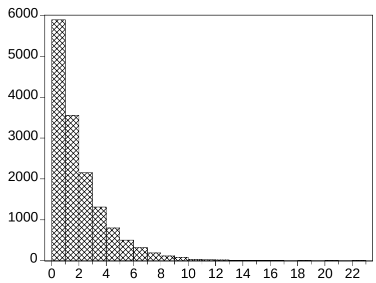
The estimated σ_t^2 values ($\left\{\hat{\sigma}^2_{t+1|t}\right\}_{t=1}^{30000}$)



Mean	0.001259
Median	0.001081
Maximum	0.012977
Minimum	0.000408
Std. Dev.	0.000670
Skewness	3.879618
Kurtosis	32.67293

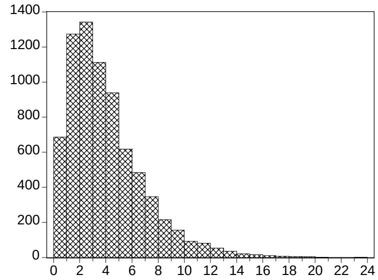
Figure 7. Histograms and descriptive statistics of $\left\{\sum_{j=t-T+1}^{t} \hat{z}^2_{j+1|j}\right\}$, for $t = T(T)30000$

$T = 2$



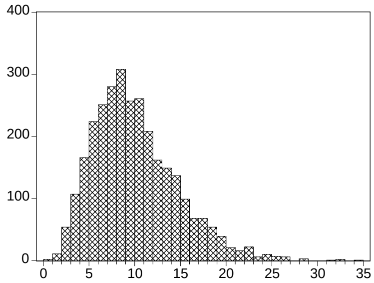
Mean	2.017960
Variance	4.128895
Observations	15000.00

$T = 4$



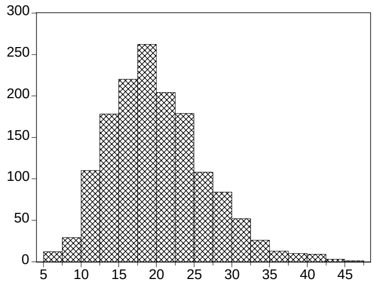
Mean	4.035919
Variance	8.334519
Observations	7500.000

$T = 10$



Mean	10.08980
Variance	21.34030
Observations	3000.000

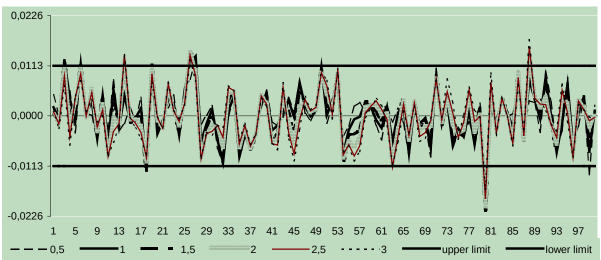
$T = 20$



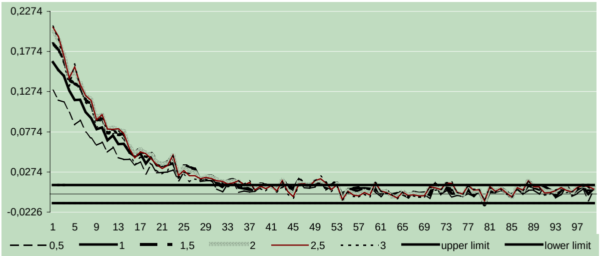
Mean	20.17960
Variance	41.31167
Observations	1500.000

Figure 8. Autocorrelation of transformations of the processes $\hat{z}_{t+1|t}$, $\hat{\varepsilon}_{t+1|t}$, $\hat{v}_{t+1|t}$, $\hat{\sigma}_{t+1|t}$

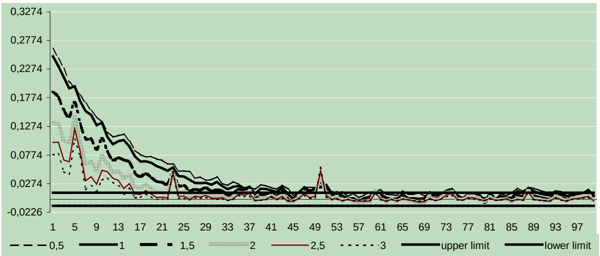
$Cor\left(\left|\hat{z}_{t+1|t}\right|^d, \left|\hat{z}_{t+\tau+1|t+\tau}\right|^d\right)$, $d = 0.5(0.5)3$, $\tau = 1(1)100$



$$Cor\left(\left|\hat{\varepsilon}_{t+1|t}\right|^{d},\left|\hat{\varepsilon}_{t+\tau+1|t+\tau}\right|^{d}\right),\ d=0.5(0.5)3,\ \tau=1(1)100$$



$$Cor\left(\left|\hat{v}_{t+1|t}\right|^{d},\left|\hat{v}_{t+\tau+1|t+\tau}\right|^{d}\right),\ d=0.5(0.5)3,\ \tau=1(1)100$$



$$Cor\left(\hat{\sigma}^{d}_{t+1|t},\hat{\sigma}^{d}_{t+\tau+1|t+\tau}\right),\ d=0.5(0.5)3,\ \tau=1(1)100$$