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A NEW METHOD FOR CONSTRUCTING CONFIDENCE INTERVALS FOR THE INDEX C_{pm}

Michael Perakis* and Evdokia Xekalaki*

***Abstract*— In the statistical literature on the study of the capability of processes through the use of indices, C_{pm} introduced by Chan et al. [2] appears to have been one of the most widely used capability indices and its estimation has attracted much interest. In this article, a new method for constructing approximate confidence intervals for this index is suggested. The method is based on an approximation of the noncentral chi-square distribution, which was proposed by Pearson [11]. Its coverage appears to be more satisfactory compared to that achieved by any of the two most widely used methods that were proposed by Boyles [1]. This is supported by the results of an extensive simulation study.**

***Index terms*— process capability indices, noncentral chi-square distribution, approximate confidence intervals.**

I. Introduction

Process capability indices are used mainly in industry in order to measure the capability of a process to produce according to some specifications. A plethora of such indices has been proposed in the last two decades. A review of them is provided in the textbooks by Kotz and Johnson [5] and Kotz and Lovelace [7] and the article by Kotz and Johnson [6]. Among the suggested indices, C_{pm} is, undoubtedly, one of the most widely used. It was initially introduced by Chan et al. [2] and since then its properties and estimation techniques have also been investigated thoroughly by various other authors, such as Boyles [1], Pearn et al. [10] and Wright [14]. It is defined as

$$C_{pm}=\frac{U-L}{6\sqrt{E(X-T)^2}}=\frac{U-L}{6\sqrt{\sigma^2+(\mu-T)^2}},$$

where L, U denote the lower and the upper specification limits, T corresponds to the target value and μ, σ refer to the mean and the standard deviation of the process, respectively.

Evidently, the assessment of the value of C_{pm} for a given process requires knowledge of both μ and σ. If these parameters are unknown, the value of the index has to be estimated. The two estimators of C_{pm} that appear most often in the literature are those proposed by Chan et al. [2] and Boyles [1]. The estimator proposed by Chan et al. [2] is defined as

$$\tilde{C}_{pm}=\frac{U-L}{6\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i-T)^2}}=\frac{U-L}{6\sqrt{S^2+\frac{n}{n-1}\left(\overline{X}-T\right)^2}}, \quad (1)$$

where X_i, i=1,...,n are the elements of a random sample taken from the examined process, $\overline{X}$ is the sample mean and S^2 is the sample variance. The estimator that Boyles [1] proposed is defined as

$$\hat{C}_{pm}=\frac{U-L}{6\sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i-T)^2}}=\frac{U-L}{6\sqrt{\frac{n-1}{n}S^2+\left(\overline{X}-T\right)^2}}. \quad (2)$$

One may observe that estimators (1) and (2) differ in the type of the estimator used for the parameter

$$\sigma'^2=\sigma^2+(\mu-T)^2.$$

More specifically, in estimator (1) σ'^2 is estimated through

$$\tilde{\sigma}'^2=\frac{1}{n-1}\sum_{i=1}^{n}(X_i-T)^2, \quad (3)$$

while, in estimator (2), σ'^2 is estimated through

$$\hat{\sigma}'^2=\frac{1}{n}\sum_{i=1}^{n}(X_i-T)^2. \quad (4)$$

According to Boyles [1], the estimator given by (4) is an unbiased estimator of σ'^2 and its mean squared error is smaller than that of the estimator given by (3). For this reason, Boyles [1] argues that estimator (2), which involves (4), is superior to (1). On the other hand, as Kotz and Lovelace [7] point out, the bias and the mean squared error of estimator (1) are smaller than those of (2). Subbaiah and Taam [13], based on simulation results, concluded that estimator (1) should be preferred for point estimation and estimator (2) is preferable when there is a need for assessing confidence intervals. It should be remarked that

* Athens University of Economics and Business

the statistical properties of the two estimators are quite similar since

$$\hat{C}_{pm} = \tilde{C}_{pm}\sqrt{\frac{n}{n-1}}.$$

Boyles [1] suggested two methods for constructing confidence limits for the actual value of C_{pm} that have become the most widely used such techniques. The first is based on the chi-square distribution, while the second, recommended for use in cases where the sample size is sufficiently large, is based on the standard normal distribution. According to Kushler and Hurley [8] and Subbaiah and Taam [13], the performance of both these methods appears to be better than that of some other methods that have been considered in the literature for the construction of confidence limits for C_{pm}.

In this paper, a modification of the first method of Boyles [1] is proposed, which appears to lead to a coverage closer to the nominal. So, in Section II, the two methods suggested by Boyles [1] are briefly discussed, while, in Section III, based on Perakis and Xekalaki [12], a new method is suggested for the construction of confidence limits through an approximation of the noncentral chi-square distribution. As demonstrated in Section IV, where the performances of the three techniques are compared via simulation, the confidence limits obtained by the new method achieve a better coverage.

II. The Methods suggested by Boyles [1]

Boyles [1] suggested two methods that enable one to construct approximate confidence intervals (or merely lower confidence limits) for the actual value of C_{pm}. These methods are based on different approximations of the noncentral chi-square distribution, which, as shown in the sequel, is related to the sampling distribution of estimators (1) and (2).

Indeed, the probability density function (pdf) of $\hat{C}_{pm}$ (and, consequently, that of $\tilde{C}_{pm}$) can be expressed in terms of the pdf of the noncentral chi-square distribution, defined by

$$f(x) = \frac{\exp\left\{-\frac{\lambda+x}{2}\right\}}{2^{v/2}}\sum_{j=0}^{\infty}\left(\frac{\lambda}{4}\right)^{j}\frac{x^{(v/2)+j-1}}{j!\,\Gamma\left(\frac{v}{2}+j\right)},$$

where λ, v denote the noncentrality parameter and the number of degrees of freedom, respectively, and $\Gamma(.)$ is the Gamma function. As one may observe, this distribution arises as a mixture of chi-squared distributed random variables with Poisson weights. (For more details on the noncentral chi-square distribution and its approximations, the interested reader is referred to Johnson and Pearson [3] and Johnson et al. [4]).

Actually, as Boyles [1] points out, the distribution of

$$\frac{n\hat{\sigma}'^2}{\sigma^2} \qquad (5)$$

is the non-central chi-square with n degrees of freedom and noncentrality parameter $n\delta$, where

$$\delta = \frac{(\mu - T)^2}{\sigma^2}.$$

The distribution of (5) can be determined by noting that, under the assumption that the process is normally distributed with mean μ and standard deviation σ, every observation X_i can be expressed as

$$X_i = \mu + \sigma U_i,$$

where U_i follows the standard normal distribution. Substituting, $\mu+\sigma U_i$ for X_i in the numerator of $\hat{\sigma}'^2$, defined in (4), we deduce that

$$\sum_{i=1}^{n}(X_i - T)^2 = \sum_{i=1}^{n}(\sigma U_i + \mu - T)^2. \qquad (6)$$

By dividing both sides of (6) by σ^{-2} we obtain

$$\frac{n\hat{\sigma}'^2}{\sigma^2} = \sum_{i=1}^{n}\left(U_i + \frac{\mu - T}{\sigma}\right)^2.$$

Taking into consideration that U_i are independent standard normal random variables and that $(\mu - T)/\sigma$ is a constant, it follows that (5) has the non-central chi-square distribution with n degrees of freedom and noncentrality parameter

$$\lambda = \sum_{i=1}^{n}\left(\frac{\mu - T}{\sigma}\right)^2 = n\left(\frac{\mu - T}{\sigma}\right)^2.$$

Therefore, the distribution of $\hat{C}_{pm}$ is given by

$$\frac{(U-L)\sqrt{n}}{6\sigma}\left\{\sqrt{\chi_n'^2(\lambda)}\right\}^{-1},$$

where $\chi_n'^2(\lambda)$ follows the non-central chi-square distribution with n degrees of freedom and noncentrality parameter

$$\lambda = n(\mu - T)^2/\sigma^2.$$

According to Patnaik [9], the noncentral chi-square distribution with v degrees of freedom and noncentrality parameter λ can generally be approximated by a scaled chi-squared distribution of the form $c\chi_f^2$, where c and f are some constants. The appropriate values of c and f can be found by equating the first two crude moments of these two distributions. Using the r-th moment of the noncentral chi-square distribution given by

$$2^r\Gamma\left(r+\frac{v}{2}\right)\sum_{j=0}^{r}\binom{r}{j}\frac{(\lambda/2)^j}{\Gamma\left(j+\frac{v}{2}\right)}, \qquad (7)$$

and the r-th moment of the chi-square distribution with n degrees of freedom given by

$$v(v+2)\cdots[v+2(r-1)], \qquad (8)$$

(see e.g., Johnson et al. [4]), Patnaik [9] found that the appropriate values of c and f are given by $(v+2\lambda)/(v+\lambda)$ and $(v+\lambda)^2/(v+2\lambda)$, respectively.

Taking advantage of this approximation, Boyles [1] concluded that the quantity

$$\hat{C}_{pm}\sqrt{\frac{\chi^2_{\hat{f},\alpha}}{\hat{f}}}, \tag{9}$$

where $\chi^2_{\hat{f},\alpha}$ denotes the $100\alpha\%$ percentile of the chi-square distribution with $\hat{f}$ degrees of freedom, constitutes a $100(1-\alpha)\%$ approximate lower confidence limit for the actual value of the index C_{pm}. In (9), the value of f is estimated by

$$\hat{f}=\frac{n\left(1+\hat{\delta}\right)^2}{1+2\hat{\delta}}, \tag{10}$$

where

$$\hat{\delta}=\left(\frac{\overline{X}-T}{\hat{\sigma}}\right)^2$$

and

$$\hat{\sigma}^2=\frac{(n-1)S^2}{n}.$$

Similarly, a $100(1-\alpha)\%$ confidence interval for C_{pm} is given by

$$\left(\hat{C}_{pm}\sqrt{\frac{\chi^2_{\hat{f},\alpha/2}}{\hat{f}}},\hat{C}_{pm}\sqrt{\frac{\chi^2_{\hat{f},1-\alpha/2}}{\hat{f}}}\right). \tag{11}$$

In cases where the value of $\hat{f}$ in (9) is greater than 100, Boyles [1] suggests the use of a normal approximation of the noncentral chi-square distribution. According to this approximation, he concluded that a $100(1-\alpha)\%$ approximate lower confidence limit for C_{pm} is given by

$$\hat{C}_{pm}\left(1-z_{1-\alpha}\sqrt{\frac{1}{2\hat{f}}}\right) \tag{12}$$

and a $100(1-\alpha)\%$ approximate confidence interval for C_{pm} is given by

$$\left(\hat{C}_{pm}\left(1-z_{1-\alpha/2}\sqrt{\frac{1}{2\hat{f}}}\right),\ \hat{C}_{pm}\left(1+z_{1-\alpha/2}\sqrt{\frac{1}{2\hat{f}}}\right)\right). \tag{13}$$

In both of these relationships, z_α denotes the $100\alpha\%$ percentile of the standard normal distribution.

III. The new Method

As pointed out in the previous section, the first method of Boyles [1] is based on an approximation of the noncentral chi-square distribution by a scaled chi-squared distribution of the form $c\chi^2_f$, proposed by Patnaik [9]. Pearson [11] proposed an improvement of this approximation, in which the noncentral chi-square distribution with v degrees of freedom and noncentrality parameter λ is approximated by a distribution of the form $c\chi^2_f$+b, where the values of c, f, and b are obtained by equating the first three moments of the noncentral chi-square distribution and $c\chi^2_f$+b. Using again the formulae for the r-th crude moments of the chi-square distribution given in (8) and the noncentral chi-square distribution given in (7), it can be found that the appropriate values of c, f and b are given by

$$\frac{v+3\lambda}{v+2\lambda},$$

$$\frac{(v+2\lambda)^3}{(v+3\lambda)^2}$$

and

$$-\frac{\lambda^2}{v+3\lambda},$$

respectively (see e.g., Johnson and Pearson [3]). As Johnson et al. [4] point out, this approximation is better than that proposed by Patnaik [9], provided that the value at which one wants to assess the cumulative distribution function of the noncentral chi-square distribution is large enough. Moreover, Johnson et al. [4] provide a table (Table 29.2 in their book), which compares the accuracy of the two approximations and reveals the superiority of that given by Pearson [11].

In the construction of confidence limits for C_{pm}, the noncentral chi-square distribution that has to be approximated has n degrees of freedom and noncentrality parameter nδ and thus the values of c, f and b can be simplified to

$$c=\frac{1+3\delta}{1+2\delta}, \tag{14}$$

$$f=\frac{n(1+2\delta)}{c^2} \tag{15}$$

and

$$b=-\frac{n\delta^2}{1+3\delta}. \tag{16}$$

To construct a $100(1-\alpha)\%$ confidence interval for C_{pm} one may note that

$$P\left(\chi'^2_{n,\alpha/2}(n\delta)<\frac{n\hat{\sigma}'^2}{\sigma^2}<\chi'^2_{n,1-\alpha/2}(n\delta)\right)=1-\alpha, \tag{17}$$

where $\chi'^2_{n,\alpha}(n\delta)$ denotes the $100\alpha\%$ percentile of the noncentral chi-square distribution with n degrees of freedom and noncentrality parameter nδ. Taking advantage of Pearson's [11] approximation of the noncentral chi-square distribution, the left hand side of (17) can be approximated by

$$P\left(c\chi^2_{f,\alpha/2}+b<\frac{n\hat{\sigma}'^2}{\sigma^2}<c\chi^2_{f,1-\alpha/2}+b\right),$$

where c, f and b are defined as in (14), (15) and (16), respectively. Taking into account the fact that

$$\frac{\hat{\sigma}'^2}{\sigma'^2}=\frac{C^2_{pm}}{\hat{C}^2_{pm}},$$

one obtains that, after some algebra, a $100(1-\alpha)\%$ approximate confidence interval for C_{pm} given by

$$\left(\hat{C}_{pm}\sqrt{\frac{\hat{c}\chi^2_{\hat{f},\alpha/2}+\hat{b}}{n\left(1+\hat{\delta}\right)}},\hat{C}_{pm}\sqrt{\frac{\hat{c}\chi^2_{\hat{f},1-\alpha/2}+\hat{b}}{n\left(1+\hat{\delta}\right)}}\right). \qquad (18)$$

where $\hat{c}$, $\hat{f}$ and $\hat{b}$ arise from (14), (15) and (16) substituting $\hat{\delta}$ for δ. Here, $\hat{\delta}$ can be either

$$\hat{\delta}_1=\left(\frac{\overline{X}-T}{\hat{\sigma}}\right)^2 \qquad (19)$$

or

$$\hat{\delta}_2=\left(\frac{\overline{X}-T}{S}\right)^2. \qquad (20)$$

Similarly, a $100(1-\alpha)\%$ approximate lower confidence limit for C_{pm} is given by

$$\hat{C}_{pm}\sqrt{\frac{\hat{c}\chi^2_{\hat{f},\alpha}+\hat{b}}{n\left(1+\hat{\delta}\right)}}. \qquad (21)$$

IV. A Simulation Study

In order to compare the performance of the constructed confidence interval in (18) and the obtained lower confidence limit in (21) to those proposed by Boyles [1] (i.e. to confidence intervals (11) and (13) and to lower confidence limits (9) and (12)), a simulation study was conducted. In this study, random samples of sizes 20 and 50 were generated from the normal distribution with the parameter combinations (μ=0, 0.5, 1, 1.5, 2 and σ=0.5, 1, 1.5) and for the specification limits (L=–3 and U=3) that were also considered by Subbaiah and Taam [13] in their simulation study. For each combination, 25000 random samples were generated and, for each of these samples, the corresponding confidence intervals and lower confidence limits were assessed using all of the methods described above. The proportion of times that each of these limits contains the actual value of the index was recorded. Moreover, in all the cases the mean range of the obtained confidence intervals was assessed for each method.

The obtained results are summarized in Tables 1 and 2. More specifically, Table 1 presents the observed coverage (OC) and the mean range (MR) of the 90% and the 95% confidence intervals as well as the OC of the lower confidence limits, when the value of δ is estimated via (19). On the other hand, Table 2 presents the corresponding values when the value of δ is estimated through (20). Each row contains the values of μ and σ, the corresponding value of C_{pm}, the OC of the confidence intervals (18) (first entry), (11) (second entry) and (13) (third entry), the mean ranges of these confidence intervals and the observed coverage of the lower confidence limits (21), (9) and (12).

The basic conclusions that may be drawn from Tables 1 and 2 are outlined in the sequel:

- the performance of the new confidence interval (18) appears to be better than that of confidence intervals (11) and (13)
- the lower confidence limit (9) seems to have the best coverage among the three confidence limits followed by (21) and (12)
- the mean range of confidence interval (18) seems to be generally greater than that of (11), but smaller than that of (13)
- the choice of the estimator of δ does not appear to affect the coverage
- the mean range seems to be larger in the case where δ is estimated via $\hat{\delta}_2$

The first two conclusions can also be established from Table 3, which summarizes the number of parameter combinations for which the new method performs better or worse than the two methods of Boyles [1]. The entries of Table 3 are of the form

$$f_{(i)}-f_{(j)}$$

and refer to the numbers $f_{(i)}$ and $f_{(j)}$ of times the confidence limits (i) and (j), respectively, achieve a coverage closer to the nominal. So, for example, 13–2, means that if n=20, δ is estimated via $\tilde{\delta}_1$ and the confidence level is 0.9, interval (18) leads to a coverage closer to the nominal than that of (11) in 13 parameter combinations, while the coverage of (11) is closer to the nominal only in 2 combinations (this can be verified from Table 1). It should be noted that the reason why the sum of the values of some entries is not equal to the total number of the examined parameter combinations, is that sometimes two or more methods result in the same observed coverage (such cases are not taken into account in the entries of Table 3).

References

[1] Boyles, R.A. (1991). The Taguchi Capability Index. *Journal of Quality Technology*, **23(1)**, 17-26.

[2] Chan, L.K., Cheng, S.W. and Spiring, F.A. (1988). A New Measure of Process Capability C_{pm}. *Journal of Quality Technology*, **20(3)**, 162-175.

[3] Johnson, N.L. and Pearson E.S. (1969). Tables of Percentage Points of Non-central χ. *Biometrika*, **56(2)**, 255-272.

[4] Johnson, N.L., Kotz, S. and Balakrishnan, N. (1995). *Continuous Univariate Distributions*, **2**, Wiley.

[5] Kotz, S. and Johnson, N.L. (1993). *Process Capability Indices*. Chapman and Hall.

[6] Kotz, S. and Johnson, N.L. (2001). Process Capability Indices - a Review, 1992-2000, *Journal of Quality Technology*. (To appear)

[7] Kotz, S. and Lovelace, C.R. (1998). *Process Capability Indices in Theory and Practice*. Arnold.

[8] Kushler, R.H. and Hurley, P. (1992) Confidence Bounds for Capability Indices. *Journal of Quality Technology*, **24(4)**, 188-195.

[9] Patnaik, P.B. (1949). The non-central χ^2- and F-distributions and their Applications. *Biometrika*, **36**, 202-232.

[10] Pearn, W.L., Kotz, S. and Johnson, N.L. (1992). Distributional and Inferential Properties of Process Capability Indices. *Journal of Quality Technology*, **24(4)**, 216-231.

[11] Pearson, E.S. (1959). Note on an Approximation to the Distribution of noncentral χ^2. *Biometrika*, **46**, 364.

[12] Perakis, M. and Xekalaki, E. (2001). "A New Method for Constructing Confidence Intervals for The Index C_{pm}". *Technical Report no 125*, Department of Statistics, Athens University of Economics and Business, March 2001.

[13] Subbaiah, P. and Taam, W. (1993). Inference on the Capability Index C_{pm}. *Communications in Statistics - Theory and Methods*, **22(2)**, 537-560.

[14] Wright, P.A. (2000). The Cumulative Distribution Function of Process Capability Index C_{pm}. *Statistics and Probability Letters*, **47**, 249-251.

Table 1. Observed Coverage of 90% and 95% confidence limits using $\hat{\delta}_1$

			n=20						n=50					
μ	σ	C_{pm}	OC (18) (11) (13)		MR (18) (11) (13)		OC (21) (9) (12)		OC (18) (11) (13)		MR (18) (11) (13)		OC (21) (9) (12)	
0	.5	2	.8994	.9466	1.0692	1.2726	.9011	.9495	.9015	.9512	.6657	.7920	.9008	.9493
			.8993	.9465	1.0690	1.2722	.9009	.9494	.9014	.9512	.6657	.7919	.9008	.9493
			.9010	.9489	1.0762	1.2828	.8892	.9471	.9021	.9516	.6675	.7945	.8934	.9475
.5	.5	1.414	.8877	.9399	.6413	.7650	.9002	.9522	.8969	.9466	.4042	.4818	.9010	.9538
			.8866	.9396	.6387	.7611	.8982	.9499	.8962	.9460	.4035	.4808	.8995	.9523
			.8879	.9409	.6418	.7658	.8893	.9474	.8970	.9465	.4043	.4819	.8942	.9508
1	.5	.894	.8810	.9345	.2755	.3286	.9010	.9510	.8912	.9450	.1757	.2092	.9012	.9537
			.8801	.9330	.2745	.3271	.8992	.9482	.8907	.9448	.1755	.2089	.9001	.9523
			.8808	.9334	.2752	.3281	.8931	.9466	.8910	.9450	.1757	.2091	.8962	.9510
1.5	.5	.632	.8779	.9308	.1398	.1669	.8947	.9474	.8900	.9445	.0899	.1070	.8993	.9528
			.8772	.9307	.1395	.1664	.8934	.9455	.8899	.9443	.0898	.1069	.8988	.9513
			.8776	.9308	.1397	.1667	.8884	.9444	.8896	.9446	.0898	.1069	.8968	.9507
2	.5	.485	.8760	.9293	.0826	.0986	.8938	.9446	.8913	.9396	.0533	.0635	.9004	.9471
			.8754	.9292	.0825	.0984	.8930	.9432	.8908	.9394	.0533	.0634	.8999	.9460
			.8752	.9294	.0826	.0985	.8901	.9424	.8908	.9395	.0533	.0634	.8984	.9457
0	1	1	.9009	.9494	.5357	.6371	.8978	.9500	.9010	.9504	.3330	.3967	.8993	.9492
			.9008	.9492	.5356	.6369	.8976	.9497	.9008	.9504	.3330	.3966	.8992	.9492
			.9021	.9511	.5392	.6423	.8885	.9470	.9012	.9511	.3339	.3979	.8818	.9475
.5	1	.894	.8912	.9425	.4637	.5515	.8955	.9472	.8963	.9494	.2903	.3453	.8985	.9506
			.8910	.9420	.4629	.5503	.8947	.9464	.8963	.9496	.2901	.3450	.8982	.9502
			.8926	.9444	.4659	.5546	.8837	.9440	.8965	.9505	.2908	.3461	.8911	.9486
1	1	.707	.8886	.9408	.3204	.3824	.9050	.9517	.8928	.9441	.2019	.2405	.9021	.9525
			.8885	.9402	.3191	.3804	.9033	.9492	.8930	.9440	.2016	.2400	.9009	.9508
			.8894	.9416	.3207	.3827	.8936	.9470	.8935	.9447	.2020	.2405	.8956	.9492
1.5	1	.555	.8891	.9352	.2076	.2475	.9048	.9541	.8934	.9469	.1316	.1568	.8997	.9533
			.8878	.9348	.2067	.2462	.9026	.9514	.8926	.9464	.1314	.1564	.8989	.9518
			.8882	.9356	.2074	.2472	.8942	.9495	.8929	.9468	.1315	.1567	.8947	.9507
2	1	.447	.8817	.9370	.1375	.1642	.9046	.9535	.8951	.9452	.0879	.1047	.8999	.9558
			.8808	.9360	.1370	.1635	.9028	.9505	.8947	.9450	.0878	.1045	.8992	.9536
			.8810	.9365	.1373	.1640	.8954	.9489	.8948	.9452	.0878	.1046	.8959	.9526
0	1.5	.667	.9001	.9495	.3564	.4245	.9008	.9502	.9002	.9500	.2219	.2643	.8975	.9502
			.9000	.9495	.3564	.4244	.9008	.9501	.9002	.9499	.2219	.2642	.8975	.9502
			.9018	.9510	.3588	.4279	.8906	.9477	.9010	.9508	.2225	.2651	.8906	.9482
.5	1.5	.632	.8930	.9489	.3345	.3971	.8959	.9493	.9012	.9514	.2087	.2487	.8990	.9502
			.8926	.9491	.3342	.3966	.8957	.9487	.9011	.9516	.2087	.2486	.8990	.9500
			.8944	.9509	.3364	.3999	.8854	.9461	.9016	.9520	.2092	.2494	.8916	.9481
1	1.5	.555	.8839	.9398	.2781	.3311	.8955	.9502	.8938	.9452	.1743	.2079	.9020	.9498
			.8833	.9401	.2773	.3300	.8944	.9486	.8936	.9453	.1741	.2077	.9012	.9488
			.8844	.9416	.2790	.3325	.8847	.9460	.8944	.9461	.1745	.2083	.8938	.9466
1.5	1.5	.471	.8886	.9406	2143	2544	.8992	.9522	.8969	.9449	.1345	.1606	.9037	.9500
			.8875	.9394	2134	2531	.8970	.9500	.8965	.9444	.1343	.1602	.9026	.9488
			.8883	.9410	.2145	.2547	.8876	.9479	.8971	.9452	.1346	.1606	.8966	.9475
2	1.5	.4	.8850	.9389	.1602	.1904	.8987	.9544	.8960	.9457	.1012	.1204	.9054	.9554
			.8826	.9384	1595	1894	.8967	.9516	.8949	.9448	.1010	.1201	.9041	.9538
			.8836	.9393	.1601	1903	.8883	.9496	.8953	.9454	.1011	.1204	.8981	.9523

Table 2. Observed Coverage of 90% and 95% confidence limits using $\hat{\delta}_2$

			n=20						n=50					
μ	σ	C_{pm}	OC (18) (11) (13)		MR (18) (11) (13)		OC (21) (9) (12)		OC (18) (11) (13)		MR (18) (11) (13)		OC (21) (9) (12)	
0	.5	2	.9008	.9487	1.0734	1.2732	.8959	.9484	.8975	.9502	.6657	.7935	.8975	.9483
			.9007	.9486	1.0732	1.2728	.8958	.9482	.8975	.9502	.6657	.7935	.8974	.9483
			.9018	.9508	1.0804	1.2835	.8846	.9454	.8986	.9507	.6675	.7961	.8916	.9465
.5	.5	1.414	.8902	.9432	.6459	.7724	.9030	.9532	.8983	.9447	.4053	.4830	.9046	.9527
			.8891	.9421	.6433	.7685	.9008	.9514	.8980	.9446	.4047	.4820	.9036	.9516
			.8898	.9435	.6466	.7733	.8918	.9491	.8985	.9453	.4055	.4832	.8981	.9502
1	.5	.894	.8936	.9423	.2809	.3350	.9048	.9557	.8934	.9481	.1771	.2110	.9027	.9555
			.8916	.9412	.2799	.3334	.9030	.9525	.8925	.9476	.1768	.2106	.9015	.9534
			.8922	.9418	.2806	.3345	.8962	.9508	.8927	.9478	.1770	.2109	.8980	.9524
1.5	.5	.632	.8891	.9376	.1431	.1705	.8992	.9495	.8937	.9451	.0906	.1081	.9032	.9513
			.8877	.9363	.1428	.1701	.8980	.9469	.8931	.9451	.0905	.1080	.9023	.9498
			.8877	.9366	.1429	.1703	.8936	.9453	.8933	.9451	.0906	.1080	.8999	.9495
2	.5	.485	.8844	.9358	.0846	.1009	.9011	.9454	.8921	.9443	.0538	.0640	.8993	.9500
			.8835	.9351	.0845	.1007	.9002	.9432	.8919	.9447	.0537	.0640	.8989	.9490
			.8840	.9353	.0846	.1008	.8962	.9422	.8920	.9448	.0538	.0640	.8972	.9488
0	1	1	.8982	.9491	.5367	.6371	.8964	.9510	.8996	.9514	.3332	.3964	.8963	.9502
			.8980	.9491	.5366	.6370	.8964	.9509	.8996	.9514	.3332	.3964	.8963	.9502
			.8996	.9510	.5402	.6423	.8856	.9486	.9002	.9522	.3341	.3977	.8884	.9484
.5	1	.894	.8938	.9435	.4635	.5537	.9017	.9470	.8954	.9485	.2905	.3452	.8976	.9518
			.8939	.9432	.4627	.5525	.9011	.9460	.8954	.9486	.2903	.3450	.8972	.9515
			.8954	.9450	.4657	.5569	.8908	.9439	.8962	.9495	.2910	.3460	.8896	.9493
1	1	.707	.8893	.9416	.3232	.3851	.9026	.9520	.9021	.9455	.2028	.2414	.9051	.9538
			.8880	.9404	.3219	.3832	.9009	.9492	.9017	.9455	.2025	.2409	.9042	.9520
			.8899	.9418	.3235	.3855	.8917	.9471	.9026	.9461	.2029	.2415	.8988	.9504
1.5	1	.555	.8916	.9410	.2107	.2505	.9027	.9552	.8924	.9466	.1323	.1579	.9030	.9546
			.8900	.9387	.2098	.2492	.9002	.9523	.8916	.9463	.1321	.1576	.9021	.9532
			.8896	.9398	.2105	.2503	.8924	.9506	.8913	.9468	.1323	.1579	.8974	.9518
2	1	.447	.8876	.9415	.1408	.1672	.8984	.9548	.8939	.9472	.0885	.1055	.9014	.9559
			.8868	.9411	.1403	.1664	.8960	.9519	.8932	.9470	.0884	.1053	.9003	.9542
			.8874	.9415	.1406	.1669	.8892	.9506	.8937	.9472	.0885	.1054	.8966	.9531
0	1.5	.667	.8982	.9488	.3573	.4246	.8980	.9482	.8991	.9513	.2223	.2643	.8974	.9499
			.8982	.9488	.3572	.4245	.8979	.9480	.8990	.9512	.2223	.2643	.8973	.9499
			.9001	.9506	.3596	.4280	.8862	.9458	.8998	.9520	.2229	.2651	.8902	.9481
.5	1.5	.632	.8944	.9456	.3341	.3983	.8988	.9480	.8975	.9468	.2090	.2486	.8971	.9469
			.8944	.9454	.3338	.3978	.8984	.9475	.8974	.9470	.2089	.2486	.8970	.9466
			.8964	.9469	.3360	.4011	.8882	.9446	.8981	.9479	.2095	.2493	.8907	.9453
1	1.5	.555	.8947	.9436	.2791	.3335	.9012	.9479	.8966	.9478	.1747	.2081	.8995	.9508
			.8940	.9432	.2784	.3325	.9002	.9460	.8961	.9475	.1745	.2078	.8990	.9501
			.8949	.9452	.2801	.3349	.8900	.9442	.8971	.9482	.1749	.2084	.8922	.9482
1.5	1.5	.471	.8933	.9454	.2159	.2563	.9027	.9575	.8987	.9462	.1350	.1610	.9026	.9527
			.8922	.9448	.2151	.2550	.9011	.9552	.8980	.9460	.1347	.1607	.9017	.9511
			.8931	.9460	.2162	.2566	.8915	.9526	.8976	.9464	.1350	.1611	.8950	.9497
2	1.5	.4	.8933	.9414	.1619	.1929	.9037	.9546	.9000	.9474	.1018	.1213	.9049	.9546
			.8916	.9404	.1612	.1918	.9016	.9516	.8988	.9464	.1016	.1210	.9037	.9527
			.8921	.9415	.1618	.1927	.8938	.9496	.8990	.9467	.1018	.1212	.8990	.9517

Table 3. Frequencies of better coverage attainments by the confidence intervals or lower confidence limits obtained by the new method in comparison to those obtained by Boyles's [1] methods

			Confidence Intervals		Lower Confidence Limits	
Estimate of δ	**Confidence Coefficient**	**Sample Size**	**(18) – (11)**	**(18) – (13)**	**(21) – (9)**	**(21) – (12)**
$\dot{\delta}_1$	90%	n=20	13–2	10–5	9–5	13–1
		n=50	9–4	10–5	6–6	13–2
	95%	n=20	12–2	4–10	8–7	11–4
		n=50	11–2	8–6	3–9	8–7
$\dot{\delta}_2$	90%	n=20	11–2	9–6	5–9	14–1
		n=50	11–1	8–7	6–8	9–6
	95%	n=20	13–0	5–9	7–8	9–6
		n=50	7–4	5–8	1–10	6–9