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ON ANALYSING DEMAND AND MAKING INVENTORY DECISIONS

Dionysia Demetrakopoulou* and Evdokia Xekalaki*

Abstract – **The questions of when to place an order for additional stock and how large a quantity to order have attracted a lot of interest in the statistical literature. In this paper, some of the inventory decision models that have been proposed with the aim of providing answers to these questions are discussed. A characteristic that all of the models considered have in common is that they regard the demand for the item as a random variable. The focus is on the case of heterogeneous demand. In particular, the first three models employ a Bayesian approach for forecast and probability revisions in the case of a heterogeneous Poisson demand. The forecasts are then incorporated into a model that determines the best order quantity. The next two models assume that the lead-time is also a random variable and determine the stock level at which to reorder when the distribution of demand is the Yule. Finally, considering that the joint distribution of the aggregate demand in two successive time periods is of a generalized Waring form, the density of the distribution of the summand variables is numerically estimated.**

Keywords and Phrases – **inventory decision models, heterogeneous Poisson demands, order quantity, reorder point, bivariate generalized Waring distribution.**

I. Introduction

All the procedures that are followed so that the best answers to the questions of when to place an order for additional material and of how much to order can be found, comprise 'inventory control' or 'stock control'. When the amount of stock is arbitrarily determined, the results may not be satisfying because a shortage of stock or surplus of stock may occur. Thus, some well studied decisions must be made since the above consequences can influence the performance of a company the years to come.

The problem of how large a quantity to order very much depends on the distribution of demand. On the other hand, the problem of when to place an order depends on the fluctuations of inventory on hand, lead time and demand. However, many researchers have presented some models which provide solutions to the above problems under certainty. The aim of this paper is to help an inventory manager to make some decisions under uncertainty which is a more realistic view. Thus, based on Demetrakopoulou and Xekalaki [2], some models are presented in which the demand is assumed to be a random variable while the other characteristics of an item may or may not fluctuate.

Section II presents three mixture models that lead to demand distributions of the Binomial, Yule or Generalized Waring form. These models consider dividing the total period of observation in two sub-periods and subsequently forecasting demand in the 2nd period on the basis of demand during the 1st period according to a Bayesian approach. All three models assume that the lead time is zero. In section III two models that address the problem of determining the best order quantity when the retail selling price, the acquisition cost, the salvage value and the lost sale cost of the item are known, are described. Section IV presents two models that determine the best reorder point. The first one assumes that the demand, the stock in hand and the lead time are all random variables. The second one also assumes that the demand varies from day to day and that the lead time is random but it doesn't take into account the characteristic 'stock in hand' at all. Both models lead to a Yule distributed demand. Section V deals with random sums where N has a Poisson distribution and presents a method which estimates the summand distribution when the distribution of demand is the bivariate generalized Waring distribution using Panjer's (1999) multivariate recursions. Finally, section VI considers some applications of the models presented in sections II, III and IV to real data.

II. Demand forecasts for items that have heterogeneous Poisson demands using mixture models.

In this section, three models will be developed according to the "*apparent contagion*" hypothesis. This hypothesis underlines the necessity to accept the role of factors that affect the placement of an order, other than pure chance and amounts to the following set of assumptions:

Items have constant but unequal Poisson probabilities to be ordered.

Heterogeneity is assumed to be averaged out in terms of a Gamma distribution.

All the three models divide the selling period [0,2t] into two equal sub-periods and use a Bayesian approach to forecast demand in period [t,2t], denoted by X_2, on the basis of demand in period [0,t], denoted by X_1. Bradford and Sugrue's [1] model leads to a Negative Binomial demand distribution as follows:

Let $$X_1|\lambda \sim (\lambda t)^x e^{-\lambda t}/x! \qquad (1)$$

Then, if λ, the parameter which represents all the non-random factors, is Gamma distributed with probability density function:

$$f(\lambda)=\frac{a^r\lambda^{r-1}e^{-a\lambda}}{(r-1)!} \qquad \lambda>0.$$

* ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS, Patission 76, 104 34, ATHENS, GREECE

the distribution of aggregate demand in period 1, that is assumed to be of an arbitrary unit length, is the Negative Binomial with parameters a, r

$$X_1 \sim \binom{r+x-1}{r-1}\left(\frac{a}{a+1}\right)^r\left(\frac{1}{a+1}\right)^x, \qquad x = 0,1,\ldots \quad (2)$$

The conditional period 2 distribution of λ given $X_1=x$ which is obtained by the Baye's theorem is

$$f_\lambda(\lambda|X_1=x)=\frac{(a+1)^{r+x}\lambda^{x+r-1}e^{-\lambda(a+1)}}{(r+x-1)!} \qquad (3)$$

Thus, the posterior distribution of λ is also a gamma distribution with updated parameters r+x and a+1.

The period 2 distribution of demand is a mixture of (1) with mixing distribution as given by (3). Therefore,

$P(X_2=y|X_1=x)=$

$$\binom{r+x+y-1}{r+x-1}\left(\frac{a+1}{a+2}\right)^{r+x}\left(\frac{1}{a+2}\right)^y, \qquad y = 0,1,\ldots$$

This is the probability function of a Negative Binomial distribution with parameters r+x and a+1.

At the beginning of period 1, when no previous knowledge on the demand $X|\lambda$ of an item exists, the best estimate of its prior mean will be the mean of the Gamma(r,a) distribution, namely $E[\lambda]=r/a$. In subsequent points in time, the forecasts of demand in period 2 are given by the expectation of the Gamma(r+x,a+1)

$$E[X_2|X_1=x]=\frac{r+x}{a+1}=\frac{r}{a+1}+\frac{x}{a+1}, \qquad x=0,1, \quad (4)$$

Xekalaki's [13] model has an advantage over Bradford and Sugrue's [1] model. It provides a deeper insight into the heterogeneity mechanism by considering dividing all non-random factors into two types, internal and external factors. The first type includes all the factors that have to do with the item's specific features and qualities that predispose the customer to buy it. The second type refers to the item's exposure in the market. The totality of internal factors are termed by Xekalaki [13] "proneness", while the external factors are termed " liability". In this paper a special case of Xekalaki's [13] model is considered which leads to the Yule distribution as the distribution of aggregate demand.

Consider items of proneness v and liability $\lambda_1|v$ over the first half of a selling period [0,2t]. Let also X be a random variable representing the demand for these items and be Poisson distributed with mean λt and probability generating function

$$G_{X|(\lambda_1,v)}(s) = e^{(\lambda_1 t|v)(s-1)}, \ \lambda_1>0. \qquad (5)$$

Assume that the liability parameter $\lambda_1|v$ for these items follows an exponential distribution with parameter $1/v$ and density function

$$f_{\lambda_1|v}(\lambda) = \frac{e^{-\lambda_1/v}}{v}, \ v>0. \qquad (6)$$

Then, for items with the same proneness but varying liabilities, the distribution of demand (X_1) over the first period [0,t] has probability generating function

$$G_{X_1|v}(s) = [1+vt(1-s)]^{-1}. \qquad (7)$$

Thus, if t is assumed to be of unit length, the probability function of the demand for items with the same proneness v is

$$P(X=x|v)=\left(\frac{v}{1+v}\right)^x\left(\frac{1}{1+v}\right), \qquad x = 0,1,\ldots \qquad (8)$$

Let us further assume that the proneness parameter v follows a Beta distribution of the second kind (Pearson type VI) with parameters 1, ρ and density function

$$f_v(v)=\frac{\Gamma(1+\rho)}{\Gamma(1)\Gamma(\rho)}(1+v)^{-(1+\rho)}. \qquad \rho>0, v>0. \qquad (9)$$

Then, the unconditional distribution of demand for the period [0,1] is the Yule distribution with parameter ρ and probability function

$$P(X=x)=\frac{\rho x!}{(\rho+1)_{(x+1)}}, \ x=0, 1, 2, \ldots \qquad (10)$$

where $\alpha_{(\beta)} = \Gamma(\alpha+\beta)/\Gamma(\alpha)$, $\alpha>0$, $\beta\in\Re$. At the beginning of period [0,1], nothing is known of the demand $X|(\lambda_1|v)$ of an item. The best guess of its prior mean would be $E(E(\lambda_1|v))=kE(v)=1/(\rho-1)$.

The predicted distribution of v for period 2 as it comes out using the Bayes' rule is a Beta distribution of a second kind with parameters 1+x and 1+ρ and probability density function

$$f_{v|(X_1=x)}(v)=\frac{\Gamma(x+\rho+2)}{\Gamma(x+1)\Gamma(\rho+1)}v^x(1+v)^{-(x+\rho+2)} \qquad (11)$$

This implies that $\lambda|(v,X_1=x)$ has a posterior distribution with probability density function

$$f_{\lambda|(v,X_1=x)}=\frac{1}{\Gamma(x+1)}\left(\frac{v}{1+v}\right)^{-(1+x)}e^{-\lambda(v+1)/v}\lambda^x \qquad (12)$$

which coincides with that of a Gamma distribution with parameters 1+x and 1+1/v. So, the conditional expected demand for period 2 given the realized demand for period 1 is

$$\frac{(x+1)^2}{\rho} \qquad (13)$$

If one is interested in repeating the above procedure so as to be in a position to find out the distribution of demand in a subsequent time interval, the a posteriori distributions (11) and (12) should be used as a priori distributions. Then, the distributions of demand in period 1 (X_1) and period 2 (X_2), can be utilized for the determination of the quantities which must be ordered at the beginning of the two periods so that the total cost will be maximized. The period 2 random distribution of demand can be obtained as the mixture

$$\text{Poisson}(\lambda t) \underset{\lambda|v,X_1=x}{\wedge} \text{Gamma}(1+x,1+1/v) \underset{v|(X_1=x)}{\wedge} \text{BetaII}(1+x,1+\rho)$$

Xekalaki [13] presented one more model which allows differences in the environmental factors affecting the same item from period to period. Let X, Y be the demand for the items of the period 1 and period 2 respectively (these periods are non-overlapping time periods) and follow a double Poisson distribution with probability generating function

$$G_{X,Y|\lambda_1,\lambda_2,v}(s,t)=\exp\{(\lambda_1|v)(s-1)+(\lambda_2|v)(t-1)\}; \ \lambda_1,\lambda_2>0. \quad (14)$$

Let also the "liability" parameters $\lambda_1|v$, $\lambda_2|v$ be independently distributed as gamma $(k;\frac{1}{v})$ and gamma $(m;\frac{1}{v})$ respectively.

Then for items with the same "proneness" the joint distribution of demand over the two periods considered has probability generating function given by

$$G_{[X,Y]|v}(s,t)=[1+v(1-s)]^{-k}[1+v(1-t)]^{-m} \qquad (15)$$

The form (15) is the probability generating function of a Bivariate Negative Binomial distribution. If "proneness" v vary from item to item according to a beta distribution of second type , then the joint distribution of demand over the two periods is the Bivariate Generalized Waring distribution with parameters a, k, m, ρ (BGWD(a;k,m;ρ)) and with probability function

$$P(X=x,Y=y)=\frac{\rho_{(k+m)}}{(a+\rho)_{(k+m)}}\frac{a_{(x+y)}k_{(x)}m_{(x)}}{(a+k+m+\rho)_{(x+y)}}\frac{1}{x!}\frac{1}{y!}, \qquad (16)$$

$$a, k, m, \rho>0, \quad x,y=0,1,\ldots$$

The conditional period 2 distribution given the observed period 1 demand is a UGWD(a+x,m;ρ+k). This means that

$$P(Y=y|X=x)= \frac{(\rho+k)_{(m)}}{(a+\rho+k)_{(m)}}\frac{(a+x)_{(y)}}{(a+\rho+k+m)_{(y)}}\frac{1}{y!} \quad y=0,1,\ldots \qquad (17)$$

Thus, the forecasts of demand in period 2 can be found by

$$E(Y|X=x)=\frac{(a+x)m}{\rho+k-1} \qquad (18)$$

which indicates that the period 2 conditional expected demand given the aggregate demand X=x at the end of period 1 is a linear function of x.

The advantage of this model is that the total variance of the observations for the total period and for each of the two subperiods can be written in the form of a sum of three components that can be expressed in terms of the parameters a, k, m, ρ as shown in Table I ([13]).

Component due to	Period 1	Period 2	Total period
Rand/ness (σ_R^2)	$\frac{ak}{\rho-1}$	$\frac{am}{\rho-1}$	$\frac{a(k+m)}{\rho-1}$
Proneness (σ_v^2)	$\frac{k^2a(a+\rho-1)}{(\rho-1)^2(\rho-2)}$	$\frac{m^2a(a+\rho-1)}{(\rho-1)^2(\rho-2)}$	$\frac{(k+m)^2a(a+\rho-1)}{(\rho-1)^2(\rho-2)}$
Liability (σ_λ^2)	$\frac{ak(a+1)}{(\rho-1)(\rho-2)}$	$\frac{am(a+1)}{(\rho-1)(\rho-2)}$	$\frac{a(k+m)(a+1)}{(\rho-1)(\rho-2)}$

Table I

The components of the variance of the Generalized Waring model.

From this table, the contribution of each of the three types of factors assumed to affect demand in any particular time period can be assessed.

III. Determination of the Best Order Quantity for Items That Have Heterogeneous Poisson Demands

In section II, some models which give the opportunity to make demand forecasts, when the selling season is divided into period 1 and period 2, were developed. If these forecasts are used for calculating the total expected profit for the entire season, the determination of the order quantity would be centered upon maximizing the total profit.

The model was considered by Bradford and Surgue [1] and is based on the following assumptions :

- The management adopts an *'order up to'* policy. This means that orders are placed only at the beginning of periods 1 and 2.
- There is no capacity restriction placed on the order size and the lead time is equal to zero.
- For the initial ordering decision, the prior distribution of aggregate demand is known by the management.
- Each item has the same retail selling price R, acquisition cost u, salvage value g and lost sale cost B. The cost of overstocking in period 1 is deferred until the end of the selling season, when any remaining stock is considered to be a total loss.

When period 1 starts, the management decides to stock H units of the item, where H can be determined on the basis of knowledge of the prior distribution of aggregate demand. Thus, following Brabford and Surgue (1990), the expected profit for the item during the period is given by

$$\Pi_1 = R\sum_{x=0}^{H}P(X_1=x)\cdot x + R\cdot H\sum_{x=H+1}^{\infty}P(X_1=x) - B\sum_{x=H+1}^{\infty}(x-H)P(X_1=x) - u\cdot H$$

where the period 1 demand, X_1, can have a Negative Binomial distribution with parameters r, a or a Yule distribution with parameter ρ.

At the end of period 1, the parameters of the distribution of aggregate demand are updated, and the conditional period 2 probabilities and forecasts are reexamined. The period 2 conditional expected profit of the item, given X_1=x, is

$$\pi_x = R\cdot\sum_{y=0}^{H_x}P(X_2=y|X_1=x)\cdot y + R\cdot H_x\cdot\sum_{y=H_x+1}^{\infty}P(X_2=y|X_1=x) - B\cdot\sum_{y=H_x+1}^{\infty}(y-H_x)\cdot P(X_2=y|X_1=x) + g\cdot\sum_{y=0}^{H_x}(H_x-y)P(X_2=y|X_1=x) - u\cdot(H_x-H+x),$$

$$x=0,1,2,\ldots,H \qquad (19)$$

where X_2 is the period 2 conditional demand, H_x is the conditional stocking level of the item in this category, and $P(X_2=y|X_1=x)$ is the period 2 conditional distribution of demand. Then, the period 2 expected profit is a weighted combination of each π_x of the form

$$\Pi_2=\sum_{x=0}^{H-1} P(X_1 = x)\cdot \pi_x + \pi_H \cdot \sum_{x=H}^{\infty} P(X_1 = x)$$

and the total expected profit for the entire season is

$\Pi_s = \Pi_1 + \Pi_2$

Hence, the optimal solution can be determined by iterating over various values of H and H_x until Π_s is maximized.

An extension of this model has been considered by Xekalaki [13] for cases where one is interested in the joint distribution of aggregate demand for two types of an item. The model is also based on the above assumptions. If the management stocks S_1 units at the beginning of period 1 and S_2 units at the beginning of period 2, then according to Xekalaki [13], the total expected profit for the entire season is

$$\Pi(S_1,S_2) = R\left\{\sum_{x=0}^{S_1} xP(X = x) + \sum_{y=0}^{S_2} yP(Y = y)\right\}$$

$$+ R\left\{S_1 \sum_{x=S_1+1}^{\infty} P(X = x) + S_2 \sum_{y=S_2+1}^{\infty} P(Y = y)\right\}$$

$$- B\left\{\sum_{x=S_1+1}^{\infty}(x - S_1)P(X = x) + \sum_{y=S_2+1}^{\infty}(y - S_2)P(Y = y)\right\}$$

$$+g\sum_{y=0}^{S_2}(S_2 - y)P(Y = y) - vS_2 - v\sum_{x=0}^{S_1} xP(X = x),$$

where (X,Y) follows the BGWD(a;k,m;ρ).

The optimal S_1, S_2 values can be determined as the maxima of $\Pi(S_1,S_2)$ by setting the partial differences

$$\Delta_{S_1}\Pi(S_1,S_2) = \Pi(S_1+1,S_2) - \Pi(S_1,S_2),$$

$$\Delta_{S_2}\Pi(S_1,S_2) = \Pi(S_1,S_2+1) - \Pi(S_1,S_2),$$

smaller than zero. As shown by Xekalaki [13], the management must stock S_1 units at the beginning of period 1 and S_2 units at the end of period 1, where S_1, S_2 are the lowest values for which

$$\Delta_{S_1}\Pi(S_1,S_2) = (R + B)P(X \geq S_1 + 1) - v(S_1 + 1)P(X = S_1 + 1) < 0$$

$$\Delta_{S_2}\Pi(S_1,S_2) = (R + B - g)P(Y \geq S_2 + 1) + g - v < 0.$$

IV. Determining Reorder Points under Uncertain Demand, Stock in Hand and Lead Time

In this section, determining reorder points is considered in the framework of two different models. The first model determines the reorder point according to a scheme that is known as the *reorder point system*. In such a system, additional items are ordered whenever stock falls to a particular value. The model has been considered in the literature with different views by different researchers. According to the view taken by Prichard and Eagle [6], only the demand of an item is a random variable, while the stock in hand and the lead time are constants. According to the more realistic view taken by Xekalaki [12], all the above characteristics are regarded as random variables. In the sequel, Xekalaki's [12] model is presented and applied to a Yule distributed demand.

Let X, Y be two non-negative integer valued random variables where X is the demand for an item in units ordered and Y is the amount (in item units) of the inventory on hand during the same lead time L. Consider L to be a random variable distributed independently of X, Y and denote by $P(X=r)=p_r$, $P(Y=r) = q_r$, $r=0,1,2,\ldots$ and $F_L(t)$, $t>0$ the probability functions of X, Y and L, respectively. As pointed out by Xekalaki and Panaretos [14], if T represents the fraction of L during which the item will be out of stock, the following relationship is valid

$$\frac{T}{L} = \frac{X - Y}{X}. \qquad (20)$$

Then, following Xekalaki [12], an order for a quantity is placed when stock reaches the value y_0 of y, where E(T) does not exceed a given length λ_0. Equation (20) and the above rule yield

$$E(T|Y = y_0) = E(L)\left\{P(X > y_0) - y_0 \sum_{x=y_0+1}^{\infty} \frac{P(X = x)}{x}\right\} \leq \lambda_0 \qquad (21)$$

Two theorems presented by Xekalaki [12] lead to a modification of the decision rule of the form:

$$P(X>y_0)-y_0(1-p_0)P(Y=y_0)\leq c, \qquad (22)$$

where $c=\lambda_0/E(\lambda)$.

The above form can be substantially simplified if the distribution of the fluctuations of demand is of the Yule type with probability function as given by (10). Xekalaki [11], obtained a result connecting the distributions of X and Y which can be presented in the form of the following theorem:

Theorem 1 (Xekalaki, 1984). Let X be a non-negative, integer valued random variable. Then X is Yule distributed with probability function given by (10) if and only if

$$P(X>r)=\frac{1}{\rho}(r+1)P(X = r), \qquad r=0,1,2,\ldots$$

Using the results of theorem 1, form (22) is modified as follows:

$$\frac{1}{\rho}(y_0+1)P(X=y_0)-y_0(1-\rho_0)P(Y=y_0)\leq c.$$

Since $X \overset{d}{=} Y$, the inventory manager must select the reorder point y_0 so that

$$\frac{\Gamma(\rho+1)}{(\rho+1)c} \leq \frac{\Gamma(\rho+y_0+1)}{\Gamma(y_0+1)}. \qquad (23)$$

The second model that is presented in the remaining of this section was developed by Keaton (1995) and utilizes Tyworth's (1992) approach for determining reorder points. According to this approach, too, both the demand and the lead time are regarded to be random variables. The particular characteristic of this approach is that it deals with the two component distributions, the conditional demand distribution and the distribution of the lead time and not just with the resulting compound distribution of demand. The lead time is assumed to follow a discrete probability distribution, while the distribution of demand per day can be either discrete or continuous.

Tyworth considered the following situation: a desired fill rate is specified (say 98%) and the reorder point required so as to achieve this in practice is to be determined. (The fill rate is the fraction of demand met from stock). Assume that daily demand follows a discrete distribution with mean μ and that the lead time L ranges from n to m days with probabilities $P(L=t)=p(t)$, $t=n,n+1,\ldots,m$. Then, the conditional demand when the lead time is t days is of the same form as the distribution of demand but only with different parameters and with mean $t\mu$. The expected number of shortages for every conditional demand distribution are determined by the formula:

$$E(R)=\sum_{x=R}^{\infty}\left((R-x)p(x|t)\right),$$

where $p(x|t)$ denotes the probability function of the demand conditional on $(L=t)$ i.e., $p(x|t)=P(X=x|L=t)$. Then, the expected number of shortages over the entire lead time, denoted by ESO(R), is simply a weighted average of the expected shortages for each conditional demand distribution, where the weights are the probabilities that the lead time takes on each possible duration, i.e.,

$$ESO(R)=\sum_{t=n}^{m}p(t)\cdot E(R) \qquad (24)$$

The ESO(R) is then compared to a target level TSO=Q(1-FR), where FR is the desired fill rate and Q is the order quantity. If the ESO(R) is far away from the TSO, another value for the reorder point must be determined. A major advantage of Tyworth's (1992) approach is that it offers great flexibility in modeling. A discrete lead time distribution using "empirical" probabilities can assume any shape. The only element that ultimately limits flexibility is the choice of the daily demand distribution.

V. The Bivariate Generalized Waring as a Demand Distribution in the Context of a Poisson Stopped Sum Model and Estimation of the Summand Distribution.

The distributions considered in the previous sections have been derived in the context of mixture models. However, they can also arise as distributions of a Poisson stopped sum of independently and identically distributed random variables (see e.g. [9]).

Consider the initial and first generations of a branching process. Let the $G_1(z)$ be the probability generating function (pgf) of the size N of the initial generation and suppose that each individual i of the initial generation independently gives rise to a random number Z_i of first generation individuals, where $Z_1, Z_2,\ldots$ have the same distribution with pgf $G_2(z)$. The total number of first generation individuals is then

$$X=Z_1+Z_2+\ldots+Z_N, \qquad (25)$$

where N and Z_i, $i=1,\ldots,N$ are mutually independent. The stopped-sum distribution has pgf $G_X=G_1(G_2(z))$. So, for example, as is well known, the negative binomial distribution can be obtained as the distribution of such a sum with N as a Poisson variable and Z_i being distributed according to a log series distribution. The Yule and, more generally, the generalized Waring distribution on the other hand, can also arise in a similar manner being self decomposable as shown by Xekalaki [10]. However, the form of the distribution of the summand variables in the Poisson stopped sum representation cannot be expressed in a closed form. It can though be determined numerically (see e.g. [4]). In the context of the inventory problems considered in this paper, N may represent the number of orders placed, while Z_i may denote the i-th order quantity (in item units).

Whenever the interest is on the joint distribution of demand for two types of an item over a given period, a bivariate extension of this model might be useful. For a Generalized Waring distributed aggregate demand, such a model could refer to the mechanism giving rise to the BGWD(a;k,m;ρ) as defined by (16) with $X\equiv X_1$, $Y\equiv X_2$ as Poisson stopped sums of independently and identically distributed random variables in the sense of (25). In particular, (16) can be regarded as the distribution of the vector

$$(X,Y)=(X_1,Y_1)+(X_2,Y_2)+\ldots+(X_N,Y_N), \qquad (26)$$

where N is the total number of orders placed in the entire period of observation, each for a certain number of items of types 1 and 2, i.e., the vector (X, Y) represents the joint aggregate demand for types 1 and 2 of an item when N orders (X_i, Y_i), $i=1,2,\ldots,N$ are placed in the entire period with X_i and Y_i being the order quantities associated with order i for type 1 and type 2, respectively. In analogy to the single type case, N is assumed to be Poisson(λ) distributed independently of the pairs (X_i,Y_i), $i=1,2,\ldots,N$ which are mutually independent and identically distributed according to same distribution with some probability function $P(X_i=s, Y_i=v)=f(s,v)$, $s=0,1,2,\ldots$; $v=0,1,2,\ldots$. The pgf of (X, Y) will obviously have the form

$$G_{X,Y}(s,t)=G_N(G_{X_i,Y_i}(s,t))=e^{\lambda(G_{X_i,Y_i}(s,t)-1)} \qquad (27)$$

In the sequel, an algorithm based on Panjer's [5] recursions is constructed to estimate the summands in (26) which leads to a probability function of aggregate demand, say g(x, y), of the BGWD form given by (16).

According to Sundt [7], when N follows the Poisson distribution with parameter λ, the above setting leads to the recurrence relationship

$$g(x,y)=\sum_{s=1}^{x}\lambda\frac{s}{x}\sum_{v=0}^{y}f(s,v)g(x-s,y-v) \qquad (28)$$

$$x=1,2,\ldots \text{ and } y=0,1,\ldots$$

Using (27), we obtain $g(0,0)=e^{-\lambda-\lambda f(0,0)}$ or, equivalently, $f(0,0)=1+\ln g(0,0)/\lambda$. Solving equation (28) in f(x,y), we arrive at the recurrence relationships:

$$f(x,y)=\frac{g(x,y)}{\lambda g(0,0)}-\frac{1}{xg(0,0)}\sum_{s=1}^{x}s\sum_{v=0}^{y-1}f(s,v)g(x-s,y-v)$$
$$-\frac{1}{xg(0,0)}\sum_{k=1}^{x-1}k\,f(k,y)g(x-k,0)$$

$$x=2,3,\ldots;\ y=1,2,\ldots \qquad (29)$$

$$f(x,0) = \frac{g(x,0)}{\lambda g(0,0)} - \frac{1}{xg(0,0)} \sum_{s=1}^{x-1} sf(s,0)g(x-s,0), \quad x = 2,3,\ldots \tag{30}$$

$$f(0,y) = \frac{g(0,y)}{\lambda g(0,0)} - \frac{1}{yg(0,0)} \sum_{v=0}^{y-1} vf(0,v)g(0,y-v), \quad y = 1,2,\ldots \tag{31}$$

$$f(1,y) = \frac{g(1,y)}{\lambda g(0,0)} - \frac{1}{g(0,0)} \sum_{v=0}^{y-1} f(1,v)g(0,y-v), \quad y = 1,2,\ldots \tag{32}$$

and (combining (16) and (28))

$$f(1,0) = \frac{g(1,0)}{\lambda g(0,0)} = \frac{ak}{(a+k+m+\rho)\lambda}.$$

Multivariate extensions of (26) (or, equivalently, of (27)) can also be considered for the description of the joint Poisson stopped aggregate demands for n items ($n \geq 2$) of a Generalized Waring form.

VI. Some applications.

In this section, the two models based on the Bivariate Generalized Waring distribution, that were considered in sections II, III and the two models based on the Yule distribution that were considered in section IV are applied to some real data sets. The first set refers to data from a small retail firm which is housed together with other retail firms in the Greek army department stores (ΕΚΕΜΣ) and sells silver icons. The data concern the demand for 200 types of icons of the same size in 1999. The total period is split into two 6-month periods, period 1 and period 2, and the BGWD model is used to forecast period 2 demand from period 1 demand. Furthermore, the model introduced by Xekalaki [13] is used to determine the quantities that must be ordered at the beginning of periods 1 and 2 for each icon, so that the total expected profit will be maximized. The data are fitted quite satisfactorily by the BGWD with parameter estimates

$$\tilde{a} = 2.612 \quad \tilde{\rho} = 10.45 \quad \tilde{k} = 1.608 \quad \tilde{m} = 1.735$$

as judged by the χ^2 goodness of fit test (degrees of freedom=4, $P(\chi^2 \geq 6.96) = 0.138$).

However, what is important is the possibility that this model allows for predicting demand for period 2 on the basis of demand in period 1. Table II contains the frequencies of actual demand in period 1, the average demand in period 2 and the forecasted demand in period 2.

As revealed by this table, the BGWD model provided satisfactorily accurate forecasts ($P(\chi^2_5 \geq 0.1382) = 0.9996$) and can thus be a useful tool for an inventory manager.

Actual Demand in Period 1	Frequency	Average Demand in Period 2	Forecasted Demand in Period 2
0	144	0.424	0.442
1	34	0.441	0.612
2	15	0.867	0.781
3	4	1	0.951
4	2	1	1.12
5	1	1	1.289

Table II

Forecasts according to the BGWD model for the icon data.

The advantage of the BGWD model over the NB model is to be sought in the features of the model that allow the partitioning of the total variance for each sub-period and for the total period into three additive components, one due to the effect of endogenous factors, one due to the effect of exogenous factors and one due to the effect of random factors. Table III contains all the estimated components in the case of the icon data.

Component due to	**Period 1**	**Period 2**	**Total Period**
Random factors	0.445 (59.9%)	0.48 (58.5%)	0.924 (51.6%)
Endogenous factors	0.108 (14.5%)	0.13 (15.9%)	0.47 (26.3%)
Exogenous factors	0.19 (25.6%)	0.21 (25.6%)	0.4 (22.4%)
Total	0.743 (100%)	0.82 (100%)	1.79 (100%)

Table III

Estimated variance components under the Generalized Waring model

The above table reveals that the random factors affect the most the demand in both periods and in the total period while the endogenous factors have the least effect on the demand in periods 1 and 2. Additionally, the contribution of the exogenous factors is about 25% in each of the three periods.

The fitted BGWD(2.612, 1.608, 1.735, 10.45) model will now be used for the determination of the quantities which must be ordered at the beginning of period 1 and period 2, respectively to replenish stock. The retail price, the acquisition cost, the salvage value and the lost sale cost per icon are 44000 drs, 32000 drs, 8000 drs and 6000 drs, respectively. Implementing Xekalaki's [13] procedure described in section III via an algorithm considered by Demetrakopoulou and Xekalaki [2], leads to the assertion that the manager of the icon firm should stock 9 items of each icon at the beginning of period 1 and 3 items at the beginning of period 2.

To illustrate Xekalaki's [12] procedure for the determination of the reorder point (section IV) we now consider the data of a Greek medical equipment company (SPIMA) on the demand for diathermic apparatus during 1999. First, the observed demand distribution is fitted by the Yule distribution which often appears to be appropriate for describing demand fluctuations in the case of slow-moving items. The parameter ρ of the Yule is estimated by $\tilde{\rho} = \frac{1}{\bar{x}} + 1 = \frac{1}{0.4426} + 1 \approx 3.26$. As judged by the χ^2 goodness of fit test the fit was quite satisfactory

($P(\chi^2_4>1.53)=0.821$). Hence, the stock level at which to reorder diathermic apparatus so that the company's warehouse does not run out of stock can be determined on the assumption of a Yule distributed daily demand with estimated mean $(\tilde{\rho}-1)^{-1}=0.44$.

The lead time is one, two or three days with probabilities 0.3, 0.4 and 0.3, respectively. Thus, the expected value of lead time is 2. (E(L)=2). The specified length of time λ_0 is assumed to be equal to 0.1. Therefore using an algorithm developed by Demetrakopoulou and Xekalaki [2] for determining the reorder point through iterative formula (23), one may assert that an order should be placed when the stock level drops at 2 units.

In the sequel, SPIMA's medical equipment demand data are used to illustrate Tyworth's approach for determining the reorder point. Using this approach the distributions of lead time and demand can easily be updated as conditions change. Thus, when the demand is Yule distributed with parameter ρ and mean μ, the conditional demand distributions when lead time equals 1 day, 2 days and 3 days have means equal to μ, 2μ and 3μ respectively. Hence, the conditional demand distribution for period 1 is the Yule(ρ), the conditional demand distribution for period 2 is the Yule((ρ+1)/2) and the conditional demand distribution for period 3 is the Yule((ρ+2)/3). So, for the data under consideration, the three estimated conditional demand distributions when the lead time (measured in days) equals 1, 2 and 3 are the Yule(3.26) with mean equal to 0.44, the Yule(2.13) with mean equal to $2\cdot 0.44=0.88$ and the Yule(1.75) with mean equal to $3\cdot 0.44=1.32$, respectively. Figure 1 depicts these conditional distributions.

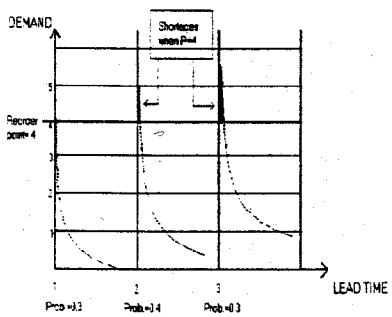

Figure 1. *Conditional distributions of demand lead time is 1, 2, or 3 days*

Then, the expected number of shortages over the entire random lead time is a weighted average of the shortages for each conditional distribution according to formula (24). Using Demetrakopoulou and Xekalaki's [12] algorithm that has been constructed for this procedure, the reorder point in the case of SPIMA's data was determined equal to R=7 units for an order quantity of Q=40 units and for a fill rate equal to 0.95 (starting values R=12, min=0 and max=$2\cdot 3\cdot\mu$=2.64).

VII. Conclusions

In this paper problems regarding when to place an order or how large a quantity to order have been considered. The models that have been presented are applicable when the demand can be regarded as a discrete variable. Analogous models can be designed in the case of a continuous demand.

The three models presented in section II are used to determine the best order quantity which maximizes the expected profit during a season. The fits that are provided by these models are similar, being thus, far from indicating appreciable differences among them. However, the BGWD model has an advantage over the other models considered as it allows the researcher to have an insight into the underlying factors which affect the placement of an order.

Moreover, the re-order point can be determined using the two different approaches presented in section IV. A characteristic that both models have in common is that the demand and the lead time are considered as random variables.

Finally, an alternative model was introduced which regards the distribution of demand as the distribution of a Poisson stopped sum of independent variables and Panjer's (1999) bivariate recursions were used to estimate the summand distribution. All the models presented in this paper can be considered in several other contexts such as accident theory and can thus be of interest to the policy makers of insurance companies.

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