Standardized Prediction Error Criterion (SPEC) Algorithm for ARCH Model Selection in Forecasting Option Prices

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- Autoregressive Conditional Heteroscedasticity (ARCH) models.
- Standardized Prediction Error Criterion (SPEC) Model Selection Algorithm.
- Black & Scholes (BS) Option Pricing Formula.
- Trading Options Based on a Set of ARCH Models.
- Trading Options Based on the SPEC Model Selection Algorithm.
- Trading Options Based on Methods of Model Selection.

Since the first decades of the 20th century, asset returns assumed to be an independently and identically distributed (i.i.d) random process with zero mean and constant variance.

$$\ln(P_t) = \ln(P_{t-1}) + \varepsilon_t$$

$$\varepsilon_t \overset{i.i.d.}{\sim} N\left(0, \sigma^2\right)$$

P_t: Price of an asset at time t

$y_t = \ln\left(P_t / P_{t-1}\right)$: Continuously compounded return series

Figure 1. S&P500 Continuously Compounded Daily Returns (2/1/1990 - 27/06/2000)

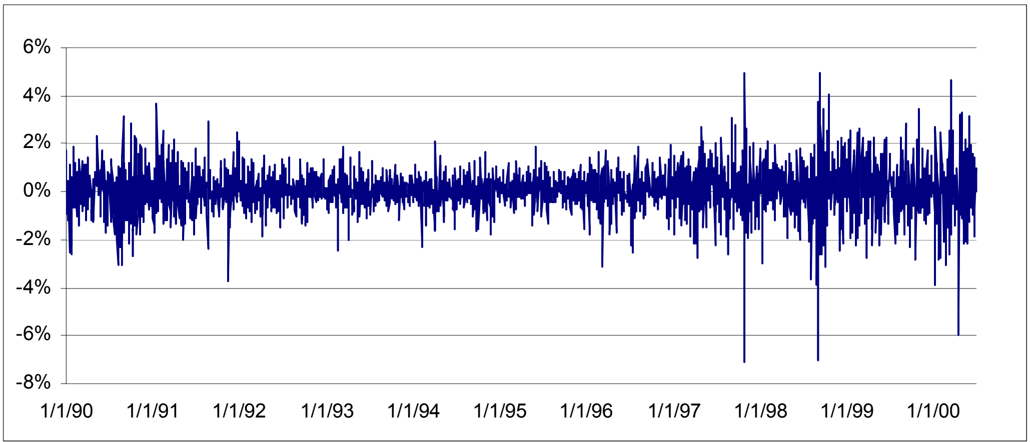
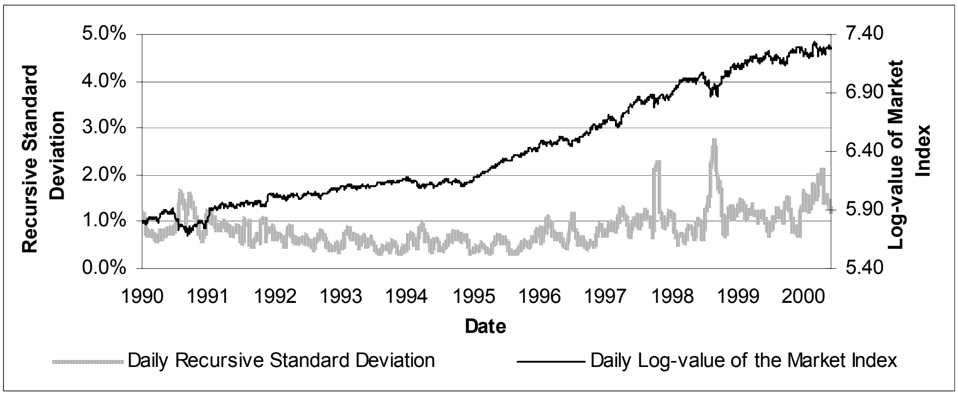


Figure 2. Daily Log-values and Recursive Standard Deviation of Returns for the S&P500 Stock Market.



The standard deviation of the 22 most recent trading days:

$$\sigma_t^{(22)} = \sqrt{\sum_{i=t-22}^{t}\left(y_i - \left(\sum_{i=t-22}^{t} y_i \Big/ 22\right)\right)^2 \Big/ 22}\,.$$

Autoregressive Conditional Heteroscedasticity (ARCH) Process

P_t: Price of an asset at time t

$y_t = \ln\left(P_t / P_{t-1}\right)$: Continuously compounded return series

$$y_t = \mu_t + z_t \sigma_t$$

$$\mu_t = c_0 + \sum_{i=1}^{\kappa} \left(c_i y_{t-i}\right)$$

$$z_t \overset{i.i.d.}{\sim} N(0,1),$$

$$\varepsilon_t = z_t \sigma_t$$

$$\sigma_t^2 = g\left(\sigma_{t-1}(\theta), \sigma_{t-2}(\theta), \ldots; \varepsilon_{t-1}(\theta), \varepsilon_{t-2}(\theta), \ldots; \upsilon_{t-1}, \upsilon_{t-2}, \ldots\right)$$

$$\varepsilon_t \mid I_{t-1} \sim N\left(0, \sigma_t^2\right)$$

Functional Forms for the Conditional Variance

The ARCH(q) model, Engle (Econometrica, 1982).

$$\sigma_t^2 = a_0 + \sum_{i=1}^{q} \left(a_i \varepsilon_{t-i}^2\right)$$

The GARCH(p,q) model, Bollerslev (Journal of Econometrics, 1986).

$$\sigma_t^2 = a_0 + \sum_{i=1}^{q} \left(a_i \varepsilon_{t-i}^2\right) + \sum_{i=1}^{p} \left(b_i \sigma_{t-i}^2\right)$$

The E-GARCH(p,q) model, Nelson (Econometrica, 1991).

$$\ln\left(\sigma_t^2\right) = a_0 + \sum_{i=1}^{q} \left(a_i \left| \frac{\varepsilon_{t-i}}{\sigma_{t-i}} \right| + \gamma_i \left(\frac{\varepsilon_{t-i}}{\sigma_{t-i}} \right) \right) + \sum_{i=1}^{p} \left(b_i \ln\left(\sigma_{t-i}^2\right)\right)$$

The TARCH(p,q) model, Glosten et al. (Journal of Finance, 1993).

$$\sigma_t^2 = a_0 + \sum_{i=1}^{q} \left(a_i \varepsilon_{t-i}^2\right) + \gamma \varepsilon_{t-1}^2 d_{t-1} + \sum_{i=1}^{p} \left(b_i \sigma_{t-i}^2\right)$$

$d_t = 1$ if $\varepsilon_t < 0$, and $d_t = 0$ otherwise.

The ARCH models that have been presented Degiannakis and Xekalaki (Qual.Tech.Quant.Manag., ***2004)***		
GARCH(p,q)	$\sigma_t^2 = a_0 + \sum_{i=1}^{q}\left(a_i\varepsilon_{t-i}^2\right) + \sum_{j=1}^{p}\left(b_j\sigma_{t-j}^2\right)$	Bollerslev (1986)
IGARCH(p,q)	$\sigma_t^2 = a_0 + \sum_{i=1}^{q}\left(a_i\varepsilon_{t-i}^2\right) + \sum_{j=1}^{p}\left(b_j\sigma_{t-j}^2\right)$, for $\sum_{i=1}^{q} a_i + \sum_{j=1}^{p} b_j = 1$	Engle and Bollerslev (1986)
EGARCH(p,q)	$\ln\left(\sigma_t^2\right) = a_0 + \left(1 + \sum_{i=1}^{q} a_i L^i\right)\left(1 - \sum_{j=1}^{p} b_j L^j\right)^{-1}$ $\left(\theta\left(\left\lvert\varepsilon_{t-1}/\sigma_{t-1}\right\rvert - E\left\lvert\varepsilon_{t-1}/\sigma_{t-1}\right\rvert\right) + \gamma\left(\varepsilon_{t-1}/\sigma_{t-1}\right)\right)$	Nelson (1991)
GJR(p,q)	$\sigma_t^2 = a_0 + \sum_{i=1}^{q}\left(a_i\varepsilon_{t-i}^2\right) + \sum_{i=1}^{q}\left(\gamma_i d\left(\varepsilon_{t-i} < 0\right)\varepsilon_{t-i}^2\right) + \sum_{j=1}^{p}\left(b_j\sigma_{t-j}^2\right)$	Glosten et al. (1993)
TGARCH(p,q)	$\sigma_t = a_0 + \sum_{i=1}^{q}\left(a_i\varepsilon_{t-i}^{+}\right) - \sum_{i=1}^{q}\left(\gamma_i\varepsilon_{t-i}^{-}\right) + \sum_{j=1}^{p}\left(b_j\sigma_{t-j}\right)$	Zakoian (1990)
AGARCH(p,q)	$\sigma_t = a_0 + \sum_{i=1}^{q} a_i\left\lvert\varepsilon_{t-i}\right\rvert + \sum_{j=1}^{p} b_j\sigma_{t-j}$	Taylor (1986) Schwert (1989a,b)
Ln-GARCH(p,q)	$\ln\left(\sigma_t^2\right) = a_0 + \sum_{i=1}^{q} a_i \ln\left(\varepsilon_{t-i}^2\right) + \sum_{j=1}^{p} b_j \ln\left(\sigma_{t-j}^2\right).$	Geweke (1986) Pantula (1986)
Stdev-ARCH(q)	$\sigma_t^2 = \left(a_0 + \sum_{i=1}^{q} a_i\left\lvert\varepsilon_{t-i}\right\rvert\right)^2$	Schwert (1990)
NARCH(p,q)	$\sigma_t^{\delta} = a_0 + \sum_{i=1}^{q} a_i\left\lvert\varepsilon_{t-i}^2\right\rvert^{\delta/2} + \sum_{j=1}^{p} b_j\sigma_{t-j}^{\delta}$	Higgins and Bera (1992)
AGARCH(p,q)	$\sigma_t^2 = a_0 + \sum_{i=1}^{q}\left(a_i\varepsilon_{t-i}^2 + \gamma_i\varepsilon_{t-i}\right) + \sum_{j=1}^{p} b_j\sigma_{t-j}^2$	Engle (1990)
NAGARCH(p,q)	$\sigma_t^2 = a_0 + \sum_{i=1}^{q} a_i\left(\varepsilon_{t-i} + \gamma_i\sigma_{t-i}\right)^2 + \sum_{j=1}^{p} b_j\sigma_{t-j}^2$	Engle and Ng (1993)
VGARCH(p,q)	$\sigma_t^2 = a_0 + \sum_{i=1}^{q} a_i\left(\varepsilon_{t-i}/\sigma_{t-i} + \gamma_i\right)^2 + \sum_{j=1}^{p} b_j\sigma_{t-j}^2$	Engle and Ng (1993)
APARCH(p,q)	$\sigma_t^{\delta} = a_0 + \sum_{i=1}^{q} a_i\left(\left\lvert\varepsilon_{t-1}\right\rvert - \gamma_i\varepsilon_{t-i}\right)^{\delta} + \sum_{j=1}^{p} b_j\sigma_{t-j}^{\delta}$	Ding et al. (1993)
GQARCH(p,q)	$\sigma_t^2 = a_0 + \sum_{i=1}^{q} a_i\varepsilon_{t-i}^2 + \sum_{i=1}^{q}\gamma_i\varepsilon_{t-i} + 2\sum_{i=1}^{q}\sum_{j=i+1}^{q} a_{ij}\varepsilon_{t-i}\varepsilon_{t-j} + \sum_{j=1}^{p} b_j\sigma_{t-j}^2$	Sentana (1995)
GQTARCH(p,q)	$\sigma_t^2 = \omega + \sum_{i=1}^{q}\sum_{j=1}^{J} a_{ij} I_j\left(\varepsilon_{t-i}\right) + \sum_{i=1}^{p} b_j\sigma_{t-j}^2$	Gouriéroux and Monfort (1992)
VSARCH(p,q):	$\sigma_t^2 = \omega + \sum_{i=1}^{q} a_i\varepsilon_{t-i}^2 + \gamma S_{t-1}\dfrac{\varepsilon_{t-1}^2}{\sigma_{t-1}^2} + \sum_{j=1}^{p} b_j\sigma_{t-j}^2$	Fornari Mele (1995)

AVSARCH(p,q)	$\sigma_t^2 = a_0 + \sum_{i=1}^{q} a_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} b_j \sigma_{t-j}^2 + \gamma S_{t-1}\varepsilon_{t-1}^2 + \delta\left(\left(\varepsilon_{t-1}^2 / \sigma_{t-1}^2\right) - k\right) S_{t-}$	Fornari and Mele (1995)
LST-GARCH(p,q)	$\sigma_t^2 = a_0 + \sum_{i=1}^{q} \left(a_i + \gamma_i F(\varepsilon_{t-i})\right)\varepsilon_{t-i}^2 + \sum_{j=1}^{p} b_j \sigma_{t-j}^2$ $F(\varepsilon_{t-i}) = \left(1 + \exp(-\theta\varepsilon_{t-i})\right)^{-1} - 0.5$	Hagerud (1996)
EST-GARCH(p,q)	$\sigma_t^2 = a_0 + \sum_{i=1}^{q} \left(a_i + \gamma_i F(\varepsilon_{t-i})\right)\varepsilon_{t-i}^2 + \sum_{j=1}^{p} b_j \sigma_{t-j}^2$ $F(\varepsilon_{t-i}) = 1 - \exp\left(-\theta\varepsilon_{t-i}^2\right)$	Hagerud (1996)
GLST-GARCH(p,q)	$\sigma_t^2 = a_0 + \sum_{i=1}^{q} \left(a_i + \gamma_i F(\varepsilon_{t-i})\right)\varepsilon_{t-i}^2 + \sum_{j=1}^{p} b_j \sigma_{t-j}^2$ $F(\varepsilon_{t-i}) = \frac{1 - \exp\left(-\theta\varepsilon_{t-i}^2\right)}{1 + \exp\left(-\theta\left(\varepsilon_{t-i}^2 - c^2\right)\right)}$	Lubrano (1998)
GEST-GARCH(p,q)	$\sigma_t^2 = a_0 + \sum_{i=1}^{q} \left(a_i + \gamma_i F(\varepsilon_{t-i})\right)\varepsilon_{t-i}^2 + \sum_{j=1}^{p} b_j \sigma_{t-j}^2$ $F(\varepsilon_{t-i}) = 1 - \exp\left(-\theta(\varepsilon_{t-i} - c)^2\right)$	Lubrano (1998)
CGARCH(1,1)	$\sigma_t^2 = q_t + a_1\left(\varepsilon_{t-1}^2 - q_{t-1}\right) + b_1\left(\sigma_{t-1}^2 - q_{t-1}\right)$ $q_t = a_0 + pq_{t-1} + \phi\left(\varepsilon_{t-1}^2 - \sigma_{t-1}^2\right)$	Engle and Lee (1993)
ACGARCH(1,1)	$\sigma_t^2 = q_t + a_1\left(\varepsilon_{t-1}^2 - q_{t-1}\right) + \gamma_1\left(d(\varepsilon_{t-1} < 0)\varepsilon_{t-1}^2 - 0.5q_{t-1}\right) + b_1\left(\sigma_{t-1}^2 - q_{t-}\right.$ $q_t = a_0 + pq_{t-1} + \phi\left(\varepsilon_{t-1}^2 - \sigma_{t-1}^2\right) + \gamma_2\left(d(\varepsilon_{t-1} < 0)\varepsilon_{t-1}^2 - 0.5\sigma_{t-1}^2\right)$	Engle and Lee (1993)
FIGARCH(p,d,q)	$\sigma_t^2 = a_0 + \left(1 - B(L) - \Phi(L)(1-L)^d\right)\varepsilon_t^2 + B(L)\sigma_t^2$	Baillie et al. (1996)
FIEGARCH(p,d,q)	$\ln\left(\sigma_t^2\right) = a_0 + \Phi(L)^{-1}(1-L)^{-d}\left(1 + A(L)\right)g(z_{t-1})$	Bollerslev and Mikkelsen (1996)
FIAPARCH(p,d,q)	$\sigma_t^{\delta} = a_0 + \left(1 - (1 - B(L))^{-1}\Phi(L)(1-L)^{-d}\right)\left(\lvert\varepsilon_t\rvert - \gamma\varepsilon_t\right)^{\delta}$	Tse (1998)
ASYMM FIFGARCH(1,d,1)	$\sigma_t^{\lambda} = \frac{k}{1-\delta} + \left(1 - \frac{(1-\varphi L)(1-L)^d}{1-\delta L}\right) f^{v}(\varepsilon_t)\sigma_t^{\lambda}$ $f(\varepsilon_t) = \left\lvert \frac{\varepsilon_t}{\sigma_t} - b \right\rvert - c\left(\frac{\varepsilon_t}{\sigma_t} - b\right),$	Hwang (2001)
ASYMM FIFGARCH(1,d,1) modified	$(1-\varphi L)(1-L)^d \frac{\sigma_t^{\lambda} - 1}{\lambda} = \omega' + a(1 + \psi L)\sigma_{t-1}^{\lambda}\left(f^{v}(z_{t-1}) - 1\right)$ $f\left(\frac{\varepsilon_t}{\sigma_t}\right) = \left\lvert \frac{\varepsilon_t}{\sigma_t} - b \right\rvert - c\left(\frac{\varepsilon_t}{\sigma_t} - b\right)$	Ruiz and Perez (2002)
R-GARCH(r,p,q)	$\sigma_t^2 = \sum_{i^*=1}^{r}\left(c_{i^*}\eta_{t-i^*}\right) + \sum_{i=1}^{q}\left(a_i\varepsilon_{t-i}^2\right) + \sum_{j=1}^{p}\left(b_j\sigma_{t-j}^2\right)$	Nowicka-Zagrajek and Weron (2001)
H-GARCH(p,n)	$\sigma_t^2 = a_0 + \sum_{i=1}^{n}\sum_{k=1}^{i} a_{ik}\left(\sum_{i^*=k}^{i}\varepsilon_{t-i^*}^2\right)^2 + \sum_{j=1}^{p}\left(b_j\sigma_{t-j}^2\right)$	Müller et al. (1997)

The Correlated Gamma Ratio (CGR) Hypothesis Testing

Degiannakis and Xekalaki (JASMBI, 2005)

Xekalaki et al. (Stochastic Musings, 2003)

Model A
$y_t = \mu_t^{(A)} + \varepsilon_t^{(A)}$
$\varepsilon_t^{(A)} = z_{1,t}\sigma_t^{(A)}$
$z_{1,t} \overset{iid}{\sim} N(0,1)$
$\sigma_t^{2(A)} = g\left(\sigma_{t-1}^{2(A)},\ldots,\sigma_{t-p}^{2(A)},\varepsilon_{t-1}^{2(A)},\ldots,\varepsilon_{t-q}^{2(A)},\upsilon_{t-1}^{(A)},\upsilon_{t-2}^{(A)},\ldots\right)$
Model B
$y_t = \mu_t^{(B)} + \varepsilon_t^{(B)}$
$\varepsilon_t^{(B)} = z_{2,t}\sigma_t^{(B)}$
$z_{2,t} \overset{iid}{\sim} N(0,1)$
$\sigma_t^{2(B)} = g\left(\sigma_{t-1}^{2(B)},\ldots,\sigma_{t-p}^{2(B)},\varepsilon_{t-1}^{2(B)},\ldots,\varepsilon_{t-q}^{2(B)},\upsilon_{t-1}^{(B)},\upsilon_{t-2}^{(B)},\ldots\right)$

Hypothesis Test

H_0 : Model (A) has equivalent predictive ability to model (B)

H_1 : Model (A) produces “better” predictions than model (B).

H_0 is rejected if $\sum_{t=1}^{T}\hat{z}_{t|t-1}^{2(B)} \Big/ \sum_{t=1}^{T}\hat{z}_{t|t-1}^{2(A)} > CGR(T/2,\rho,\alpha)$

where $\hat{z}_{t|t-1}^{(.)} = \left(y_t - \hat{\mu}_{t|t-1}^{(.)}\right)\big/\hat{\sigma}_{t|t-1}^{(.)}$, $\rho = Cor\left(\hat{z}_{t|t-1}^{(A)},\hat{z}_{t|t-1}^{(B)}\right)$ and $CGR(T/2,\rho,a)$ is the $100(1-\alpha)$ percentile of the CGR distribution.

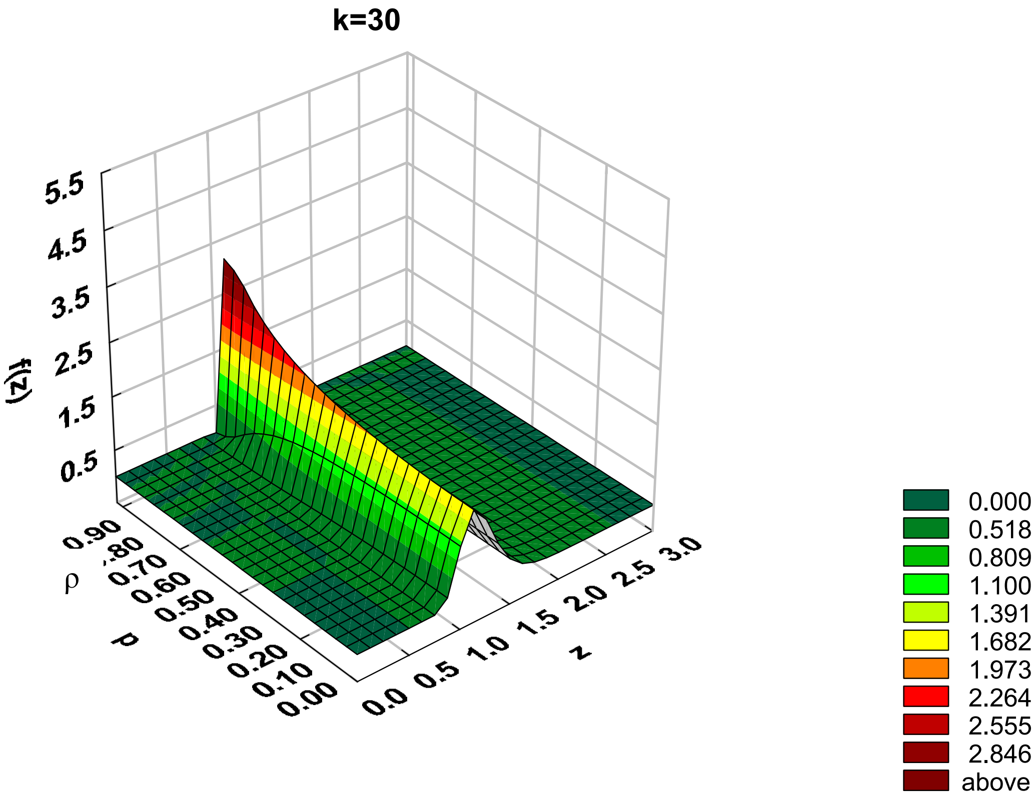
Cumulative Function of the CGR Distribution

$$\Phi(z)=\int_0^z \frac{\left(1-\rho^2\right)^{\frac{T}{2}}}{B\left(\frac{T}{2},\frac{T}{2}\right)} x^{\frac{T}{2}-1}(1+x)^{-T}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{T+1}{2}} dx = 1-a,$$

where $B\left(\frac{T}{2},\frac{T}{2}\right)=\Gamma\left(\frac{T}{2}\right)^2 \Big/ \Gamma(T)$.

The probability density function of the CGR Distribution

$$f(z)=\frac{\left(1-\rho^2\right)^k}{B(k,k)} z^{k-1}(1+z)^{-2k}\left[1-\left(\frac{2\rho}{z+1}\right)^2 z\right]^{-\frac{2k+1}{2}}$$, for $z \geq 0$, $0 \leq \rho < 1$, $k = 30$.



The Minimum Multivariate Gamma (MMG) Hypothesis Testing

Degiannakis and Xekalaki (Technical Report 214, 2004a)

Hypothesis Test

H_0 : Models m_i are of equivalent predictive ability

H_1 : Model $m_{(1)}$ has the highest predictive ability.

H_0 is rejected if $X_{(1)} \equiv \min_{m_i}\left(2^{-1}\sum_{t=1}^{T} \hat{z}_{t|t-1}^{2(m_i)} \right) > F_{X_{(1)}}\left(T/2, C_{12...n}, a\right)$

where $\hat{z}_{t|t-1}^{(.)} = \left(y_t - \hat{\mu}_{t|t-1}^{(.)}\right) / \hat{\sigma}_{t|t-1}^{(.)}$, $C_{12...n} = \begin{bmatrix} 1 & \rho_{12} & ... & \rho_{1n} \\ \rho_{12} & 1 & ... & \rho_{2n} \\ ... & ... & & ... \\ \rho_{1n} & \rho_{2n} & ... & 1 \end{bmatrix}$,

$\rho_{ij} \equiv Cor\left(\hat{z}_{t|t-1}^{(i)}, \hat{z}_{t|t-1}^{(j)}\right)$ and $F_{X_{(1)}}\left(T/2, C_{12...n}, a\right)$ is the $100(1-\alpha)$ percentile of the MMG distribution.

Cumulative Function of the MMG Distribution

$$F_{X_{(1)}}(x;a,C_{12\dots n}) = \sum_{j=1}^{n}(-1)^{j-1}\sum_{n \quad j} F_{X_{i_1},X_{i_2},\dots,X_{i_j},}(x,x,\dots,x) =$$

$$= \sum_{j=1}^{n}(-1)^{j-1}\sum_{n \quad j}\ \sum_{0\le r_{i_1},r_{i_1 i_2},\dots,r_{i_1 i_2\dots i_j}<\infty}\left(a_{\left(\sum\limits_{j=1}^{n}\sum\limits_{n \quad j} r_{i_1 i_2\dots i_j}\right)} \left(\prod_{j=1}^{n}\prod_{n \quad j}\frac{C_{i_1 i_2\dots i_j}^{r_{i_1 i_2\dots i_j}}}{r_{i_1 i_2\dots i_j}!}\right)\prod_{k=1}^{n}\frac{I_{\sum\limits_{j=1}^{n}\Sigma_k r_{i_1 i_2\dots i_j}}(x,a)}{a_{\left(\sum\limits_{j=1}^{n}\Sigma_k r_{i_1 i_2\dots i_j}\right)}}\right)$$

where,

- $\sum\limits_{0\le r_1,r_2,\dots,r_{12\dots n}<\infty}$ contains 2^n-1 terms, i.e. for $n=3$, the terms are

$r_1, r_2, r_3, r_{12}, r_{13}, r_{23}, r_{123},$

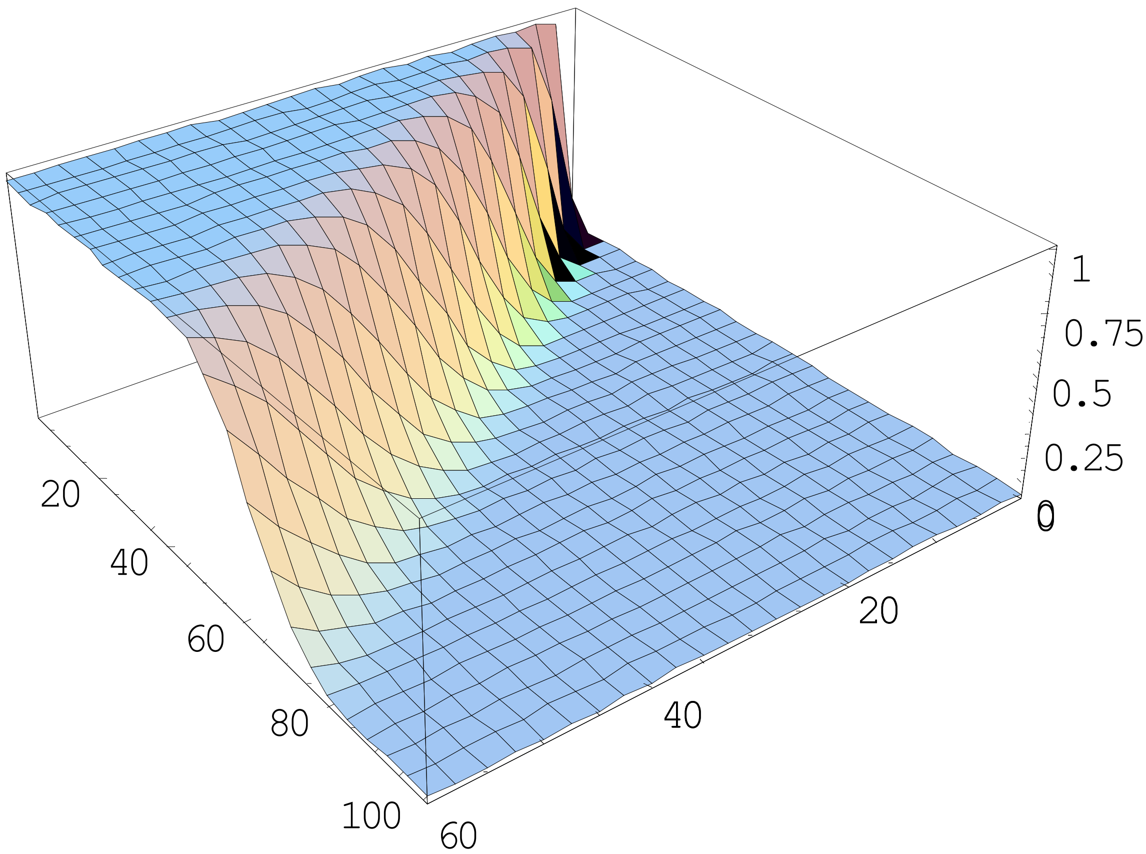
- $I_r(x_k,a) = \int_0^{x_k}\frac{t^{a-1}e^{-t}L_r(t,a)}{\Gamma(a)}dt = \frac{(-1)^r a_{(r)}}{\Gamma(a)}\sum_{l=0}^{r}\frac{(-r)_{(l)}}{a_{(l)}l!}\left(\Gamma(a+l)-\Gamma_{x_k}(a+l)\right)$
- $\Gamma(a)=\int_0^{\infty}e^{-t}t^{a-1}dt$ and $\Gamma_x(a)=\int_x^{\infty}e^{-t}t^{a-1}dt$
- $\prod\limits_{n \quad j} x_{i_1 i_2\dots i_j} = \prod\limits_{i_1=1}^{n\wedge n-j+1}\ \prod\limits_{i_2=i_1+1}^{n\wedge n-j+2}\cdots\prod\limits_{i_m=i_{m-1}+1}^{n\wedge n-j+m}\cdots\prod\limits_{i_j=i_{j-1}+1}^{n} x_{i_1 i_2\dots i_j}$
- $(n\wedge n-j+k)\equiv \min(n,n-j+k)$
- Σ_k denotes summation over the set

$\{i_1=1,2,\dots,k;\, i_m=i_{m-1}+1, i_{m-1}+2,\dots,n-j+m, m=2,3,\dots,j\}$

- $C_{12\dots n}=\begin{bmatrix} 1 & \rho_{12} & \dots & \rho_{1n} \\ \rho_{12} & 1 & \dots & \rho_{2n} \\ \dots & \dots & & \dots \\ \rho_{1n} & \rho_{2n} & \dots & 1 \end{bmatrix}$, $\rho_{ij}\equiv Cor\left(\hat{z}_{t|t-1}^{(i)},\hat{z}_{t|t-1}^{(j)}\right)$

Figure 3. The cumulative density function of the tri-variate minimum multivariate gamma distribution, for $60 \geq x \geq 0$, $100 \geq a > 0$, and $\rho_{1,2} = 60\%$, $\rho_{1,3} = 95\%$ and $\rho_{2,3} = 95\%$ the non-diagonal elements of C_{123}.

$$F_{X_{(1)}}(x; a, C_{123}) = 3F_{X_1}(x) - \sum_{i_1=1}^{2}\sum_{i_2=2}^{3} F_{X_{i_1}, X_{i_2}}(x, x) + F_{X_1, X_2, X_3}(x, x, x)$$



The SPEC Model Selection Algorithm

The estimation steps required at time k for each model m by the SPEC model selection algorithm. At time k $(k = T, T+1, \ldots)$, select the model m with the minimum value for the sum of the squares of the T most recent standardized one-step-ahead prediction errors, $\sum_{t=k-T+1}^{k} \hat{z}_{t|t-1}^{2(m)}$.

Model	**Time** $k = T$	$k = T+1$	$\ldots$	$k = T+j$	$\ldots$
$m = 1$	$\sum_{t=1}^{T} \hat{z}_{t\|t-1}^{2(1)}$	$\sum_{t=2}^{T+1} \hat{z}_{t\|t-1}^{2(1)}$	$\ldots$	$\sum_{t=j+1}^{T+j} \hat{z}_{t\|t-1}^{2(1)}$	$\ldots$
$m = 2$	$\sum_{t=1}^{T} \hat{z}_{t\|t-1}^{2(2)}$	$\sum_{t=2}^{T+1} \hat{z}_{t\|t-1}^{2(2)}$	$\ldots$	$\sum_{t=j+1}^{T+j} \hat{z}_{t\|t-1}^{2(2)}$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	
$m = i$	$\sum_{t=1}^{T} \hat{z}_{t\|t-1}^{2(i)}$	$\sum_{t=2}^{T+1} \hat{z}_{t\|t-1}^{2(i)}$	$\ldots$	$\sum_{t=j+1}^{T+j} \hat{z}_{t\|t-1}^{2(i)}$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	
$m = M$	$\sum_{t=1}^{T} \hat{z}_{t\|t-1}^{2(M)}$	$\sum_{t=2}^{T+1} \hat{z}_{t\|t-1}^{2(M)}$	$\ldots$	$\sum_{t=j+1}^{T+j} \hat{z}_{t\|t-1}^{2(M)}$	$\ldots$

Evaluating the Forecast Performance of the SPEC Algorithm

- Part A Statistical Loss Functions.

Degiannakis and Xekalaki (Technical Report 133, 2001b)

- Part B Economic Loss Functions – Options Trading.

Degiannakis and Xekalaki (Technical Report 131, 2001a)

- Part C Economic Loss Functions – Simulated Options Trading.

Xekalaki and Degiannakis (Comp. Stat. Data Anal., 2005 and Technical Report 205, 2004b)

Evaluating the Volatility Forecast Performance Part B Economic Loss Functions & Real Options Data

Options

- S&P500 stock index daily returns (March 14th, 1996 to June 2nd, 2000).

- S&P500 index options (March 11th, 1998 to June 2nd, 2000).

- The **option** is a security that gives its owner the right, not the obligation, to buy or sell an asset at a fixed price (**exercise price**) within a specified period of time.

- A **call option** is the right to buy a number of shares, of the underline asset, at a fixed price on or before the maturity day.

- A **put option** is a right to sell a number of shares, of the underline asset, at a fixed price on or before the maturity day.

- A **straddle option** is the purchase (or sale) of both a call and a put option, of the underline asset, with the same maturity day.

- The **maturity (expiration) day** is the latest date that the option can be exercised.

- If the option can be exercised only at the maturity day it is called **European** option, whereas an **American** option can be exercised on or before the expiration day.

Black & Scholes (Journal of Political Economy, 1973) Option Pricing Formula

Pricing call and put options at time $t+1$ given the information available at time t:

$$C_{t+1|t}^{(\tau)} = S_t e^{-\gamma_t \tau} N(d_1) - K e^{-rf_t \tau} N(d_2)$$

$$P_{t+1|t}^{(\tau)} = -S_t e^{-\gamma_t \tau} N(-d_1) + K e^{-rf_t \tau} N(-d_2)$$

$$d_1 = \frac{\ln\left(S_t / K \right) + \left(rf_t - \gamma_t + 1/2 \left(\sigma_{t+1|t}^{(\tau)} \right)^2 \right)\tau}{\sigma_{t+1|t}^{(\tau)} \sqrt{\tau}}$$

$$d_2 = d_1 - \sigma_{t+1|t}^{(\tau)} \sqrt{\tau}$$

$C_{t+1|t}^{(\tau)}$: the forecasted price of a call option, at time $t+1$, given the information at time t, with τ days to maturity,

$P_{t+1|t}^{(\tau)}$: the forecasted price of a put option, at time $t+1$, given the information at time t, with τ days to maturity,

S_t: the daily closing stock price at time t, as a forecast of S_{t+1},

τ: the remain life of the option in days (time to maturity),

rf_t: the annual continuously compounded risk free interest rate (i.e. three-month treasury bill),

γ_t: the dividend yield of S_t,

K: the exercise (or strike) price at maturity day τ,

$N(.)$: the cumulative normal distribution function,

$\sigma_{t+1|t}^{(\tau)} = \left(\tau^{-1} \sum_{i=2}^{\tau+1} \hat{\sigma}_{t+i|t}^2 \right)^{1/2}$: the volatility during the life of the option.

Trading Rules

If $C_{t+1|t}^{(\tau)} + P_{t+1|t}^{(\tau)} > P_t^{(\tau)} + C_t^{(\tau)} \Rightarrow$ The straddle is bought at time t.

If $C_{t+1|t}^{(\tau)} + P_{t+1|t}^{(\tau)} < P_t^{(\tau)} + C_t^{(\tau)} \Rightarrow$ The straddle is sold at time t.

The rate of return from trading an option is:

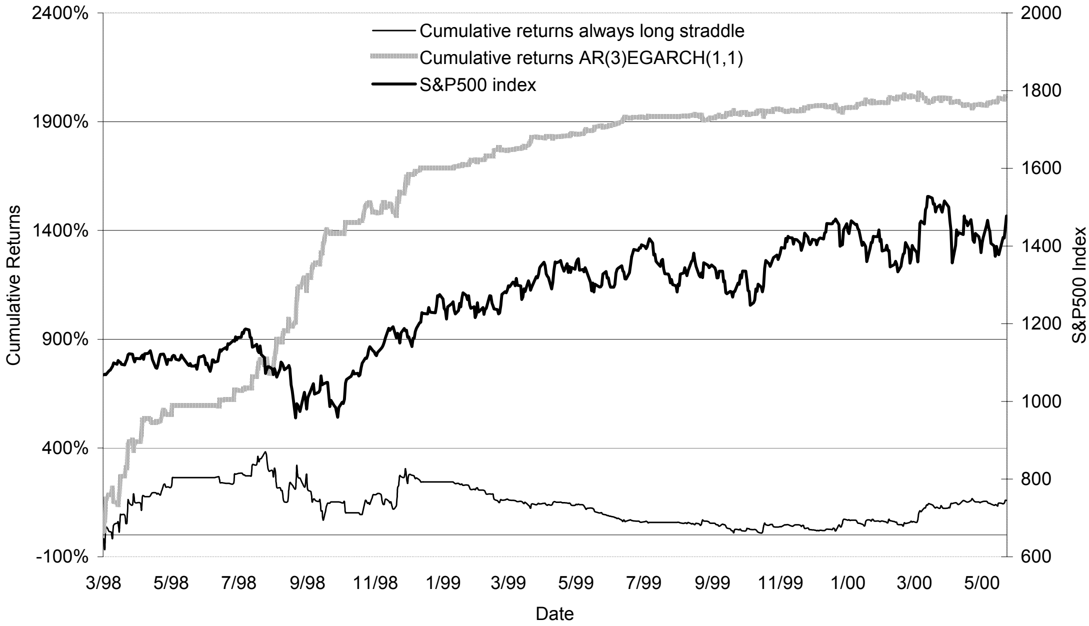
$$RT_t = \frac{C_t + P_t - C_{t-1} - P_{t-1}}{C_{t-1} + P_{t-1}}, \text{ on buying a straddle,}$$

$$RT_t = \frac{-C_t - P_t + C_{t-1} + P_{t-1}}{C_{t-1} + P_{t-1}}, \text{ on sort-selling a straddle.}$$

Daily rate of return from trading straddles on the S&P500 index based on the 85 ARCH volatility forecasts (11 March 1998 – 2 June 2000).

Rank	Model	Mean	Stand. Dev.	t- test	Days
1	AR(3)EGARCH(1,1)	4.42%	17.75%	5.32	456
2	AR(2)EGARCH(1,1)	4.40%	17.75%	5.29	456
3	AR(2)EGARCH(2,1)	4.40%	17.76%	5.29	456
4	AR(0)TARCH(1,2)	4.39%	17.78%	5.27	455
5	AR(4)EGARCH(1,1)	4.39%	17.76%	5.28	456
6	AR(1)EGARCH(1,1)	4.33%	17.77%	5.2	456
7	AR(4)EGARCH(2,1)	4.33%	17.77%	5.21	456
8	AR(3)EGARCH(2,1)	4.32%	17.77%	5.19	456
9	AR(1)EGARCH(2,1)	4.31%	17.78%	5.18	456
10	AR(3)TARCH(1,1)	4.26%	17.79%	5.11	456
	…	…	…	…	…
76	AR(4)EGARCH(0,1)	3.37%	17.98%	4	456
77	AR(2)EGARCH(0,1)	3.35%	17.98%	3.98	456
78	AR(2)EGARCH(1,2)	3.27%	18.00%	3.88	456
79	AR(4)TARCH(0,1)	3.27%	18.00%	3.88	456
80	AR(0)EGARCH(1,2)	3.16%	18.02%	3.74	456
81	AR(1)EGARCH(0,2)	2.90%	18.06%	3.43	456
82	AR(2)EGARCH(0,2)	2.89%	18.06%	3.41	456
83	AR(3)EGARCH(0,2)	2.86%	18.07%	3.38	456
84	AR(4)EGARCH(0,2)	2.86%	18.07%	3.38	456
85	AR(0)EGARCH(0,2)	2.53%	18.12%	2.98	456

Figure 4. Cumulative rate of return of i) the AR(3)EGARCH(1,1) agent and ii) an agent who takes a long position on every trading day from trading straddles on the S&P500 index. (11 March 1998 – 2 June 2000).



- An agent trades a contract has to pay a transaction cost, X, of \$2, which reflects the bid – ask spread.
- Straddles are traded only when the absolute difference between forecasted and today's option price exceeds the amount of the filter, F, yielding a net rate of return of:

$$NRT_t = \begin{cases} \dfrac{C_t + P_t - C_{t-1} - P_{t-1} - X}{C_{t-1} + P_{t-1}} & \text{, if } C_{t+1|t}^{(\tau)} + P_{t+1|t}^{(\tau)} - C_{t-1} - P_{t-1} > F_{il} \\ \dfrac{C_{t-1} + P_{t-1} - C_t - P_t - X}{C_{t-1} + P_{t-1}} + r_{f_t} & \text{, if } C_{t-1} + P_{t-1} - C_{t+1|t}^{(\tau)} - P_{t+1|t}^{(\tau)} > F_{il} \\ rf_t & \text{, otherwise} \end{cases}$$

ARCH models that yield the highest rate of return from trading straddles on the S&P500 index (11 March 1998 – 2 June 2000), after a \$2 transaction cost and various values of filters.

Filter	Model	Mean	St.Dev	t-ratio	p-value	Trading Days	Total Returns
\$1.25	AR(2)EGARCH(1,1)	0.84%	16.66%	1.08	0.28	421	385%
\$1.75	AR(0)GARCH(2,2)	0.91%	16.96%	1.14	0.25	385	413%
\$2.00	AR(0)GARCH(1,2)	0.93%	16.94%	1.18	0.24	381	425%
\$2.25	AR(0)GARCH(1,2)	1.09%	16.49%	1.41	0.16	362	496%
\$2.75	AR(4)GARCH(0,2)	1.23%	15.34%	1.71	0.09	346	559%
\$3.50	AR(3)GARCH(0,2)	1.35%	15.24%	1.89	0.06	322	614%

Trading options based on the SPEC algorithm

- In order to apply the SPEC model selection method, the sum of squared standardized one-step-ahead prediction errors, $\sum_{t=1}^{T} \hat{z}_{t|t-1}^{2}$, was estimated considering various values for T, and, in particular, $T = 5(5)80$.

Daily rate of return from trading straddles on the S&P500 index based on the ARCH models selected by the SPEC model selection method (11 March 1998 – 2 June 2000).

Model Selection Method	$2 transaction cost $3.50 filter			
	Mean	Stand. Dev.	t-test	Days
SPEC(5)	1.46%	15.85%	1.97	329
SPEC (10)	1.14%	16.03%	1.51	334
SPEC (15)	1.06%	15.76%	1.44	331
SPEC (20)	0.99%	15.80%	1.33	338
SPEC (25)	1.19%	15.70%	1.62	342
SPEC (30)	1.16%	15.71%	1.57	342
SPEC (35)	1.19%	15.82%	1.61	338
SPEC (40)	1.30%	15.77%	1.76	330
SPEC (45)	1.18%	15.81%	1.60	334
SPEC (50)	1.06%	15.97%	1.42	337
SPEC (55)	1.13%	15.98%	1.51	338
SPEC (60)	0.93%	16.04%	1.24	341
SPEC (65)	1.17%	15.85%	1.57	339
SPEC (70)	1.09%	15.81%	1.47	333
SPEC (75)	1.05%	15.75%	1.42	327
SPEC (80)	0.69%	16.14%	0.92	334

Daily rate of return from trading straddles (11 March 1998 – 2 June 2000).

Trans. Cost - Filter	Model	Mean	St.Dev	t-ratio	p-value	Trading Days	Total Returns
$2.00 - $3.50	AR(3)GARCH(0,2)	1.35%	15.24%	1.89	0.06	456	615.6%
$2.00 - $3.50	SPEC(5)	1.46%	15.85%	1.97	0.05	456	665.8%

Trading Options Based on Methods of Model Selection.

$$AIC = l_T\left(\hat{\theta}\right) - n$$
$$SBC = l_T\left(\hat{\theta}\right) - 2^{-1} n \ln\left(T\right).$$

$l_T\left(\hat{\theta}\right)$ is the maximized value of the log-likelihood function of a model, where $\hat{\theta}$ is the maximum likelihood estimator of the parameter vector θ based on a sample of size T and n denotes the dimension of θ.

ARCH model selection methods: $\sigma^2_{t(N)} = N^{-1}\sum_{i=1}^{N}\hat{\sigma}^2_{t+i|t}$ denotes the forecasting variance over an N day period measured at day t and $s^2_{t(N)} = N^{-1}\sum_{i=1}^{N} y^2_{t+i}$ denotes the realized variance over the same period.

1. Square Error of Conditional Variance (SEVar):

$$\sum_{t=1}^{T}\left(\left(\sigma^2_{t(N)} - s^2_{t(N)}\right)^2\right)$$

2. Absolute Error of Conditional Variance (AEVar):

$$\sum_{t=1}^{T}\left(\left|\sigma^2_{t(N)} - s^2_{t(N)}\right|\right)$$

3. Square Error of Conditional Standard Deviation (SEDev):

$$\sum_{t=1}^{T}\left(\left(\sigma_{t(N)} - s_{t(N)}\right)^2\right)$$

4. Absolute Error of Conditional Standard Deviation (AEDev):

$$\sum_{t=1}^{T}\left(\left|\sigma_{t(N)} - s_{t(N)}\right|\right)$$

5. Heteroscedasticity Adjusted Squared Error of Cond. Variance (HASEVar):

$$\sum_{t=1}^{T}\left(\left(1 - s^2_{t(N)}\big/\sigma^2_{t(N)}\right)^2\right)$$

6. Heteroscedasticity Adjusted Absolute Error of Cond. Variance (HAAEVar):

$$\sum_{t=1}^{T}\left(\left|1 - s^2_{t(N)}\big/\sigma^2_{t(N)}\right|\right)$$

7. Heteroscedasticity Adjusted Squared Error of Cond. St. Deviation (HASEDev):

$$\sum_{t=1}^{T}\left(\left(1 - s_{t(N)}\big/\sigma_{t(N)}\right)^2\right)$$

8. Heteroscedasticity Adjusted Absolute Error of Cond. St. Deviation (HAAEDev):

$$\sum_{t=1}^{T}\left(\left|1 - s_{t(N)}\big/\sigma_{t(N)}\right|\right)$$

9. Logarithmic Error of Conditional Variance (LEVar):

$$\sum_{t=1}^{T}\left(\ln\left(s^2_{t(N)}\big/\sigma^2_{t(N)}\right)^2\right)$$

Daily rate of return from trading straddles on the S&P500 index based on the ARCH models selected by the AIC and SBC model selection methods (11 March 1998 – 2 June 2000).

Model Selection Method	Mean	Stand. Dev.	t-test	Days
AIC	1.21%	18.22%	1.21	333
SBC	1.41%	18.49%	1.4	338

Daily rate of return from trading straddles on the S&P500 index based on the ARCH models selected by a set of model selection methods (11 March 1998 – 2 June 2000).

Model Selection Method	Sample size	Mean	Stand. Dev.	t-test	Days
SEVar	T = 40	0.78%	18.69%	0.78	349
AEVar	T = 60	0.99%	18.17%	1.01	346
SEDev	T = 60	0.93%	18.41%	0.95	357
AEDev	T = 60	1.02%	18.17%	1.06	352
HASEVar	T = 10	1.45%	18.47%	1.45	341
HAAEVar	T = 40	1.66%	18.74%	1.63	337
HASEDev	T = 20	1.17%	18.68%	1.17	348
HAAEDev	T = 30	1.38%	18.40%	1.43	365
LEVar	T = 80	0.96%	18.25%	0.98	347

Scope for Further Research

- SPEC algorithm application in ARCH models with non-normally distributed conditional innovations. Approaches similar to Politis (University of California 2003b, 2004) may add power in the applicability of the SPEC method.
- SPEC algorithm performance with a set of more flexible conditional variance specifications.
- Artificial neural networks, chaotic dynamical systems, nonlinear parametric and nonparametric models are some examples from the literature dealing with conditional mean predictions.
- Value-at-Risk (VaR) at a given probability level a, is the predicted amount of financial loss of a portfolio over a given time horizon. The forecasting of the VaR number is another area of applied financial statistics that the added value of the SPEC method should be explored.
- However, the SPEC method can be compared to models that are based on intra-day datasets, like the ARFIMA methodology (Granger, Journal of Econometrics, 1980). A future application of the SPEC model selection method on inter-day and intra-day models would be interested.
- The SPEC algorithm should be applied in more data sets such as stocks, stock indices, bonds, commodities and exchange rates.
- In further research, it may be interested to investigate whether the selection of specific models is related with any economic factors.
- The MMG hypothesis testing, although a complicate methodology, provides the researchers with a tool that takes into account the forecasting ability of all the candidate models. It should not be considered only as the theoretical justification of the SPEC algorithm in a multivariate framework. In a future work, we plan to study the added value of the MMG test in empirical applications.

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