COMMUN. STATIST.-THEOR. METH., 12(21), 2503-2509 (1983)

HAZARD FUNCTIONS AND LIFE DISTRIBUTIONS IN DISCRETE TIME

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Key Words and Phrases; Life distribution; reliability; survival distribution; hazard function; geometric distribution; Waring distribution; negative hypergeometric distribution.

ABSTRACT

The problem of studying lifelength distributions in discrete time is considered for certain forms of hazard functions. A class of life distributions that consists of the geometric, the Waring and the negative hypergeometric distributions is shown to result when the hazard function is inversely proportional to some linear function of time.

1. INTRODUCTION

Let T be a nonnegative random variable (r.v.) which represents some lifetime. In practice, if only because of limitations of measuring devices, it may be appropriate to consider models where T is a discrete r.v. Various authors (e.g., Cox (1972), Kalbfleisch and Prentice (1980, p. 7)) have touched this problem and provided discrete analogues for the basic concepts used in reliability theory.

Denote by P_t the probability $P(T=t)$, $t=0,1,\cdots$. Let $F(t) = P(t \le t) = \sum_{i=0}^{t} P_i$, $t=0,1,\cdots$, be the lifetime distribution function and $R(t) = P(T \ge t) = 1-F(t-1)$, $t=0,1,\cdots$ the corresponding survivor function. The hazard function $\lambda(t)$ is given by

$$\lambda(t) = P(T=t \mid T>t-1) = P_t/R(t), \quad t=0,1,\cdots.$$

The main result of this paper characterizes lifetime distributions whose hazard function is of the form $1/(a+bt)$. Specifically, such distributions must be either geometric, negative hypergeometric, or Waring distributions. (For information about Waring distributions see Irwin (1975) and Xekalaki (1981).) This result is established in Section 2. The final section discusses the relationship with continuous time results.

2. THE MAIN RESULTS

We first prove the following lemma.

Lemma: Consider a r.v. T defined on the set $\{0,1,2,\cdots,m\}$, $m \in \{0,1,\cdots\} \cup \{+\infty\}$ and such that $0 < P_0 < 1$ and

$$\lambda(r) = 1/(a+br), \quad a,b \in R, \quad a > 0, \quad r=0,1,\cdots,m. \tag{2.1}$$

Then, (i) $a > 1$ and (ii) T has a distribution with finite support $(m < +\infty)$ if and only if $b < 0$ and in this case $b = (1-a)/m$.

Proof: (i) From (2.1) for $r = 0$ we have that $a = 1/P_0$. Hence $a > 1$.

(ii) Let T have a distribution with finite support. Then we have from (2.1), for $r = m$, $P(T \ge m) = (a+bm)\cdot P_m$ which implies that $a+bm = 1$. Thus, since $a > 1$, $b = (1-a)/m < 0$.

To show the converse of this result, assume that $b < 0$ and set $m = \sup\{t : R(t) > 0\}$. Then, by (2.1), $m = \sup\{t : P_t > 0$ and $a+bt > 0\} \le \sup\{t : a+bt > 0\} < -\frac{a}{b} < +\infty$.
This completes the proof of (ii) and therefore the lemma has been established.

The following theorem can now be proved.

Theorem 2.1: Let T be a r.v. taking values in the set $\{0,1,2,\cdots,m\}$, $m \in \{0,1,\cdots\} \cup \{+\infty\}$. Assume that $0 < P_0 < 1$. Then,

$$\lambda(t) = 1/(a+bt), \quad t=0,1,2,\cdots,m, \tag{2.2}$$

for $a > 0$ and real b if and only if

(i) T is geometric with probability function

$$P_t = \frac{1}{a}\left(\frac{a-1}{a}\right)^t, \quad t=0,1,2,\cdots, \tag{2.3}$$

or (ii) T is Waring with probability function

$$P_t = \left(\frac{a-1}{b}\right)_{(t)} \Big/ a\left(\frac{a}{b}+1\right)_{(t)}, \quad t=0,1,2,\cdots, \tag{2.4}$$

where $\alpha_{(t)} = \Gamma(\alpha+t)/\Gamma(\alpha)$, $\alpha > 0$, $t=0,1,2,\cdots$,

or (iii) T is negative hypergeometric with probability function

$$P_t = \binom{-1}{t}\binom{1/b}{m-t}\Big/\binom{-1+1/b}{m}, \quad t=0,1,\cdots,m, \quad m=(1-a)/b. \tag{2.5}$$

(These cases correspond to $b = 0$, $b > 0$ and $b < 0$ respectively).

Proof: We first prove the "only if" part of the theorem. Let (2.2) hold. It follows then that

$$P(T \ge r) = (a+br)\cdot P_r, \quad r=0,1,\cdots,m. \tag{2.6}$$

Specializing (2.6) for $r = t$ and $r = t+1$ and subtracting the resulting equations we obtain

$$P_{t+1}-(a-1+bt)P_t/(a+b+bt) = 0, \quad t=0,1,\cdots,m. \tag{2.7}$$

This is a first order difference equation in P_t. To solve this equation, we consider the cases: (i) $b=0$, (ii) $b>0$ and (iii) $b<0$.

Case (i): Let $b = 0$. Then, from the lemma, $m = +\infty$ and hence (2.7) becomes $P_{t+1}-(a-1)P_t/a = 0$, $t=0,1,\cdots$, whose unique solution subject to $\sum_{t=0}^{\infty} P_t = 1$ is given by (2.3). (Recall $a > 1$).

Case (ii): Let $b > 0$. Then, from (2.7) we have

$$P_t = P_0\left(\frac{a-1}{b}\right)_{(t)} \Big/ \left(\frac{a+b}{b}\right)_{(t)}, \quad t=0,1,2,\cdots,m.$$

Also, from the lemma it follows that $m = +\infty$. Hence

$$P_0^{-1} = \sum_{t=0}^{\infty} \left(\frac{a-1}{b}\right)_{(t)} \Big/ \left(\frac{a+b}{b}\right)_{(t)} = a.$$

Therefore, the probability distribution of T is given by (2.4).

Case (iii): Let $b < 0$ (in which case $m < +\infty$). Then (2.7) yields

$$P_t = P_0\left(\frac{1-a}{b}\right)^{(t)} \Big/ \left(-\frac{a+b}{b}\right)^{(t)}, \quad t = 0,1,\cdots,m$$

where $\alpha^{(t)} = \alpha(\alpha-1)\cdots(\alpha-t+1)$, $t = 0,1,\cdots,m$; ($\alpha^{(0)} = 1$). From the lemma, $(1-a)/b = m > 0$, i.e., $(1-a)/b$ is a positive integer. Hence $P_0^{-1} = \sum_{t=0}^{m} m^{(t)}/(-1-a/b)^{(t)} = (-1+1/b)^{(m)}/(1/b)^{(m)}$. Therefore, for $t = 0,1,2,\cdots,m$,

$$\begin{aligned} P_t &= (1/b)^m m^{(t)}/(-1+1/b)^{(m)}(-1-a/b)^{(t)} \\ &= (-1)^t\binom{1/b}{m-t}\Big/\binom{-1+1/b}{m} \end{aligned}$$

which since $\binom{-1}{t} = (-1)^t$, leads to (2.5).

To prove the "if" part of the theorem first assume that T has the negative hypergeometric distribution with parameters $n > 0$, 1 and $\nu > 0$. Then,

$$\begin{aligned} P(T \geq t) &= \sum_{r=t}^{n} \binom{-1}{r}\binom{-\nu}{n-r}\Big/\binom{-1-\nu}{n} = \sum_{r=t}^{n} \frac{n!}{(n-t)!}\,\frac{\nu_{(n-r)}}{(\nu+1)_{(n)}} \\ &= \frac{(n-t+1)_{(t)}}{(n-t+\nu+1)_{(t)}} \sum_{r=0}^{n-t} \frac{(n-t)!}{(n-r-t)!}\,\frac{\nu_{(n-r-t)}}{(\nu+1)_{(n-t)}} \\ &= \frac{n^{(t)}}{(n+\nu)^{(t)}} \sum_{r=0}^{n-t} \binom{-1}{r}\binom{-\nu}{n-t-r}\Big/\binom{-1-\nu}{n-t} \\ &= \frac{n+\nu-t}{\nu}\,\frac{n!}{(n-t)!}\,\frac{\nu_{(n-t)}}{(\nu+1)_{(n)}} = (n+\nu-t)\binom{-1}{t}\binom{-\nu}{n-t}\Big/\nu\binom{-1-\nu}{n} \\ &= (n+\nu-t)P_t/\nu. \end{aligned}$$

Therefore, $\lambda(t)$ is of the form (2.2) for $a = 1+n/\nu > 1$ and $b = -\nu^{-1} < 0$.

The geometric case is trivial. Also, the Waring case follows from Irwin (1975). Hence the theorem has been established.

Notice that when $b = -1$ the distribution in (2.5) reduces to the discrete uniform on $\{0,1,\cdots,a-1\}$. Hence the following corollary holds true.

Corollary: Let T be defined as in the theorem. Then, $\lambda(t) = 1/(a-t)$, $t=0,1,\cdots,a-1$ if and only if $P_t = 1/a$, $t=0,1,\cdots,a-1$.

Obvious modifications in the argument used to prove Theorem 2.1 lead to similar conclusions concerning lifelength distributions when there exists a $t_0 \neq 0$ such that $F(t) = 0$ for $t < t_0$.

3. CONNECTION WITH CONTINUOUS TIME RESULTS

The results that have been obtained bear striking similarities to results concerning lifelength distributions in continuous time obtained under similar hypotheses. Notice, for instance, that both the geometric distribution and its continuous analogue, the exponential distribution arise on the assumption of a constant hazard rate. On the other hand, as it is shown below (Theorem 3.1), a hazard rate of the form (2.2) with $t \in [0,+\infty)$ and $b > 0$ gives rise to the Pearson type VI (beta of the second type) distribution that has been shown by Irwin (1975) to be the continuous analogue of the Waring distribution. Moreover, when $b < 0$, the continuous lifelength distribution that arises is beta of the first kind (Pearson's type I) which is rather interesting since the negative hypergeometric that arises when t is discrete can be regarded as a discrete approximation to the beta distribution. This can be seen using the following argument.

Let $f_r = \binom{-1}{r}\binom{-\nu}{n-r}\Big/\binom{-1-\nu}{n}$. Then the slope to mean ordinate ratio at $x = r-\frac{1}{2}$ defined by $(f_r-f_{r-1})/\frac{1}{2}(f_r+f_{r-1})$ satisfies the relationship

$$\frac{f_r-f_{r-1}}{\frac{1}{2}\{f_r+f_{r-1}\}} = \frac{1-\nu}{n+\frac{\nu+1}{2}-r}, \quad r=1,2,\cdots.$$

Let $g(x)$ be the density function of the continuous distribution that $\{f_r, r=0,1,\cdots,n\}$ approximates. Then, equating the slope to mean ordinate ratio at $x = r-\frac{1}{2}$ to $\left(\frac{1}{g(x)}\frac{d}{dx}g(x)\right)_{x=r-\frac{1}{2}}$ and putting $x = y-\frac{1}{2}$ we have

$$d \log g(y)/dy = (1-\nu)/[n-y+(\nu+1)/2]$$

which by integration yields a beta density given by

$$g(y) = c\left(n+\frac{\nu+1}{2}-y\right)^{\nu-1}, 0 < y < n+\frac{\nu+1}{2}, \tag{3.1}$$

where c is the normalizing constant.

The following theorem will now be shown.

Theorem 3.1: Let X be a continuous r.v. defined on $(0,+\infty)$ and denoting some lifelength. Then the hazard function will be given by

$$\lambda(x) = 1/(\alpha+\beta x), \quad \alpha,\beta \in R, \quad \alpha > 0, \quad x \in (0,+\infty) \tag{3.2}$$

if and only if the distribution of X belongs to the Pearson family of continuous distributions with a density of the form

$$f(x) = \frac{1}{\alpha}\left(1+\frac{\beta}{\alpha}x\right)^{-\frac{1}{\beta}-1}, \quad x \in (0,+\infty). \tag{3.3}$$

Proof: It can be observed that (3.2) holds if and only if $R(x) = \exp\{-\int_0^x \frac{1}{\alpha+\beta t}dt\}$, i.e., if and only if $f(x) = -\frac{d}{dx}R(x)$ $= \frac{1}{\alpha+\beta x}\left(\frac{\alpha+\beta x}{\alpha}\right)^{-\frac{1}{\beta}}$ which implies that (3.3) is true.

One may notice now that if $\beta > 0$ then (3.3) represents the density function of the Pearson type VI distribution with parameters 1 and β^{-1} and scale parameter β/α. On the other hand, if $\beta < 0$ then (3.3) represents the density function of the beta distribution with parameters 1, β^{-1} and scale parameter $-\beta/\alpha$. Note that in the case $\beta = -1$, the lifelength distribution is the uniform in $(0,\alpha)$. Finally, if $\beta \to 0$ then (3.3) reduces to $f(x) = \frac{1}{\alpha}e^{-x/\alpha}$, $x > 0$, leading to the well-known result for the exponential life distribution.

The above analysis shows that the results of Section 2—apart from providing an insight to applied reliability problems where time is actually measured discretely—may also be useful in approximating the usually used continuous lifelength distributions whenever the corresponding hazard function is of the appropriate form.

ACKNOWLEDGEMENT

I would like to thank a referee for comments and suggestions that made the shortening of an earlier version of this paper an enjoyable task.

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Received by Editorial Board member July, 1982; Revised December, 1982 and March 1983.

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Refereed by R. K. Milne University of Western Australia, Nedlands, Australia.