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ON CHARACTERIZING THE BIVARIATE POISSON, BINOMIAL AND NEGATIVE BINOMIAL DISTRIBUTIONS

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1. INTRODUCTION

KORWAR [1] characterized the distribution of a non-negative r.v. X as Poisson, binomial and negative binomial when for another non-negative r.v. Y, the conditional distribution of Y given X is binomial and the regression of X on Y is linear. Here we state his result.

THEOREM 1.1 (KORWAR [1]). *Let* X *be a discrete r.v. on* $\{0,1,\ldots,m\}$ $(m\in I^{+}\cup\{+\infty\})$. *Assume that* $\mathrm{E}(X) < +\infty$. *Let* Y *be another non-negative discrete r.v. such that*

$$\mathrm{P}(Y = y \mid X = x) = \binom{x}{y} p^{y} q^{x-y} \tag{1.1}$$

$$(0 < p < 1,\ q=1-p,\ y=0,1,2,\ldots,x).$$

Then

(1.2) $\quad E(X|Y = y) = ay + b, \quad (a,b$ constants)

if and only if (iff)

$$X \sim \begin{cases} \text{Poisson } (b/q) & (a = 1), \\ \text{binomial } (b/1-a,\ (1-a)/(1-ap)) & (0 < a < 1), \\ \text{negative binomial } (b/a-1,\ (1-ap)/aq) & (a > 1). \end{cases}$$

The proof is based on the following theorem.

THEOREM 1.2 (KORWAR [1]). *Let* X, Y *be as in Theorem 1.1. Assume that (1.2) holds. Then (i)* $b > 0$, *(ii)* X *is bounded iff* $0 < a < 1$. *Also if* X *is bounded then* $b = m(1-a)$, *(iii)* $0 < a < p^{-1}$.

Obviously, Theorem 1.2 ensures the positivity of the Poisson, binomial and negative binomial parameters.

In Section 2 we consider an extension of Theorem 1.1 to the bivariate case which provides a characterization for the double Poisson, binomial and negative binomial distributions. A characterization of the double Poisson using RAO and RUBIN's [3] condition has been given by TALWALKER [4].

We go on in Section 3 to provide characterizations for the bivariate binomial and negative binomial distributions with p.g.f.'s $(p_{11} + p_{10}s + p_{01}t)^n$ and $p_{11}^k(1 - p_{10}s - p_{01}t)^{-k}$, respectively. The case of the bivariate Poisson with p.g.f. of the form $\exp[\lambda_1(s - 1) + \lambda_2(t - 1) + \lambda_{12}(st - 1)]$ is also discussed.

2. CHARACTERIZATION OF THE DOUBLE POISSON, BINOMIAL AND NEGATIVE BINOMIAL DISTRIBUTIONS

THEOREM 2.1. *Let* $\underline{X} = (X_1, X_2)$ *be a discrete random vector on* $\{0,1,\ldots,m_1\} \times \{0,1,\ldots,m_2\}$ $(m_i \in I^+ \cup \{+\infty\},$ $i=1,2)$. *Assume that* $E(X_i) < +\infty$ $(i=1,2)$. *Let* $\underline{Y} = (Y_1, Y_2)$ *be another non-negative random vector such that*

$$P(\underline{Y} = \underline{y} | \underline{X} = \underline{x}) = \prod_{i=1}^{2} \binom{x_i}{y_i} p_i^{y_i} q_i^{x_i - y_i} \tag{2.1}$$

$$(0 < p_i < 1,\ q_i = 1-p_i,\ y_i = 0,1,\ldots,x_i,\ i=1,2).$$

Then

$$E(X_i | \underline{Y} = \underline{y}) = a_i y_i + b_i, \quad (a_i, b_i \text{ constants}, \; i=1,2). \tag{2.2}$$

iff

$$X \sim \begin{cases} \text{double Poisson } (\underline{b}/\underline{q}) \quad (a_i = 1,\ i=1,2), \\ \text{double binomial } (\underline{b}/\underline{1}-\underline{a};\ (\underline{1}-\underline{a})/(\underline{1}-\underline{ap})) \\ \qquad (0 < a_i < 1,\ i=1,2), \\ \text{double negative binomial } (\underline{b}/(\underline{a}-\underline{1}); \\ (\underline{1}-\underline{ap})/\underline{aq}) \quad (a_i > 1,\ i=1,2) \end{cases}$$

where $\underline{u}/\underline{v} = (u_1/v_1, u_2/v_2)$, $\underline{uv} = (u_1 v_1, u_2 v_2)$.

PROOF. Necessity follows immediately.

For sufficiency we observe that using (2.1) and the identity

$$x\binom{x}{y} = (y+1)\binom{x}{y+1} + y\binom{x}{y} \tag{2.3}$$

we obtain

$$(2.4) \qquad E(X_i \mid \underline{Y} = \underline{y}) = y_i + (y_i+1) q_i \times$$

$$\times P(Y_i = y_i+1,\ Y_j = y_j) / (p_i P(\underline{Y} = \underline{y})) \qquad (i,j=1,2,\ i \neq j).$$

Hence from (2.2), (2.4) we have

$$\frac{q_i}{p_i} (y_i+1) P(Y_i = y_i+1,\ Y_j = y_j) =$$

$$= (a_i-1) y_i P(\underline{Y} = \underline{y}) + b_i P(\underline{Y} = \underline{y}) \quad (i,j=1,2,\ i \neq j).$$

Taking p.g.f.'s we obtain

$$(2.5) \qquad \frac{q_i}{p_i} \frac{\partial}{\partial t_i} G_{\underline{Y}}(\underline{t}) = t_i (a_i - 1) \frac{\partial}{\partial t_i} G_{\underline{Y}}(\underline{t}) + b_i G_{\underline{Y}}(\underline{t}) \qquad (i=1,2)$$

where $G_{\underline{Z}}(\underline{t})$ denotes the p.g.f. of $\underline{Z}$.

But it is known (RAO [2]) that

$$(2.6) \qquad G_{\underline{Y}}(\underline{t}) = G_{\underline{X}}(\underline{pt} + \underline{q}).$$

Then equation (2.5) can be written in terms of $G_{\underline{X}}(\underline{t})$ as

$$(2.7) \qquad \frac{\partial}{\partial t_i} \log G_{\underline{X}}(\underline{t}) = \frac{b_i}{q_i - (a_i - 1)(t_i - q_i)} \qquad (i=1,2).$$

(i) For $a_1 = a_2 = 1$, (2.7) reduces to

$$\frac{\partial}{\partial t_i} \log G_{\underline{X}}(\underline{t}) = \frac{b_i}{q_i} \qquad (i=1,2)$$

therefore

$$(2.8)\qquad G_{\underline{X}}(\underline{t}) = \exp[(b_1/q_1)(t_1-1)+(b_2/q_2)(t_2-1)].$$

From Theorem 1.2 we have $b_i > 0$ $(i=1,2)$. Hence (2.8) represents the p.g.f. of the double Poisson $(\underline{b}/\underline{q})$.

(ii) For $a_i \neq 1$ $(i=1,2)$ we obtain from (2.7) by integration

$$(2.9)\qquad G_{\underline{X}}(\underline{t}) = \prod_{i=1}^{2} (1-a_i p_i)^{\frac{b_i}{a_i-1}} \{q_i-(a_i-1)(t_i-q_i)\}^{-\frac{b_i}{a_i-1}}.$$

From Theorem 1.2 if $a_i < 1$ $(i=1,2)$ it follows that $b_i/(1-a_i)$ is an integer and hence (2.9) represents the p.g.f. of the double binomial $(\underline{b}/(\underline{1}-\underline{a});$ $(\underline{1}-\underline{a})/(\underline{1}-\underline{ap}))$.

If, on the other hand $a_i > 1$ $(i=1,2)$, (2.9) is the p.g.f. of the double negative binomial $(\underline{b}/\underline{a}-\underline{1};$ $(\underline{1}-\underline{ap})/\underline{aq})$.

Hence the theorem is established.

COROLLARY 1. *Let* $\underline{X} = (X_1,X_2)$, $\underline{Y} = (Y_1,Y_2)$ *be as in Theorem 2.1. Then (2.2) holds iff* $\mathrm{P}(\underline{X} = \underline{x}) = \mathrm{P}(X_1 = x_1)\ \mathrm{P}(X_2 = x_2)$ *where*

(i) $X_i \sim$ Poisson (b_i/q_i), $X_j \sim$ binomial $(b_j/1-a_j;$ $(1-a_j)/(1-a_jp_j))$ *for* $a_i=1$, $a_j < 1$ $(i\neq j;\ i,j=1,2)$,

(ii) $X_i \sim$ Poisson (b_i/q_i), $X_j \sim$ negative binomial $(b_j/a_j-1;$ $(1-a_jp_j)/a_jq_j)$ *for* $a_i=1$, $a_j > 1$ $(i\neq j;\ i,j = 1,2)$,

(iii) $X_i \sim$ binomial $(b_i/(1-a_i);$ $(1-a_i)/(1-a_ip_i))$, $X_j \sim$ negative binomial $(b_j/a_j-1;$ $(1-a_jp_j)/a_jq_j)$ *for* $a_i < 1$, $a_j > 1$ $(i\neq j;\ i,j=1,2)$.

3. CHARACTERIZATION OF THE BIVARIATE (DEPENDENT) BINOMIAL, NEGATIVE BINOMIAL AND POISSON DISTRIBUTIONS

Before proving the main result, we need to show the following

THEOREM 3.1. *Let* $\underline{X} = (X_1,X_2)$, $\underline{Y} = (Y_1,Y_2)$ *be as in Theorem 2.1. Assume that for some constants* a_i, b_i *such that* $a_i \neq 1$, $b_i/a_i - 1 = b_j/a_j - 1 = h$ $(i \neq j)$ *we have*

$$(3.1) \qquad E(X_i \mid \underline{Y} = \underline{y}) = a_i y_i + (a_i - 1) y_j + b_i, \quad (i \neq j,\ i,j = 1,2).$$

Then

(i) $b_i > 0$ $(i=1,2)$,

(ii) $\underline{X}$ *is bounded iff* $0 < a_i < 1$ $(i=1,2)$. *Moreover if* $\underline{X}$ *is bounded then* $b_i = (m_1+m_2)(1-a_i)$ $(i=1,2)$,

(iii) $0 < a_i < p_i^{-1}$ $(i=1,2)$.

PROOF.

(i) Letting $y_1=y_2=0$ equation (3.1) becomes (since $\underline{Y} \le \underline{X}$)

$$0 \le E(X_i \mid \underline{Y} = \underline{0}) = b_i \qquad (i=1,2).$$

But equality cannot hold since it would imply that

$$\sum_{x_1,x_2} x_i q_1^{x_1} q_2^{x_2} P(\underline{X} = \underline{x})/P(\underline{Y} = \underline{0}) = 0 \quad \text{i.e. } P(\underline{X} = \underline{x}) = 0$$

for all $\underline{x}$. But $\underline{X}$ is non-degenerate. Hence $b_i > 0$ $(i=1,2)$.

(ii) Let $\underline{X}$ be bounded. Then from (3.1) since

$\underline{X} \geq \underline{Y}$ we have

$$m_i = \mathrm{E}(X_i | \underline{Y} = \underline{m}) = a_i m_i + m_j(a_i - 1) + b_i$$

$$(i \neq j,\ i,j=1,2),$$

i.e.

$$(3.2) \qquad b_i = (m_1+m_2)(1-a_i) \qquad (i=1,2).$$

From the positivity of b_i it follows that $a_i < 1$ $(i=1,2)$. Also from (3.1) we have

$$(3.3) \qquad m_i \geq \mathrm{E}(X_i | \underline{Y} = \underline{0}) = b_i \qquad (i=1,2).$$

Hence from (3.2), (3.3) it follows that $a_i > 0$ $(i=1,2)$.

So, if $\underline{X}$ is bounded then $0 < a_i < 1$ $(i=1,2)$. The converse is also true since if for $0 < a_i < 1$ $\underline{X}$ were unbounded we would have from (3.1) that

$$y_i \leq a_i y_i + (a_i - 1) y_j + b_i \qquad (i \neq j,\ i,j=1,2),$$

i.e.

$$(1-a_i)(y_1+y_2) \leq b_i \qquad (i=1,2).$$

But it holds for all $\underline{y}$ only if $a_i \geq 1$ which is a contradiction.

(iii) It has been proved that either $0 < a_i < 1$ or $a_i > 1$. In the latter case, from (3.1), (2.1) we have

$$\mathrm{E}(X_i)(1-a_i p_i) = b_i + (a_i - 1) p_j \mathrm{E}(X_j) \quad (i \neq j,\ i,j=1,2).$$

From the finiteness of $E(X_i)$ $(i=1,2)$ it follows that $a_i < p_i^{-1}$ $(i=1,2)$.

Also $a_i < 1$ implies $a_i < p_i^{-1}$. Hence $0 < a_i < p_i^{-1}$ $(i=1,2)$. This completes the proof of the theorem.

THEOREM 3.2. *Let* $\underline{X} = (X_1, X_2)$, $\underline{Y} = (Y_1, Y_2)$ *be as in Theorem 2.1. Then (3.1) holds iff*

$$\underline{X} \sim \begin{cases} \text{bivariate binomial } (-h;\ (1-a_1)q_2/c, \\ \qquad (1-a_2)q_1/c) \quad (a_i > 1,\ i=1,2) \\ \text{bivariate negative binomial} \\ (h;\ (a_1-1)/q_1(a_1+a_2-1), \\ \ (a_2-1)/q_2(a_1+a_2-1)) \quad (a_i < 1,\ i=1,2) \end{cases}$$

where $c = (a_1+a_2-1)q_1q_2+(1-a_1)q_2+(1-a_2)q_1$.

PROOF. Necessity follows immediately.

Sufficiency. From (2.4) and (3.1) we obtain

$$P(Y_i = y_i+1,\ Y_j=y_j)q_i = p_i[(a_i-1)(y_1+y_2) + b_i]/ \tag{3.4}$$

$$/(y_i+1)P(\underline{Y} = \underline{y}) \quad (i\neq j,\ i,j=1,2),$$

i.e.

$$P(\underline{Y} = \underline{y}) = P(\underline{Y} = \underline{0}) \prod_{i=0}^{y_1-1} h_1(i,0) \prod_{j=0}^{y_2-1} h_2(y_1,j) \tag{3.5}$$

where $h_i(y_1,y_2) = P(Y_i = y_i+1,\ Y_j = y_j)/P(\underline{Y} = \underline{y})$ $(i,j=1,2,\ i\neq j)$. Hence

$$P(\underline{Y} = \underline{y}) = c_1 \frac{\Gamma(h+y_1+y_2)}{\Gamma(h)y_1!y_2!}[p_1(a_1-1)/q_1]^{y_1}[p_2(a_2-1)/q_2]^{y_2}$$

where $c_1 = (1 - p_1(a_1-1)/q_1 - p_2(a_2-1)/q_2)^h$. Then from (2.6) we have

$$G_{\underline{X}}(\underline{pt}+\underline{q}) = c_1(1-p_1(a_1-1)t_1/q_1-p_2(a_2-1)t_2/q_2)^{-h},$$

i.e.

$$(3.6) \qquad G_{\underline{X}}(\underline{t}) = c_2(1-(a_1-1)t_1/q_1(a_1+a_2-1)-(a_2-1)t_2/q_2(a_1+a_2-1))^{-h}$$

where $c_2 = (1-(a_1-1)/q_1(a_1+a_2-1)-(a_2-1)/q_2(a_1+a_2-1))^h$.

Using Theorem 3.1 it follows that if $a_i < 1$ $(i=1,2)$, h is a negative integer and hence (3.6) represents a bivariate binomial $(-h;\ (1-a_1)q_2/c,$ $(1-a_2)q_1/c)$. Also if $a_i > 1$ $(i=1,2)$, (3.6) represents the bivariate negative binomial $(h;\ (a_1-1)/q_1(a_1+a_2-1),$ $(a_2-1)/q_2(a_1+a_2-1))$.

NOTE. If we allow $a_i=1$ $(i=1,2)$ then (3.1) reduces to the necessary and sufficient condition for $\underline{X}$ to be double Poisson (Theorem 2.1).

The case of the bivariate Poisson is more complicated as the regression of X_i on $\underline{Y}$ is not linear. However, if we observe that its p.d.f. has the form

$$\exp\{-\lambda_1-\lambda_2-\lambda_{12}\}\lambda_1^{x_1}\lambda_2^{x_2}\,{}_2F_0(-x_1,-x_2;\ ;\ \lambda_{12}/\lambda_1\lambda_2)/x_1!x_2!$$

where ${}_2F_0\ (a,b;\ ;z) = \sum_r a_{(r)}b_{(r)}z^r/r!$

$$a_{(r)} = a(a+1)\dots(a+r-1) \quad (r=0,1,\dots;\ a_{(0)}=1)$$

$$\lambda_1,\ \lambda_2,\ \lambda_{12} > 0$$

a characterization can be obtained as follows.

THEOREM 3.3. *Let* $\underline{X} = (X_1,X_2)$, $\underline{Y} = (Y_1,Y_2)$ *be as in Theorem 2.1. Then*

$$(3.7)\qquad E(X_i \mid \underline{Y}=\underline{y}) = y_i + a_i\ {}_2F_0(-y_i-1,\ -y_j;;\ c)/{}_2F_0(-y_1,\ -y_2;;\ c)$$

$$(i \neq j,\ i,j=1,2)$$

where a_i, c *are constants such that* $c>0$, $a_i q_i c/p_i <$ < 1 $(i=1,2)$ *iff*

$$\underline{X} \sim \text{bivariate Poisson } \left(\frac{a_1}{p_1} - \frac{a_1 a_2 c q_2}{p_1 p_2},\right.$$

$$\left.\frac{a_2}{p_2} - \frac{a_1 a_2 c q_1}{p_1 p_2};\ \frac{a_1 a_2 c}{p_1 p_2}\right).$$

PROOF. The "necessary" part is straightforward. *Sufficiency.* From (2.4) and (3.7) we obtain

$$P(Y_i = y_i+1,\ Y_j=y_j)/P(\underline{Y}=\underline{y}) =$$

$$= p_i a_i\ {}_2F_0(-y_i-1,-y_j;\ ;\ c)/q_i(y_i+1)\,{}_2F_0(-y_1,-y_2;\ ;\ c)$$

$$(i\neq j,\ i,j=1,2).$$

Applying formula (3.5) we have

$$P(\underline{Y}=\underline{y}) = c^* a_1^{y_1} a_2^{y_2}\ {}_2F_0(-y_1,-y_2;\ ;\ c)/y_1!\,y_2!$$

where $c^* = \exp(-a_1-a_2-a_1a_2c)$.

Hence from (2.6)

$$(3.8)\qquad G_{\underline{X}}(\underline{t}) = \exp\left[\frac{a_1p_2-a_1a_2cq_2}{p_1p_2}(t_1-1)\right]\times$$

$$\times \exp\left[\frac{a_2p_1-a_1a_2cq_1}{p_1p_2}(t_2-1) + \frac{a_1a_2c}{p_1p_2}(t_1t_2-1)\right].$$

It can be easily proved using a method similar to that employed to prove part (i) of Theorem 3.1 that $a_i > 0$ $(i=1,2)$. Therefore the parameters of (3.8) are positive. Hence the result.

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