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A CHARACTERISTIC PROPERTY OF CERTAIN DISCRETE DISTRIBUTIONS

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1. INTRODUCTION

A shifted univariate distribution has a probability generating function (p.g.f.) of the form $s^k G(s)$ where $G(s)$ is the p.g.f. of a distribution on the integers $0,1,2,\dots$. The distribution with p.g.f. $G(s)$ is said to be shifted k units to the right or left according as k is a positive or negative integer.

In the bivariate case a shifted distribution will have p.g.f. of the form $s^k t^m G(s,t)$, where $G(s,t)$ represents the p.g.f. of a distribution on $\{0,1,\dots\} \times \{0,1,\dots\}$ and k,m are integers.

Consider now two distrete random variables (r.v.'s) X and Y. Assume that $G_Y(s) = s^k G_X(s)$, k integer. Then it can be shown that the factorial moments of Y relate to the factorial moments of X thus

$$\mathrm{E}(Y^{(r)}) = \sum_{i=0}^{r} \binom{r}{i} k^{(i)} \mathrm{E}(X^{(r-i)}) \quad (r=0,1,2,\dots) \tag{1.1}$$

where $z^{(r)} = z(z-1)\dots(z-r+1)$, $z^{(0)}=1$.

Analogous is the expression for the factorial moments of the vectors $\underline{X} = (X_1, X_2)$ and $\underline{Y} = (Y_1, Y_2)$ with $G_{\underline{Y}}(s,t) = s^k t^m G_{\underline{X}}(s,t)$ (k, m integers), i.e.

$$(1.2) \qquad \mathrm{E}(Y_1^{(r)} Y_2^{(\ell)}) = \sum_{i=0}^{r} \sum_{j=0}^{\ell} \binom{r}{i}\binom{\ell}{j} k^{(i)} m^{(j)} \mathrm{E}(X_1^{(r-i)} X_2^{(\ell-j)})$$

$$(r=0,1,2,\ldots, \quad \ell=0,1,2,\ldots).$$

For certain types of discrete distributions the relationships between the factorial moments of their original and shifted forms reduce to expressions which can be shown to constitute a unique property.

In the sequel, such properties will be used to provide characterizations for some well-known discrete univariate and bivariate distributions. Specifically, in Section 2 we provide a characterization for the geometric which subsequently is extended to characterize the class of distributions which consists of the Poisson, binomial and negative binomial distributions. A characterization of the Hermite distribution is also given.

Section 3 extends the results to obtain characterizations for some bivariate distributions whose marginals are independent. Finally, Section 4 considers the case of certain bivariate dependent distributions.

2. CHARACTERIZATION OF SOME UNIVARIATE DISCRETE DISTRIBUTIONS

THEOREM 2.1. *Let* X, Y *be non-negative discrete r.v.'s such that*

$$(2.1) \qquad G_Y(s) = sG_X(s)$$

where $G_W(s)$ *denotes the p.g.f. of* W. *Then the condition*

tion

$$(2.2) \qquad \mathrm{E}(Y^{(r)}) = c\mathrm{E}(X^{(r)}) \qquad (c > 1, \quad r=1,2,\ldots)$$

is necessary and sufficient for X *to be geometric with parameter* c^{-1}.

PROOF. *Necessity* follows immediately.

Sufficiency. From (1.1) we have for $k=1$

$$(2.3) \qquad \mathrm{E}(Y^{(r)}) = \mathrm{E}(X^{(r)})+r\mathrm{E}(X^{(r-1)}) \quad (r=1,2,\ldots).$$

Hence (2.2) holds if and only if (iff)

$$c\mathrm{E}(X^{(r)}) = \mathrm{E}(X^{(r)})+r\mathrm{E}(X^{(r-1)}) \qquad (r=1,2,\ldots),$$

i.e. iff

$$\mathrm{E}(X^{(r+1)})-\frac{r+1}{c-1}\mathrm{E}(X^{(r)}) = 0 \qquad (r=0,1,2,\ldots),$$

which implies that

$$\mathrm{E}(X^{(r)}) = r!(c-1)^{-r} \qquad (r=0,1,2,\ldots).$$

But this is the r-th factorial moment about the origin of the geometric distribution with parameter $q=c^{-1}$. Hence the theorem is established.

Note. In the context of stochastic processes, the characteristic property (2.2) is equivalent to the well--known lack-of-memory property (see PARZEN [3], p.123).

It has just been proved that the geometric distribution is uniquely determined by

$$\mathrm{E}((X+1)^{(r)}) = c\mathrm{E}(X^{(r)}) \quad (r=1,2,\ldots; \ c > 1).$$

One may ask what other distributions can be characterized by similar properties. Consider for example the more general case where c is not a constant but instead it is a function of r. Specifically, let X be a non-negative discrete r.v. with the property that

$$E((X+1)^{(r)}) = c(r)E(X^{(r-m)}) \quad (r=m,m+1,\ldots)$$

for some positive integer m. Consider the simple case $m=1$ and $c(r)=ar+b$, $a,b>0$, i.e.

$$(2.4) \quad E((X+1)^{(r)}) = (ar+b)E(X^{(r-1)}) \quad (r=1,2,\ldots).$$

What distributions can be characterized by this property?

By the following theorem it turns out that (2.4) uniquely determines the class of distributions which contains precisely the Poisson, binomial and negative binomial distributions.

THEOREM 2.2 (univariate case). *Let* X,Y *be as in Theorem 2.1. Then the condition*

$$(2.5) \quad E(Y^{(r)}) = (ar+b)E(X^{(r-1)}) \quad (r=1,2,\ldots;\ a,b>0)$$

holds iff X *has one of the following distributions*

(i) *Poisson with parameter* b *for* $a=1$.

(ii) *binomial with parameters* $p=1-a$, $n=-1+\frac{b}{1-a}$ *for* $a<1$,

(iii) *negative binomial with parameters* $q=(a-1)/a$ *and* $k=1+\frac{b}{a-1}$ *for* $a>1$.

PROOF. *Necessity* follows immediately.

Sufficiency. From (2.3) we have that (2.5) holds iff

$$E(X^{(r)})-[(a-1)r+b]\,E(X^{(r-1)}) = 0 \quad (r=1,2,\ldots),$$

i.e. iff

$$E(X^{(r+1)})-[(a-1)r+a+b-1]\,E(X^{(r)}) = 0 \quad (r=0,1,2\ldots) \tag{2.6}$$

Case $a=1$. Then (2.6) becomes

$$E(X^{(r+1)})-bE(X^{(r)}) = 0 \qquad (r=0,1,2,\ldots).$$

Solving we obtain

$$E(X^{(r)}) = b^r \qquad (r=0,1,2,\ldots)$$

which implies that $X \sim$ Poisson (b).

Case $a \neq 1$. We have from (2.6)

$$E(X^{(r+1)})-(a-1)(r+\frac{b}{a-1}+1)E(X^{(r)}) = 0$$
$$(r=0,1,2,\ldots).$$

Solving we find that

$$E(X^{(r)}) = (a-1)^r(\frac{b}{a-1}+1)_{(r)} \qquad (r=0,1,2,\ldots) \tag{2.7}$$

where $z_{(r)} = z(z+1)\ldots(z+r-1)$, $z_{(0)}=1$.

Obviously, for $a > 1$, (2.6) represents the r-th factorial moment of the negative binomial distribution with parameters $q = (a-1)/a$ and $k = 1+\frac{b}{a-1}$.

If now $a < 1$ we have from (3.4) for $r=0$ that $1 < E(X)+1 = a+b$ or $\frac{b}{1-a} > 1$. Then (2.7) becomes

$$(2.8) \qquad E(X^{(r)}) = \begin{cases} (1-a)^r \left(\frac{b}{1-a} - 1\right)^{(r)} & \text{for } \ 0 \le r \le \left[\frac{b}{1-a}\right] - 1 \\ 0 & \text{otherwise} \end{cases}$$

where $[w]$ denotes the integral part of w.

Therefore, the distribution of X is terminating, i.e. there exists an integer $m > 0$ such that $P[X=r] = 0$ for every $r > m$. Then, we have from (2.6) for $r=m$

$$E(X^{(m+1)}) - [(a-1)m+a+b-1]\,E(X^{(m)}) = 0$$

which implies that

$$(a-1)m+a+b-1 = 0$$

or equivalently

$$(2.9) \qquad \frac{b}{1-a} - 1 = m$$

which implies that $\frac{b}{1-a}$ is a positive integer.

Hence (2.8) represents the r^{th} factorial moment of the binomial distribution with parameters $n = \frac{b}{1-a} - 1$ and $p=1-a$.

Note. It can be seen from (2.9) that when X is bounded

$$\frac{b}{1-a} > 0$$

which (since $b > 0$) implies $a < 1$. Hence X is bounded iff $a < 1$. This shows that the class of distributions characterized by (2.5) contains precisely the Poisson, binomial and negative binomial distributions.

LAHA and LUKACS [2] provided characterizations of the Poisson, binomial and negative binomial among other distributions by the quadratic regression of the statistic

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} X_i X_j + \sum_{j=1}^{n} b_j X_j$$

on $S = n\overline{X}$.

Since all the distributions they have got are uniquely determined by their moments their result can alternatively be obtained by a method analogous to that of the previous theorem. This is so, because under their assumptions concerning the finiteness of the second moment and the validity of the regression equation, the distributions have all their moments to be finite; this implies that they satisfy certain recurrence equations which will lead us to the moments of the distributions in question.

To some extent, our results bear also an analogy to those obtained by SHANBHAG [4].

By Theorem 2.2 the univariate Poisson distribution has been characterized. It is of interest now to examine whether similar characterizations can be derived for generalized Poisson distributions, i.e. for distributions with p.g.f. of the form $\exp\{\lambda(g(s)-1)\}$, where $\lambda > 0$ and $g(s)$ valid p.g.f.

Specifically, we turn our attention to the particular case where $g(s) = \lambda_1(s-1)+\lambda_2(s^2-1)$ $(\lambda_i > 0,\ i=1,2)$.

The distribution defined by

$$(2.10) \quad G(s) = \exp\{a_1(s-1)+a_2(s^2-1)\} \qquad (a_i > 0, \quad i=1,2)$$

is known in the literature as the univariate Hermite distribution and was introduced by C.D. KEMP and A.W.

KEMP [1]. It is a special case of the Poisson-binomial distribution ($n=2$) and may be regarded as either the distribution of the sum of two dependent Poisson variables or that of the sum of a Poisson and an independent Poisson "doublet" variable.

The following theorem provides a chracteristic property for this form of generalized Poisson distribution.

THEOREM 2.3. *Let* X, Y *be as in Theorem 2.1. Then the condition*

$$\mathrm{E}(Y^{(r)}) = a\mathrm{E}(X^{(r+1)}) + b\mathrm{E}(X^{(r)}) \tag{2.11}$$

$$(a > 0,\ b < 0;\ r=0,1,2,\dots)$$

holds iff X *has the Hermite distribution with parameters* $a_1 = -\frac{b}{a}$ *and* $a_2 = \frac{1}{2a}$.

PROOF. *Necessity.* It has been shown (C.D. KEMP and A.W. KEMP [1]) that if X is Hermite (a_1, a_2) then

$$\mathrm{E}(X^{(r)}) = (2a_2)^{r/2} H_r^*\left[(2a_2)^{\frac{1}{2}} + a_1(2a_2)^{-\frac{1}{2}}\right] \tag{2.12}$$

$$(r=0,1,2,\dots)$$

where

$$H_n^*(x) = \sum_{j=0}^{[n/2]} \frac{n!\,x^{n-2j}}{(n-2j)!\,j!\,2^j} \quad (n=0,1,2,\dots;\ H_0^*(x)=1).$$

Moreover

$$\mathrm{E}(X^{(r+1)}) = (2a_2)^{\frac{1}{2}}\left[(2a_2)^{\frac{1}{2}} + a_1(2a_2)^{-\frac{1}{2}}\right]\mathrm{E}(X^{(r)}) + \tag{2.13}$$

$$+ 2a_2 r\mathrm{E}(X^{(r-1)}) \qquad (r=1,2,\dots).$$

Combining (2.3), (2.12) and (2.13) we find that $E(Y^{(r)})$ satisfies a relationship of the form (2.11) with $a = (2a_2)^{-1}$ and $b = -a_1(2a_2)^{-1}$. Obviously $a > 0$ and $b < 0$.

Sufficiency. From (2.3) it follows that (2.11) holds iff

$$E(X^{(r)})+rE(X^{(r-1)}) = aE(X^{(r+1)})+bE(X^{(r)}) \qquad (r=1,2,\ldots),$$

i.e. iff

$$E(X^{(r+1)}) = \frac{1-b}{a} E(X^{(r)})+ \frac{1}{a} rE(X^{(r-1)}) \qquad (r=1,2,\ldots).$$

But this is the recurrence relationship that the factorial moments of the Hermite distribution with parameters $-b/a$ and $1/2a$ satisfy. Hence the result.

Note. The Poisson "doublet" distribution or the distribution $(P(X=2r)=e^{-\lambda}\lambda^{r}/r!,\ P(X=2r+1) = 0,\ r = 0,1,\ldots))$ can also be characterized by Theorem 2.3 if we allow b to take on the value 0.

3. CHARACTERIZATION OF SOME BIVARIATE DISTRIBUTIONS WITH INDEPENDENT COMPONENTS

We now turn to the problem of providing characterizations for bivariate versions of the distributions examined in the previous section. We first consider the simplest case of having a bivariate form with independent marginals. In what follows a bivariate distribution whose marginals are independent and of the same form will be called "double" (e.g. double Poisson).

Indeed, by arguments which are analogous to those used in Secion 2 the following theorems can be proved to hold for double distributions.

THEOREM 3.1 (characterization of the double geometric distribution). *Let* $\underline{X} = (X_1, X_2)$, $\underline{Y} = (Y_1, Y_2)$, $\underline{Z} = (Z_1, Z_2)$ *be random vectors with non-negative integer-valued components. Assume that*

$$(3.1)\qquad \begin{aligned} G_{\underline{Y}}(s,t) &= s\, G_{\underline{X}}(s,t), \\ G_{\underline{Z}}(s,t) &= t\, G_{\underline{X}}(s,t). \end{aligned}$$

Then the conditions

$$(3.2)\qquad \begin{aligned} \mathrm{E}(Y_1^{(r)} Y_2^{(\ell)}) &= c_1 \mathrm{E}(X_1^{(r)} X_2^{(\ell)}), \\ \mathrm{E}(Z_1^{(r)} Z_2^{(\ell)}) &= c_2 \mathrm{E}(X_1^{(r)} X_2^{(\ell)}) \end{aligned}$$

$$(r=1,2,\ldots;\ \ell=1,2,\ldots;\ c_1, c_2 > 1)$$

are necessary and sufficient for $\underline{X}$ *to have the double geometric distribution with parameters* c_1^{-1}, c_2^{-1}.

THEOREM 3.2 (characterization of the double Poisson, binomial and negative binomial distributions). *Let* $X = (X_1, X_2)$, $\underline{Y} = (Y_1, Y_2)$ *and* $\underline{Z} = (Z_1, Z_2)$ *be as in Theorem 3.1. Then the conditions*

$$(3.3)\qquad \begin{aligned} \mathrm{E}(Y_1^{(r)} Y_2^{(\ell)}) &= (a_1 r + b_1) \mathrm{E}(X_1^{(r-1)} X_2^{(\ell)}) \\ \mathrm{E}(Z_1^{(r)} Z_2^{(\ell)}) &= (a_2 \ell + b_2) \mathrm{E}(X_1^{(r)} X_2^{(\ell-1)}) \end{aligned}$$

$$(r=1,2,\ldots;\ \ell=1,2,\ldots;\ a_i, b_i > 0,\ i=1,2)$$

are necessary and sufficient for $\underline{X}$ *to have one of the distributions*

(i) *double Poisson with parameters* (b_1,b_2) *if* $a_i = 1$, $(i=1,2)$,

(ii) *double binomial with parameters* $p_i = 1-a_i$, $n_i = -1+b_i/1-a_i$, *if* $a_i < 1$ $(i=1,2)$,

(iii) *double negative binomial with parameters* $q_i = (a_i-1)/a_i$ *and* $k_i = 1+b_i/(a_i-1)$ *if* $a_i > 1$ $(i=1,2)$.

An immediate consequence of Theorem 3.2 is the following theorem which enables us to characterize bivariate distributions whose marginals are not necessarily of the same form.

THEOREM 3.3. *Let* $\underline{X} = (X_1,X_2)$, $\underline{Y} = (Y_1,Y_2)$, $\underline{Z} = (Z_1,Z_2)$ *be as in Theorem* 3.1. *Then the conditions* (3.3) *hold iff* $\mathrm{P}(\underline{X} = \underline{x}) = \mathrm{P}(X_1 = x_1)\mathrm{P}(X_2 = x_2)$ *where*

(i) $X_i \sim$ Poisson (b_i), $X_j \sim$ binomial $(-1+b_j/1-a_j; 1-a_j)$ *for* $a_i = 1$, $a_j < 1$, $(i \neq j;\ i,j=1,2)$,

(ii) $X_i \sim$ Poisson (b_i), $X_j \sim$ neg. bin. $(1+\frac{b_j}{a_j-1}; (a_j-1)/a_j)$ *for* $a_i = 1$, $a_j > 1$ $(i \neq j;\ i,j=1,2)$,

(iii) $X_i \sim$ binomial $(-1+\frac{b_i}{1-a_i}; 1-a_i)$, $X_j \sim$ neg. bin. $(1+\frac{b_j}{a_j-1}; \frac{a_{j-1}}{a_j})$ *for* $a_i < 1$, $a_j > 1$ $(i \neq j;\ i,j=1,2)$.

THEOREM 3.4 (characterization of the double Hermite). *Let* $\underline{X} = (X_1,X_2)$, $\underline{Y} = (Y_1,Y_2)$, $\underline{Z} = (Z_1,Z_2)$ *be as in Theorem* 3.1. *Then the conditions*

$$(3.4)\quad \begin{aligned} \mathrm{E}(Y_1^{(r)}Y_2^{(\ell)}) &= a_1\mathrm{E}(X_1^{(r+1)}X_2^{(\ell)})+b_1\mathrm{E}(X_1^{(r)}X_2^{(\ell)}) \\ \mathrm{E}(Z_1^{(r)}Z_2^{(\ell)}) &= a_2\mathrm{E}(X_1^{(r)}X_2^{(\ell+1)})+b_2\mathrm{E}(X_1^{(r)}X_2^{(\ell)}) \end{aligned}$$

$$(a_i > 0,\ b_i < 0;\ i=1,2;\ r=0,1,2,\ldots;\ \ell=0,1,2,\ldots)$$

hold iff $\underline{X}$ *has the double Hermite distribution with parameters* $-b_1/a_1,\ 1/2a_1,\ -b_2/a_2,\ 1/2a_2$.

4. CHARACTERIZATION OF SOME BIVARIATE DISCRETE DISTRIBUTIONS WITH DEPENDENT COMPONENTS

Let us now consider the problem of characterizing dependent forms of bivariate distributions. We restrict ourselves to the case of the bivariate binomial and bivariate negative binomial with p.g.f.'s of the form $(p_{11}+p_{10}s+p_{01}t)^n$ and $p_{11}^k(1-p_{10}s-p_{01}t)^{-k}$ respectively.

A change in the characterizing conditions (3.8) is necessary as it is seen in the following theorem.

THEOREM 4.1 (characterization of the bivariate binomial and negative binomial). *Let* $\underline{X} = (X_1,X_2)$, $\underline{Y} = (Y_1,Y_2)$, $\underline{Z} = (Z_1,Z_2)$ *be as in Theorem* 3.1. *Then the conditions*

$$\text{(4.1)}\qquad \begin{aligned} E(Y_1^{(r)}Y_2^{(\ell)}) &= [a_1r+(a_1-1)\ell+b_1]\,E(X_1^{(r-1)}X_2^{(\ell)}),\\ E(Z_1^{(r)}Z_2^{(\ell)}) &= [(a_2-1)r+a_2\ell+b_2]\,E(X_1^{(r)}X_2^{(\ell-1)}) \end{aligned}$$

$$(r=1,2,\ldots;\ \ell=1,2,\ldots;\ a_i,b_i > 0,\ a_i\neq 1,\ i=1,2;\ \frac{b_1}{a_1-1} = \frac{b_2}{a_2-1} = h)$$

are necessary and sufficient for $\underline{X}$ *to have one of the distributions*

(i) bivariate binomial with parameters $n=-h-1$, $p_{10}=1-a_1$, $p_{01}=1-a_2$ *for* $a_i < 1$ $(i=1,2)$,

(ii) bivariate negative binomial with parameters $k=h+1$, $p_{10}=(a_1-1)/(a_1+a_2-1)$, $p_{01}=(a_2-1)/(a_1+a_2-1)$ *for* $a_i > 1$ $(i=1,2)$.

PROOF. *Necessity* follows immediately.

Sufficiency. From (1.2) for $k=m=1$ we have that the conditions (4.1) hold iff

$$E(X_1^{(r)}X_2^{(\ell)})-(a_1-1)(r+\ell+b_1/a_1-1)E(X_1^{(r-1)}X_2^{(\ell)}) = 0,$$
$$E(X_1^{(r)}X_2^{(\ell)})-(a_2-1)(r+\ell+b_2/a_2-1)E(X_1^{(r)}X_2^{(\ell-1)}) = 0$$
$$(r=1,2,\ldots;\ \ell=1,2,\ldots),$$

i.e. iff

$$\begin{aligned} &E(X_1^{(r+1)}X_2^{(\ell)})-(a_1-1)(r+\ell+h+1)E(X_1^{(r)}X_2^{(\ell)}) = 0,\\ (4.2)\qquad &E(X_1^{(r)}X_2^{(\ell+1)})-(a_2-1)(r+\ell+h+1)E(X_1^{(r)}X_2^{(\ell)}) = 0\\ &\qquad (r=0,1,2,\ldots;\ \ell=0,1,2,\ldots). \end{aligned}$$

The solution is given by

$$\begin{aligned} &E(X_1^{(r)}X_2^{(\ell)})=E(X_1^{(0)}X_2^{(0)})\prod_{i=0}^{r-1}(a_1-1)(i+h+1)\prod_{j=0}^{\ell-1}(a_2-1)(r+j+h+1)=\\ (4.3)\qquad &= (a_1-1)^r(a_2-1)^{\ell}(h+1)_{(r)}(r+h+1)_{(\ell)} =\\ &= (a_1-1)^r(a_2-1)^{\ell}(h+1)_{(r+\ell)}\\ &\qquad (r=0,1,2,\ldots;\ \ell=0,1,2). \end{aligned}$$

In the case $a_i < 1$ $(i=1,2)$ (4.3) represents the (r,ℓ)-factorial moment of the bivariate negative binomial with parameters $k=h+1$ and $p_{10}=(a_1-1)/(a_1+a_2-1)$, $p_{01}=(a_2-1)/(a_1+a_2-1)$.

Assume now that $a_i > 1$ $(i=1,2)$. Then from (4.2) we have for $r=\ell=0$ that $1 < E(X_i)+1 = b_i+a_i$ $(i=1,2)$

iff $-h > 1$.

Then (4.3) becomes

$$(4.4) \qquad E(X_1^{(r)} X_2^{(\ell)}) = \begin{cases} (1-a_1)^r (1-a_2)^\ell (-h-1)^{(r+\ell)} \\ \qquad\qquad (0 \le r \le [-h]-1,\ 0 \le \ell \le [-h]-1), \\ 0 \qquad\qquad \text{otherwise}. \end{cases}$$

That is when $a_i < 1$ $(i=1,2)$ the distribution of $\underline{X}$ is terminating i.e. there exists a vector $\underline{m} = (m_1, m_2)$ with non-negative integer-valued components such that $P(\underline{X} = \underline{x}) = 0$ whenever $x_1 \ge m_1+1$ and also whenever $x_2 \ge m_2+1$.

Then, we have from (3.10) for $r=m_1$, $\ell=m_2$

$$E(X_i^{(m_i+1)} X_j^{(m_j)}) + (1-a_i)(m_1+m_2+h+1) E(X_i^{(m_i)} X_j^{(m_j)}) = 0$$

$$(i \ne j; i,j=1,2)$$

which because $1-a_i \ne 0$ $(i=1,2)$ implies that

$$m_1+m_2+h+1 = 0,$$

i.e.

$$(4.5) \qquad h+1 = -(m_1+m_2).$$

This shows that $-h-1$ is a positive integer.

Then (4.4) represents the (r,ℓ)-factorial moment of the bivariate binomial distribution with parameters $n=-h-1$ and $p_{10}=1-a_1$, $p_{01}=1-a_2$. Hence the theorem is established.

Note 1. The relationship (4.5) tells something more. It shows that when $\underline{X}$ has a terminating distribution then

$$\frac{b_i}{a_i - 1} < 0 \qquad (i=1,2)$$

which since $b_i > 0$ $(i=1,2)$ implies that $a_i < 1$ $(i=1,2)$. Hence $\underline{X}$ has a terminating distribution iff $a_i < 1$ $(i=1,2)$.

Moreover, since $b_1/(a_1-1)=b_2/(a_2-1)$ and $b_i > 0$ $(i=1,2)$ it follows that the differences a_1-1 and a_2-1 have the same sign. Hence the class of distributions chracterized by the conditions (4.1) contains precisely the bivariate binomial and negative binomial distributions.

Note 2. If we allow a_i $(i=1,2)$ to take the value 1, then the conditions (4.1) reduce to the characterizing conditions of the double Poisson (Theorem 3.2).

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