GARCH option pricing: A review

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Introduction

- This presentation surveys the theory and empirical evidence on **GARCH option valuation models**.
- GARCH models are used extensively in all area of finance capturing the heteroskedasticity of returns.
- We focus on the implementation of this class of models for the valuation of derivative contracts, in particular of options.
- Our treatment includes:
  - Modeling issues including possible volatility dynamics and non-normal innovations.
  - Different pricing kernels typically used.

Introduction

- Why GARCH models in option pricing?
  - There is extensive empirical evidence that *time-varying volatility* and *volatility clustering* is important to model index and equity returns, and thus index and equity options.
  - Discrete-time GARCH models are straightforward to implement because:
    - In GARCH models the volatility is readily observable from the history of asset returns. This is not true with continuous-time models in which it is impossible to exactly filter it from discrete-time past observations. This facilitates the estimation of the model (through maximum likelihood) and it is particularly important when conducting out-of-sample option valuation exercises.
    - For several GARCH models (which dominate the literature) a closed-form option valuation formula exists. For others (less used) one can apply a Monte Carlo simulation approach.
- The survey will present, in chronological order, the most important studies in the field focusing on their innovations, similarities and differences.

Duan (MF, 1995) I

The model

- Duan (MF, 1995) is among the first papers that establish this field of research.
- For the reminder of this presentation define as $r_t = \ln\left(S_t / S_{t-1}\right)$ the one-period log-return of a financial asset and $h_t = Var^P_{t-1}\left(r_t\right)$ its conditional (on $\mathcal{F}_{t-1}$) variance.
- Duan assumes a simple GARCH(1,1) model under the physical measure P:

$$
\begin{aligned}
r_t &= r + \lambda\sqrt{h_t} - \frac{1}{2}h_t + \varepsilon_t, \ \ \varepsilon_t \left|\mathcal{F}_{t-1} \right. \sim N\left(0, h_t\right) \\
h_t &= w + bh_{t-1} + a\varepsilon^2_{t-1}
\end{aligned}
$$

where r is the one-period continuously compounded risk-free rate and λ is the price of risk.

Duan (MF, 1995) II

The model

- $E^P_{t-1}(r_t) \equiv m_t = r + \lambda\sqrt{h_t} - \frac{1}{2}h_t$
- $Var^P_{t-1}(r_t) = h_t$ which is $\mathcal{F}_{t-1}$-measurable.
- $Cov^P_{t-1}(r_t, h_{t+1}) = aE^P_{t-1}\left(\varepsilon^3_t\right) = 0$, due to the normality assumption. *So, variance and returns are not correlated*. This is not in line with empirical evidences indicating that these two variables are negatively correlated. When returns decrease, market volatility increases.
- Also, to see why term $-\frac{1}{2}h_t$ appear in the mean equation and why λ is the price of risk write the conditional expected gross rate of return as:

$$E^P_{t-1}\left(\frac{S_t}{S_{t-1}}\right) = E^P_{t-1}\left(e^{r_t}\right) = e^{r+\lambda\sqrt{h_t}-\frac{1}{2}h_t+\frac{1}{2}h_t} = e^{r+\lambda\sqrt{h_t}}$$

Duan (MF, 1995) I

Probability measure transformation

- To valuation of derivative contracts lies on the transformation of the physical probability measure to the risk-neutral one.
  - The physical measure P summarizes the actual probabilities of occurrence of future events.
  - The risk-neutral measure summarizes the probabilities of occurrence of future events as if investors were risk-neutral (i.e., they do not ask for a reward to take on risk).
  - Derivative contracts are valued under the risk-neutral measure (mainly for convenience). This generates a **risk-neutral valuation relationship (RNVR)** which relates the value of the derivative to the value of the underlying asset and other variables.
- However, to move from the physical to the risk-neutral measure we need either:
  - to specify the risk preferences of a representative investor.

Duan (MF, 1995) II

Probability measure transformation

- or to assume a class of Radon-Nikodym derivatives and specify an **equivalent martingale measure** (EMM) in this class.

- Duan followed the first approach assuming the existence of a representative investor with a utility function of constant relative risk aversion coefficient and the consumption growth $\Delta c_t = \ln(C_t / C_{t-1})$ is normally distributed with constant mean and variance.
- Under this assumption the stochastic discount factor (or pricing kernel) is given as:

$$M\left(C_{t-1}, C_t\right) = e^{-\rho} \frac{U'\left(C_t\right)}{U'\left(C_{t-1}\right)} = e^{-\rho} e^{-\gamma \Delta c_t}$$

where ρ is the impatience factor and γ is the relative risk aversion coefficient.

Duan (MF, 1995) III

Probability measure transformation

- The standard expected utility maximization argument leads to the following equation:

$$S_{t-1} = E_{t-1}^{P}\left(M\left(C_{t-1}, C_{t}\right) S_{t}\right)$$

for the risky financial asset and

$$e^{-r} = E_{t-1}^{P}\left(M\left(C_{t-1}, C_{t}\right)\right)$$

for the risk-free asset.

Duan (MF, 1995) IV

Probability measure transformation

- Based on this stochastic discount factor (SDF) define the Radon-Nikodym derivative:

$$\frac{dQ}{dP}\,|\mathcal{F}_t = \exp\left(rt + \sum_{j=1}^{t} \ln M\left(C_{j-1}, C_j\right)\right)$$

so that:

$$E_{t-1}^{Q}\left(S_t\right) = e^r E_{t-1}^{P}\left(M\left(C_{t-1}, C_t\right) S_t\right) = e^r S_{t-1}$$

indicating that under measure Q the (discounted) asset price follows a martingale.

- The new measure Q is known as the risk-neutral or equivalent martingale measure.

- The next step is to examine the dynamics of the return and variance processes under Q.
- To do so we consider the conditional moment generating function (MGF) of r_t under Q:

$$
\begin{aligned}
& E_{t-1}^{Q}\left(e^{ur_t}\right) \\
= \; & e^{r} E_{t-1}^{P}\left(e^{ur_t} M\left(C_{t-1}, C_t\right)\right) \\
= \; & e^{r} E_{t-1}^{P}\left(e^{ur_t-\rho-\gamma \ln\left(C_t / C_{t-1}\right)}\right) \\
= \; & e^{r} e^{-\rho} E_{t-1}^{P}\left(e^{ur_t-\gamma \ln\left(C_t / C_{t-1}\right)}\right) \\
= \; & e^{r} e^{-\rho} \exp\left(um_t + \frac{u^2}{2} h_t - \gamma\mu_c + \frac{\gamma^2}{2}\sigma_c^2 - u\gamma Cov_{t-1}\left(r_t, \Delta c_t\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \underbrace{\exp\left(r - \rho - \gamma\mu_c + \frac{\gamma^2}{2}\sigma_c^2\right)}_{=1} \\
& \exp\left(u\left(m_t - \gamma Cov_{t-1}\left(r_t, \Delta c_t\right)\right) + \frac{u^2}{2}h_t\right) \\
= \quad & \exp\left(u\left(m_t - \gamma Cov_{t-1}\left(r_t, \Delta c_t\right)\right) + \frac{u^2}{2}h_t\right).
\end{aligned}
$$

Set $u = 1$, then,

$$
e^r = E_{t-1}^Q\left(e^{r_t}\right) = \exp\left(m_t - \gamma Cov_{t-1}\left(r_t, \Delta c_t\right) + \frac{1}{2}h_t\right)
$$

so that,

$$
\exp\left(r - \frac{1}{2}h_t\right) = \exp\left(m_t - \gamma Cov_{t-1}\left(r_t, \Delta c_t\right)\right)
$$

Substituting in the last formula yields:

$$E_{t-1}^{Q}\left(e^{ur_t}\right) = \exp\left(u\left(r - \frac{1}{2}h_t\right) + \frac{u^2}{2}h_t\right).$$

- This result indicate that r_t is (conditional) normally distributed under Q with mean $r - \frac{1}{2}h_t$ and variance h_t.
- So we can write the return process as:

$$r_t = r - \frac{1}{2}h_t + \left(\varepsilon_t + \lambda\sqrt{h_t}\right) = r - \frac{1}{2}h_t + \varepsilon_t^*,\ \ \varepsilon_t^* \left|\mathcal{F}_{t-1}\right. \sim N\left(0, h_t\right)$$

and

$$h_t = w + bh_{t-1} + a\left(\varepsilon_{t-1}^* - \lambda\sqrt{h_t}\right)^2$$

- Implications:

Duan (MF, 1995) IV

The model under the risk-neutral measure

- The conditional variance remains the same under both measures.
- The unconditional mean of the variance is now equal to $E^Q(h_t) = w / \left(1 - \left(1 + \lambda^2\right) a - b\right)$. Compared to the unconditional mean under P, $E^Q(h_t) = w/(1 - a - b)$, it is now larger.
- The preference parameter λ does not disappear completely under risk neutralization. It appears in the variance process albeit not in the mean equation. Duan call that **local risk neutralization**.
- $Cov_{t-1}^{Q}(r_t, h_{t+1}) = -2a\lambda h_t^{3/2}$ so that if $\lambda > 0$, $Cov_{t-1}^{Q}(r_t, h_{t+1}) < 0$.

Duan (MF, 1995) I

Option pricing

- Unfortunately, this model does not generate an analytic option pricing formula.
- This is due to the fact that the conditional distribution of returns over more than one period cannot be analytically derived.
- In that case, a Monte Carlo simulation approach is used.

Duan (MF, 1995) I

Alternative characterization of the transformation

- Christoffersen, Jacobs and Ornthanalai (2012) indicate that one could get the same results by defining the Radon-Nikodym derivative as:

$$\frac{dQ}{dP}|\mathcal{F}_t = \exp\left(-\sum_{j=1}^{t}\left(\frac{\varepsilon_j}{\sqrt{h_j}}\lambda + \frac{1}{2}\lambda^2\right)\right)$$

- This corresponds to the second approach in which we do not specify a particular economic environment but a suitable transformation that makes the (discounted) asset price to be a martingale.

Duan (MF, 1995) II

Alternative characterization of the transformation

- Measure Q is an EMM since:

$$
\begin{aligned}
& E^Q_{t-1}\left(\frac{S_t}{S_{t-1}}\right) \\
= \quad & E^Q_{t-1}\left(e^{r_t}\right) = E^P_{t-1}\left(e^{r_t}\exp\left(-\left(\frac{\varepsilon_t}{\sqrt{h_t}}\lambda + \frac{1}{2}\lambda^2\right)\right)\right) \\
= \quad & E^P_{t-1}\left(\exp\left(r + \lambda\sqrt{h_t} - \frac{1}{2}h_t + \varepsilon_t - \left(\frac{\varepsilon_t}{\sqrt{h_t}}\lambda + \frac{1}{2}\lambda^2\right)\right)\right) \\
= \quad & \exp\left(r + \lambda\sqrt{h_t} - \frac{1}{2}h_t - \frac{1}{2}\lambda^2\right) E^P_{t-1}\left(\exp\left(1 - \frac{\lambda}{\sqrt{h_t}}\right)\varepsilon_t\right) \\
= \quad & \exp\left(r + \lambda\sqrt{h_t} - \frac{1}{2}h_t - \frac{1}{2}\lambda^2\right)\exp\left(\left(1 - \frac{\lambda}{\sqrt{h_t}}\right)^2\frac{h_t}{2}\right) = e^r
\end{aligned}
$$

Duan (MF, 1995) III

Alternative characterization of the transformation

- Examining once more the MGF of r_t under Q we can uncover its distribution:

$$
\begin{aligned}
& E_{t-1}^{Q}\left(e^{ur_t}\right) \\
= \;& E_{t-1}^{P}\left(e^{ur_t}\exp\left(-\left(\frac{\varepsilon_t}{\sqrt{h_t}}\lambda+\frac{1}{2}\lambda^2\right)\right)\right) \\
= \;& \exp\left(u\left(r+\lambda\sqrt{h_t}-\frac{1}{2}h_t\right)-\frac{1}{2}\lambda^2\right)E_{t-1}^{P}\left(\exp\left(u-\frac{\lambda}{\sqrt{h_t}}\right)\varepsilon_t\right) \\
= \;& \exp\left(u\left(r+\lambda\sqrt{h_t}-\frac{1}{2}h_t\right)-\frac{1}{2}\lambda^2\right)\exp\left(\left(u-\frac{\lambda}{\sqrt{h_t}}\right)^2\frac{h_t}{2}\right) \\
= \;& \exp\left(u\left(r-\frac{1}{2}h_t\right)+\frac{u^2}{2}h_t\right)
\end{aligned}
$$

Duan (MF, 1995) IV

Alternative characterization of the transformation

which indicates once more that $r_t \left| \mathcal{F}_{t-1} \right. \sim N\left(r - \frac{1}{2}h_t, h_t\right)$ under Q, thus reaching exactly the same results to Duan.

- Heston and Nandi (RFS, 2000) pointed out two deficiencies of Duan model: (1) Zero correlation between returns and variance under the physical measure, (2) No closed-form option pricing formula.
- They propose a GARCH model which accounts for these two issues:

$$
\begin{aligned}
r_t &= r + \lambda h_t + \underbrace{\sqrt{h_t} z_t}_{=\varepsilon_t}, \;\; z_t \left| \mathcal{F}_{t-1} \right. \sim N(0,1) \\
h_t &= w + b h_{t-1} + a \left(z_{t-1} - \gamma \sqrt{h_{t-1}} \right)^2
\end{aligned}
$$

  where r is the one-period continuously compounded risk-free rate and λ is the price of risk.
- $E^P_{t-1}(r_t) \equiv m_t = r + \lambda h_t$.

- $Var^P_{t-1}(r_t) = h_t$ which is $\mathcal{F}_{t-1}$-measurable. The unconditional variance is $E^P(h_t) = (w + a) / \left(1 - b - a\gamma^2\right)$.
- $Cov^P_{t-1}(r_t, h_{t+1}) = -2a\gamma h_t$. If $\gamma > 0$ then $Cov^P_{t-1}(r_t, h_{t+1}) < 0$, *so, variance and returns are now negatively correlated*. This result implies that the distribution of multi-period ahead returns deviate from normality. In fact, the non-zero covariance between returns and variance generates skewness for the multi-period ahead returns.
- The model has a continuous-time limit. As the time interval goes to zero it converges weakly to Heston (1993) stochastic volatility model.

Heston and Nandi (RFS, 2000) I

Probability measure transformation

- In transforming the model from the physical to the risk-neutral measure, Heston and Nandi followed Duan (1995), though they did not model explicitly the economic environment. All they assumed is that innovation process preserves normality under the risk-neutral measure.
- Christoffersen, Jacobs and Ornthanalai (2012) indicate that their approach is equivalent to assuming the Radon-Nikodym derivative:

$$\frac{dQ}{dP}\,|\mathcal{F}_t = \exp\left(-\sum_{j=1}^{t}\left(\lambda+\frac{1}{2}\right)\varepsilon_j+\frac{1}{2}\left(\lambda+\frac{1}{2}\right)^2 h_j\right)$$

Heston and Nandi (RFS, 2000) II

Probability measure transformation

- Measure Q is an equivalent martingale measure since:

$$
\begin{aligned}
& E^Q_{t-1}\left(\frac{S_t}{S_{t-1}}\right) \\
= \quad & E^Q_{t-1}\left(e^{r_t}\right) = E^P_{t-1}\left(e^{r_t}\exp\left(-\left(\left(\lambda+\frac{1}{2}\right)\varepsilon_t+\frac{1}{2}\left(\lambda+\frac{1}{2}\right)^2 h_t\right)\right)\right) \\
= \quad & E^P_{t-1}\left(\exp\left(r+\lambda h_t+\varepsilon_t-\left(\left(\lambda+\frac{1}{2}\right)\varepsilon_t+\frac{1}{2}\left(\lambda+\frac{1}{2}\right)^2 h_t\right)\right)\right) \\
= \quad & \exp\left(r+\lambda h_t-\frac{1}{2}\left(\lambda+\frac{1}{2}\right)^2 h_t\right) E^P_{t-1}\left(\exp\left(1-\lambda-\frac{1}{2}\right)\varepsilon_t\right) \\
= \quad & \exp\left(r+\lambda h_t-\frac{1}{2}\left(\lambda+\frac{1}{2}\right)^2 h_t\right)\exp\left(\left(\frac{1}{2}-\lambda\right)^2\frac{h_t}{2}\right) = e^r
\end{aligned}
$$

Heston and Nandi (RFS, 2000) I

The model under the risk-neutral measure

- Examining once more the MGF of r_t under Q we can uncover its distribution:

$$
\begin{aligned}
& E^Q_{t-1}\left(e^{ur_t}\right) \\
= \; & E^P_{t-1}\left(e^{ur_t}\exp\left(-\left(\left(\lambda+\frac{1}{2}\right)\varepsilon_t+\frac{1}{2}\left(\lambda+\frac{1}{2}\right)^2 h_t\right)\right)\right) \\
= \; & \exp\left(u\left(r+\lambda h_t\right)-\frac{1}{2}\left(\lambda+\frac{1}{2}\right)^2 h_t\right) E^P_{t-1}\left(\exp\left(u-\lambda-\frac{1}{2}\right)\varepsilon_t\right) \\
= \; & \exp\left(u\left(r+\lambda h_t\right)-\frac{1}{2}\left(\lambda+\frac{1}{2}\right)^2 h_t\right)\exp\left(\left(u-\lambda-\frac{1}{2}\right)^2\frac{h_t}{2}\right) \\
= \; & \exp\left(u\left(r-\frac{1}{2}h_t\right)+\frac{u^2}{2}h_t\right)
\end{aligned}
$$

Heston and Nandi (RFS, 2000) II

The model under the risk-neutral measure

- This result indicate that r_t is (conditional) normally distributed under Q with mean $r - \frac{1}{2}h_t$ and variance h_t.
- So we can write the return process as:

$$r_t = r - \frac{1}{2}h_t + \underbrace{\left(\varepsilon_t + \left(\lambda + \frac{1}{2}\right) h_t\right)}_{=\varepsilon_t^*}, \; \varepsilon_t^* \left| \mathcal{F}_{t-1} \sim N\left(0, h_t\right)\right.$$

and $z_t^* = \varepsilon_t^* / \sqrt{h_t} = \left(\varepsilon_t + \left(\lambda + \frac{1}{2}\right) h_t\right) / \sqrt{h_t} = z_t + \left(\lambda + \frac{1}{2}\right) \sqrt{h_t}$ so that:

$$\begin{aligned} h_t &= w + bh_{t-1} + a\left(z_{t-1}^* - \left(\lambda + \frac{1}{2}\right)\sqrt{h_{t-1}} - \gamma\sqrt{h_{t-1}}\right)^2 \\ &= w + bh_{t-1} + a\left(z_{t-1}^* - \gamma^*\sqrt{h_{t-1}}\right)^2 \end{aligned}$$

with $\gamma^* = \gamma + \lambda + \frac{1}{2}$. As $\lambda > 0$ this implies that $\gamma^* > \gamma$.

Heston and Nandi (RFS, 2000) III

The model under the risk-neutral measure

- Implications:
  - The conditional variance remains the same under both measures.
  - The unconditional mean of the variance is now equal to $E^{Q}(h_t) = (w + a) \,/\, \left(1 - b - a(\gamma^*)^2\right)$. Compared to the unconditional mean under P, it is now larger.
  - Since $\gamma^* > \gamma$ the covariance between returns and variance is stronger under the risk-neutral measure.
- Heston and Nandi pointed out that in this environment the Black-Scholes model may give misleading indications to investors.
- As the unconditional mean of h_t is higher under the risk-neutral measure than under the physical one, this means that the implied volatility will be typically higher than the expected future variance.

Heston and Nandi (RFS, 2000) IV

The model under the risk-neutral measure

- In this case an investor using the Black-Scholes formula will incorrectly perceive an investment opportunity from selling and delta-hedging volatility sensitive option positions (e.g., long-term at-the-money straddles).
- However, in the event of a sharp market downturn, the variance would rise significantly (due to the negative correlation) causing the short option position to lose considerable value.

- In contrast to Duan's model this model has a closed-form option pricing formula.
- This valuation formula is based on the MGF of the logarithm of asset price $x_T = \ln S_T$ over more than one period ahead.
- Consider a European call option with strike price K and maturity period T. Standard no-arbitrage arguments imply that the current price of the option is given as:

$$
\begin{aligned}
& C_t(K,T) \\
= \;& e^{-r(T-t)} E_t^Q\left(\max\left(S_T - K, 0\right)\right) \\
= \;& e^{-r(T-t)} \int_{\ln K}^{\infty} \left(e^{x_T} - K\right) f_t^Q(x_T)\, dx_T \\
= \;& e^{-r(T-t)} \int_{\ln K}^{\infty} e^{x_T} f_t^Q(x_T)\, dx_T - e^{-r(T-t)} K \int_{\ln K}^{\infty} f_t^Q(x_T)\, dx_T
\end{aligned}
$$

- Define $g_t^Q(x_T) = e^{x_T} \ f_t^Q(x_T) \, / \, E_t^Q(e^{x_T})$ so that we can write the first integral as:

$$
\begin{aligned}
C_t(K, T) &= e^{-r(T-t)} E_t^Q(e^{x_T}) \int_{\ln K}^{\infty} g_t^Q(x_T) dx_T - \\
&\quad - e^{-r(T-t)} K \int_{\ln K}^{\infty} f_t^Q(x_T) \, dx_T
\end{aligned}
$$

- Define the (conditional) MGF of $f_t^Q(x_T)$ under Q as $\varphi(u) \equiv \varphi(t, T; u) = E_t^Q(e^{ux_T})$, then the MGF of $g_t^Q(x_T)$ is given as:

$$\begin{aligned} \varphi^*(u) &= \int e^{ux_T} g_t^Q(x_T) dx_T = \frac{1}{\varphi(1)} \int e^{(u+1)x_T} f_t^Q(x_T)\, dx_T \\ &= \frac{\varphi(u+1)}{\varphi(1)} \end{aligned}$$

- Kendall and Stuart (1977) shows that we can calculate probabilities by inverting the characteristic function:

$$\int_{\ln K}^{\infty} f_t^Q(x_T)\, dx_T = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \mathrm{Re}\left[\frac{e^{-iu \ln K} \varphi(iu)}{iu} \right] du$$

so that we can write the option pricing formula as:

$$
\begin{aligned}
& C_t(K, T) \\
= \quad & e^{-r(T-t)}\varphi(1)\left(\frac{1}{2}+\frac{1}{\pi}\int_0^{\infty}\mathrm{Re}\left[\frac{e^{-iu\ln K}\varphi(iu+1)}{iu\varphi(1)}\right]du\right) - \\
& -e^{-r(T-t)}K\left(\frac{1}{2}+\frac{1}{\pi}\int_0^{\infty}\mathrm{Re}\left[\frac{e^{-iu\ln K}\varphi(iu)}{iu}\right]du\right)
\end{aligned}
$$

- The last formula is very general and can be applied to any model for which we have a closed-form solution of the MGF of the log asset price at expiration.
- The major contribution of Heston and Nandi is that their model has this property and thus option valuation is straightforward.

Heston and Nandi (RFS, 2000) I

The MGF of the log asset price at expiration

- First note that the MGF of one period ahead log asset price is:

$$\varphi(t, t+1; u) = E_t^Q\left(e^{ux_{t+1}}\right) = \exp\left(u\left(x_t + r - \frac{1}{2}h_{t+1}\right) + \frac{u^2}{2}h_{t+1}\right)$$

  so it is exponential affine on the state variables x_t and h_{t+1}.
- We can also show (after tedious calculations) that the MGF of two periods ahead log asset price x_{t+2} also exponential affine on x_t and h_{t+1}.
- Based on that we guess that the MGF of x_T is also exponential affine on x_t and h_{t+1} and takes the form:

$$\varphi(t, T; u) = E_t^Q\left(e^{ux_T}\right) = \exp\left(ux_t + A(t, T; u) + B(t, T; u)h_{t+1}\right)$$

- The problem now is to solve for the coefficients A and B.

Heston and Nandi (RFS, 2000) II

The MGF of the log asset price at expiration

- For $t = T$ we have that $A(T, T; u) = B(T, T; u) = 0$.

- Applying the law of iterated expectations to $\varphi(t, T; u)$ we get:

$$
\begin{aligned}
& \varphi(t, T; u) \\
= \; & E_t^Q \left(E_{t+1}^Q \left(e^{u x_T} \right) \right) = E_t^Q \left(\varphi(t+1, T; u) \right) \\
= \; & E_t^Q \left(\exp \left(u x_{t+1} + A(t+1, T; u) + B(t+1, T; u) h_{t+2} \right) \right) \\
= \; & E_t^Q \left(\exp \left(u \left(x_t + r - \frac{h_{t+1}}{2} + \sqrt{h_{t+1}} z_{t+1}^* \right) + A(t+1, T; u) \right. \right. \\
& \left. \left. + B(t+1, T; u) \left(w + b h_{t+1} + a \left(z_{t+1}^* - \gamma^* \sqrt{h_{t+1}} \right)^2 \right) \right) \right) \\
= \; & \exp \left(u \left(x_t + r - \frac{h_{t+1}}{2} \right) + A(t+1, T; u) \right. \\
& \left. + B(t+1, T; u) \left(w + b h_{t+1} \right) \right) \\
& E_t^Q \left(\exp \left(u \sqrt{h_{t+1}} z_{t+1}^* + B(t+1, T; u) a \left(z_{t+1}^* - \gamma^* \sqrt{h_{t+1}} \right)^2 \right) \right)
\end{aligned}
$$

Heston and Nandi (RFS, 2000) IV

The MGF of the log asset price at expiration

$$
\begin{aligned}
&= \exp\left(u\left(x_t + r - \frac{h_{t+1}}{2}\right) + A(t+1, T; u)\right. \\
&\quad + B(t+1, T; u)\,(w + bh_{t+1})) \\
&\quad \exp\left(\frac{u^2 - 4ua\gamma^* B(t+1, T; u) + 2a(\gamma^*)^2 B(t+1, T; u)}{2(1 - 2B(t+1, T; u)a)} h_{t+1}\right. \\
&\quad \left. - \frac{1}{2}\ln(1 - 2aB(t+1, T; u))\right) \\
&= \exp\left(ux_t + ur + A(t+1, T; u) + B(t+1, T; u)w\right. \\
&\quad - \frac{1}{2}\ln(1 - 2aB(t+1, T; u)) \\
&\quad + \left(-\frac{u}{2} + B(t+1, T; u)b\right. \\
&\quad \left.\left. + \frac{u^2 - 4ua\gamma^* B(t+1, T; u) + 2a(\gamma^*)^2 B(t+1, T; u)}{2(1 - 2B(t+1, T; u)a)}\right) h_{t+1}\right)
\end{aligned}
$$

- The last formula implies that:

$$A(t,T;u) = ur + A(t+1,T;u) + B(t+1,T;u)w - \frac{1}{2}\ln\left(1 - 2aB(t+1,T;u)\right)$$

$$B(t,T;u) = -\frac{u}{2} + B(t+1,T;u)b + \frac{u^2 - 4ua\gamma^* B(t+1,T;u) + 2a\left(\gamma^*\right)^2 B(t+1,T;u)}{2(1-2B(t+1,T;u)a)}$$

- This system of difference equations can be solved backwards using the terminal conditions $A(T,T;u) = B(T,T;u) = 0$.

Heston and Nandi (RFS, 2000) VI

The MGF of the log asset price at expiration

- Note here that the original expressions of Heston and Nandi (2000) contain some typos. Please refer to Christoffersen, Jacobs and Ornthanalai (2012) for a revised version.
- Why Duan's model does not generate a closed-form option pricing formula? *Because it is a non-affine model.*
- A simple way to see that is to write the conditional variance of h_{t+2}. Under the GARCH(1,1) model this is equal to $Var_t^P(h_{t+2}) = 2a^2 h_{t+1}^2$ which is quadratic (not linear) on h_{t+1}.
- In contrast for the Heston-Nandi model it is given as $Var_t^P(h_{t+2}) = 2a^2(1 + 2\gamma^2 h_{t+1})$ which is linear on h_{t+1}.

Generalization of risk-neutralization I

- Christoffersen, Elkamhi, Feunou and Jacobs (RFS, 2010) specify a class of Radon-Nikodym derivatives and derive restrictions that ensure the existence of an EMM.
- They assume the following model under the physical measure P:

$$r_t = \mu_t - \frac{1}{2}h_t + \varepsilon_t, \ \ \varepsilon_t \left| \mathcal{F}_{t-1} \right. \sim N\left(0, h_t\right)$$

where μ_t and h_t are $\mathcal{F}_{t-1}$-measurable. The mean correction factor $-\frac{1}{2}h_t$ ensures that that the conditional expected gross rate of return is equal to:

$$E^P_{t-1}\left(\frac{S_t}{S_{t-1}}\right) = e^{\mu_t}$$

- Note here that we do not specify a particular variance process so that this framework of analysis can fit in both models.

Generalization of risk-neutralization II

- Define the Radon-Nikodym derivative as:

$$\frac{dQ}{dP}|\mathcal{F}_t = \exp\left(-\sum_{j=1}^{t}\frac{\mu_j - r}{h_j}\varepsilon_j + \frac{1}{2}\left(\frac{\mu_j - r}{h_j}\right)^2 h_j\right)$$

Generalization of risk-neutralization III

- Measure Q is an EMM since:

$$
\begin{aligned}
& E^Q_{t-1}\left(\frac{S_t}{S_{t-1}}\right) \\
= \quad & E^Q_{t-1}\left(e^{r_t}\right) = E^P_{t-1}\left(e^{r_t}\exp\left(-\left(\frac{\mu_t - r}{h_t}\varepsilon_t + \frac{1}{2}\left(\frac{\mu_t - r}{h_t}\right)^2 h_t\right)\right)\right) \\
= \quad & E^P_{t-1}\left(\exp\left(\mu_t - \frac{1}{2}h_t + \varepsilon_t - \left(\frac{\mu_t - r}{h_t}\varepsilon_t + \frac{1}{2}\left(\frac{\mu_t - r}{h_t}\right)^2 h_t\right)\right)\right) \\
= \quad & \exp\left(\mu_t - \frac{1}{2}h_t - \frac{1}{2}\left(\frac{\mu_t - r}{h_t}\right)^2 h_t\right) E^P_{t-1}\left(\exp\left(\left(1 - \frac{\mu_t - r}{h_t}\right)\varepsilon_t\right)\right) \\
= \quad & \exp\left(\mu_t - \frac{1}{2}h_t - \frac{1}{2}\left(\frac{\mu_t - r}{h_t}\right)^2 h_t\right) \exp\left(\left(1 - \frac{\mu_t - r}{h_t}\right)^2 \frac{h_t}{2}\right) \\
= \quad & e^r
\end{aligned}
$$

Generalization of risk-neutralization IV

- We can write the return process as:

$$r_t = r - \frac{1}{2}h_t + \underbrace{(\varepsilon_t + \mu_t - r)}_{=\varepsilon_t^*}, \;\; \varepsilon_t^* \,|\mathcal{F}_{t-1} \sim N\left(0, h_t\right)$$

- Both models are special cases of this general framework:
  - In Duan's model $\mu_t = r + \lambda\sqrt{h_t}$ so that $\frac{\mu_t - r}{h_t} = \lambda / \sqrt{h_t}$.
  - In Heston and Nandi's model $\mu_t = r + \lambda h_t + \frac{1}{2}h_t$ so that $\frac{\mu_t - r}{h_t} = \lambda + \frac{1}{2}$.

Extensions I

- The previous two models set the ground for this field of research. Various extensions of these models were proposed in the following years accounting for:
    - Non-normal innovations
    - Extending the pricing kernel accounting for other sources of risk (volatility-dependent)
    - Adding extra components to provide a richer structure for the conditional variance

- Christoffersen, Heston and Jacobs (JoE, 2006) introduce a GARCH model with inverse Gaussian innovations:

$$r_t = r + \left(\zeta + \eta^{-1}\right) h_t + \varepsilon_t$$

where $\varepsilon_t = \eta y_t - \eta^{-1} h_t$ and $y_t \sim IG\left(h_t / \eta^2\right)$.

- Note that $E\left(y_t\right) = h_t / \eta^2$, $Var\left(y_t\right) = h_t / \eta^2$, $Skew(y_t) = 3\eta / \sqrt{h_t}$ and $Exc.Kurt(y_t) = 15\eta^2 / h_t$.
- These properties imply that:
  - $E^P_{t-1}\left(r_t\right) = r + \left(\zeta + \eta^{-1}\right) h_t$
  - $Var^P_{t-1}(r_t) = h_t$
  - $Skew^P_{t-1}(r_t) = 3\eta / \sqrt{h_t}$
  - $Exc.Kurt^P_{t-1}(r_t) = 15\eta^2 / h_t$

Non-normal innovations II

The model

- Parameter η controls for the sign of the skewness. If $\eta < 0$ then the conditional skewness of the return process is negative.
- The major difference between this model and that of Heston and Nandi (2000) is that it assumes non-normality for the one period ahead log-return distribution. Note, however here, that both models generate non-normality for multi-period ahead log-return distributions.
- Thus, the GARCH model with inverse Gaussian innovations captures skewness in the short-term which is particularly important for valuing short-term options.
- The conditional variance process is defined as:

$$h_t = w + bh_t + cy_t + ah_t^2 / y_t$$

The model also implies a non-zero covariance between returns and variance as $Cov_{t-1}^{P}(r_t, h_{t+1}) = (c/\eta - \eta^3 a)h_t$. Thus a negative covariance requires that $c/\eta - \eta^3 a < 0$.

Non-normal innovations I

Probability measure transformation

- Christoffersen, Heston and Jacobs (JoE, 2006) originally assume that the pricing kernel is an exponential function of log-returns motivated by the power utility function.
- Later on, Christoffersen, Elkamhi, Feunou and Jacobs (RFS, 2010) specify a class of Radon-Nikodym derivatives and derive restrictions that ensure the existence of an EMM even when innovations are non-normal.
- They assume the following model under the physical measure P:

$$r_t = \mu_t - \gamma_t + \varepsilon_t, \ \varepsilon_t \left| \mathcal{F}_{t-1} \sim D\left(0, h_t\right)\right.$$

Non-normal innovations II

Probability measure transformation

where μ_t and h_t are $\mathcal{F}_{t-1}$-measurable and D is an (unspecified) distribution. The mean correction factor $\exp(\gamma_t) = E^P_{t-1}\left(e^{\varepsilon_t}\right)$ ensures that that the conditional expected gross rate of return is equal to:

$$E^P_{t-1}\left(\frac{S_t}{S_{t-1}}\right) = e^{\mu_t}$$

- The candidate Radon-Nikodym derivative takes the form:

$$\frac{dQ}{dP}|\mathcal{F}_t = \exp\left(-\sum_{j=1}^{t}(v_j\varepsilon_j + \Psi_j(v_j))\right)$$

where $\exp\left(\Psi_t(u)\right) = E^P_{t-1}\left(e^{-u\varepsilon_t}\right)$, so that $\Psi_t(u)$ is defined as the logarithm of the MGF.

- Now $\gamma_t = \Psi_t(-1)$.

Non-normal innovations III

Probability measure transformation

- They proved that measure Q defined by the previous Radon-Nikodym derivative is an EMM if and only if:

$$\Psi_t\left(v_t - 1\right) - \Psi_t(v_t) - \Psi_t(-1) + \mu_t - r = 0$$

- This result implies that we can construct an EMM by the choosing the process v_t to make the previous condition to hold.
- In the case of the inverse Gaussian innovations we have that:

$$\Psi_t(u) = \left(u + \frac{1 - \sqrt{1 + 2u\eta}}{\eta}\right)\frac{h_t}{\eta}$$

and the EMM condition is now solved for the constant:

$$v = v_t = \frac{1}{2\eta}\left(\frac{(2 + \zeta\eta^3)^2}{4\zeta^2\eta^2} - 1\right)$$

- Again we can specify the return distribution by examining the MGF under measure Q. We have:

$$
\begin{aligned}
& E^Q_{t-1}\left(e^{ur_t}\right) \\
= \; & E^P_{t-1}\left(e^{ur_t}\exp\left(-(v\varepsilon_t + \Psi_t(v))\right)\right) \\
= \; & E^P_{t-1}\left(\exp\left(ur + u\left(\zeta + \eta^{-1}\right)h_t + u\varepsilon_t - v\varepsilon_t - \Psi_t(v)\right)\right) \\
= \; & \exp\left(ur + u\left(\zeta + \eta^{-1}\right)h_t - \Psi_t(v)\right) E^P_{t-1}\left(e^{(u-v)\varepsilon_t}\right) \\
= \; & \exp\left(ur + u\left(\zeta + \eta^{-1}\right)h_t - \Psi_t(v)\right)\exp\left(\Psi_t(v-u)\right) \\
= \; & \exp\left(ur + u\left(\zeta + \eta^{-1}\right)h_t - \Psi_t(v) + \Psi_t(v-u)\right) \\
= \; & \exp\left(ur + u\zeta h_t + \frac{h_t}{\eta^2}\sqrt{1+2v\eta}\left(1 - \sqrt{1 - \frac{2u\eta}{1+2v\eta}}\right)\right)
\end{aligned}
$$

Non-normal innovations II

The model under the risk-neutral measure

Set

$$\eta^* = \frac{\eta}{1 + 2v\eta}$$

$$\frac{h_t^*}{(\eta^*)^2} = \frac{h_t}{\eta^2}\sqrt{1 + 2v\eta} \Rightarrow h_t^* = \frac{h_t}{(1 + 2v\eta)^{3/2}}$$

and

$$\zeta^* = \zeta \frac{h_t}{h_t^*} = \zeta (1 + 2v\eta)^{3/2}$$

so that

$$
\begin{aligned}
& E_{t-1}^{Q}\left(e^{ur_t}\right) \\
= \quad & \exp\left(ur + u\zeta^* h_t^* + \frac{h_t^*}{(\eta^*)^2}\left(1 - \sqrt{1 - 2u\eta^*}\right)\right) \\
= \quad & \exp\left(ur + u\left(\zeta^* + (\eta^*)^{-1}\right) h_t^* + \underbrace{\left(-u + \frac{1 - \sqrt{1 - 2u\eta^*}}{\eta^*}\right)\frac{h_t^*}{\eta^*}}_{(*)}\right)
\end{aligned}
$$

where

$$
(*) = E_{t-1}^{Q}\left(e^{u\varepsilon_t^*}\right), \ \varepsilon_t^* = \eta^* y_t^* - (\eta^*)^{-1} h_t^* \text{ and } y_t^* \sim IG\left(h_t^*/(\eta^*)^2\right)
$$

indicating that under the EMM measure the return follows an inverse Gaussian distribution and the risk-neutral process can be written as:

$$r_t = r + \left(\zeta^* + (\eta^*)^{-1}\right) h_t^* + \varepsilon_t^*$$

thus taking the same form as the physical process.

- Note here that in contrast to the two previous models (with normal innovations) the conditional variance under the risk-neutral measure h_t^* is different to the variance h_t under the physical measure. In fact if $1/(1+2v\eta)^{3/2} = (\eta^*/\eta)^{3/2} > 1$ then $h_t^* > h_t$ as often observed empirically.

Non-normal innovations I

Option pricing

- The model has a closed-form option pricing formula.
- To derive it we follow the Heston and Nandi (2000) approach specifying the MGF of the log asset price at expiration. This is exponential affine to the state variables.
- This MGF includes two coefficients which again are solved backwards. For more details please refer to Christoffersen, Heston and Jacobs (JoE, 2006).

Further extensions I

Volatility-dependent pricing kernel

- Christoffersen, Heston and Jacobs (RFS, 2013) adopted a volatility-dependent pricing kernel which is equivalent to the following Radon-Nikodym derivative:

$$\frac{dQ}{dP}|\mathcal{F}_t = \exp\left(-\sum_{j=1}^{t}\left(v_1\varepsilon_i + v_2\varepsilon_i^2 + g\left(v_1, v_2, h_i\right)\right)\right)$$

- This is a more general case of a quadratic rather than linear EMM.
- The new parameter v_2 accounts for the market price of volatility risk (investors demand an extra reward for assets exposed to volatility).
- Christoffersen, Heston and Jacobs (RFS, 2013) combined this new pricing kernel with Heston and Nandi model.

Further extensions I

Two-volatility component models

- Christoffersen, Jacobs, Ornthanalai and Wang (JFE, 2008) propose a GARCH option valuation model with two volatility components.
- It has more structure than the previous models since volatility evolves around a stochastic long-run mean.
- In Heston-Nandi the unconditional variance is $E^P(h_t) \equiv \sigma^2 = (w + a) / \left(1 - b - a\gamma^2\right)$. Using that to substitute out w from the variance process we obtain:

$$h_t = \sigma^2 + b\left(h_{t-1} - \sigma^2\right) + a\left(\left(z_{t-1} - \gamma\sqrt{h_{t-1}}\right)^2 - \left(1 + \gamma^2\sigma^2\right)\right)$$

Further extensions II

Two-volatility component models

- A natural extension is to specify σ^2 as time-varying. Denoting this new time-varying component as q_t the model is now written as:

$$h_t = q_t + b\left(h_{t-1} - q_{t-1}\right) + a\left(\left(z_{t-1} - \gamma_1\sqrt{h_{t-1}}\right)^2 - \left(1 + \gamma_1^2 q_{t-1}\right)\right)$$

where

$$q_t = w + \rho q_{t-1} + \varphi\left(\left(z_{t-1}^2 - 1\right) - 2\gamma_2\sqrt{h_{t-1}}z_{t-1}\right)$$

- This model aims to improve the shortcomings of previous models to capture the path of conditional variance and the variance term structure.
- It has been observed that the variance autocorrelations are too high at longer lags to be explained by Heston and Nandi model, unless the process is extremely persistent.

Further extensions III

Two-volatility component models

- However, this extreme persistence may impact negatively on other aspects of option valuation, such as the valuation of short-maturity options.

Empirical evidence I

- A number of papers examined the performance of the previous models to fit option data both in- and out-of-sample.
- Non-normal innovation models, volatility-dependent pricing kernel and two-volatility component models outperform the Heston and Nandi model.
- All these model features lead to significant model improvements with the volatility-dependent pricing kernel being the most important.
- However, the results indicate that these three features are complements rather than substitutes.

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