Optimal Transport and Risk Management:
Applications in Finance & Natural Resources Management

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Risk Measures

Risk measures are the tools in risk management theory and practice for the risk quantification of financial positions of portfolios.

Historically, **variance** is the first statistical index that has been used as a tool for measuring risk. However, due to its inability to distinguish between positive and negative deviance from the mean it can be applied successfully only for the case of symmetric risk factors.

Value at Risk (VaR), is probably the most popular risk measure and is widely used in practice. However, this risk measure is not subadditive in general, with an exception the case of elliptically distributed risk factors. The lack of this property, makes this risk measure rather unreliable in general.

Subadditivity:
Given two positions $X_1, X_2 \in \mathcal{X}$ and a risk measure $\rho(\cdot)$, this measure is subadditive if it holds that

$$\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2).$$

Obviously, the lack of this property indicates that we cannot have an upper bound for the risk value.

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An Example for the insufficiency of VaR

Consider the case that we have $d = 100$ independent **defaultable** corporate bonds with equal default probability $p = 0.02$ and all bonds with initial value equal to 100 euros.

Let L_i denote the loss occurring from each bond, which can be represented as

$$L_i := 100Y_i - 5(1 - Y_i) = 105Y_i - 5,$$

where Y_i is the indicator of the default, and assuming that $P(L_i = 100) = 0.02$ while $P(L_i = -5) = 0.98$.

There are two different portfolios, A and B, both of total value equal to 10000 euros, where portfolio A consists of 100 bonds of type 1 and portfolio B is fully diversified. The corresponding loss variables are defined as:

$$L_A := 100L_1, \qquad L_B := \sum_{i=1}^{100} L_i.$$

One, would except the portfolio B to be more **robust** to risk since it is not concentrated only to one bond type, like portfolio A.

An Example for the insufficiency of VaR

However, according to VaR for $q = 0.95$ we have that:

- $P(L_1 \leq -5) = 0.98 \geq 0.95$ and $P(L_1 \leq \ell) = 0 < 0.95$ for $\ell < -5$, therefore $VaR_{0.95}(L_1) = -5$. As a result

  $$\boxed{VaR_{0.95}(L_A) = 100 VaR_{0.95}(L_1) = 100 \cdot (-5) = -500}$$

  This means that portfolio A remains acceptable even if we **remove** 500 euros.

- Denoting $M := \sum_{i=1}^{100} Y_i$, we have that $M \sim Bin(100, 0.02)$, $P(M \leq 5) = 0.984 \geq 0.95$ and $P(M \leq 4) = 0.949 < 0.95$, i.e. $VaR_{0.95}(M) = 5$. Therefore,

  $$\boxed{VaR_{0.95}(L_B) = 105 VaR_{0.95}(M) - 500 = 25}$$

  This means that the fully diversified portfolio needs to **add** a capital of 25 euro in order to become acceptable.

Obviously, such a behaviour from a risk measure is NOT acceptable, and on particular for VaR this happens due to the lack of subadditivity property in the general case. Therefore, there is the need to set some qualifications for risk measures in order to avoid such strange behaviours.

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The Class of Convex Risk Measures

A more appropriate concept for risk quantification is the class of **convex risk measures**. Let $X : \Omega \to \mathbb{R}$ be the position of a firm, affected by the states of the world $\omega \in \Omega$.

For the purposes of this work, we assume that the states of the world are described by a vector of random factors $Z = (Z_1, ..., Z_d)'$ i.e. we consider $\Omega = \mathbb{R}^d$. As a result, the position of the firm depends on the state of Z through a risk mapping $\Phi_0 : \mathbb{R}^d \to \mathbb{R}$, i.e. $-X = \Phi_0(Z)$.

Given that the distribution of random factors Z is known, let us denote it by $\mu \in \mathcal{P}(\mathbb{R}^d)$, the risk of the position X is quantified by a mapping $\rho : \mathcal{X} \to \mathbb{R}$ where $\mathcal{X}$ is an appropriate space for X.

Such a mapping is called a **convex risk measure** if it satisfies the properties (axioms):

(i) For $X, Y \in \mathcal{X}$ such that $X \leq Y$ then $\rho(X) \geq \rho(Y)$ (**Monotonicity**)

(ii) For $X \in \mathcal{X}$ and $m \in \mathbb{R}$ it holds that $\rho(X + m) = \rho(X) - m$ (**Translation Invariance**)

(iii) For $X, Y \in \mathcal{X}$ and for any $\lambda \in (0, 1)$ it holds that $\rho(\lambda X + (1 - \lambda Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$ (**Convexity**)

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Robust Representation Property

A very important property of the class of convex risk measures is their general **robust representation** property according to which, any convex risk measure admits the representation:

$$\rho(X) = \sup_{\mu \in \mathcal{P}(\Omega)} \{\mathbb{E}_\mu[-X] - \alpha(\mu)\} \tag{1}$$

where $\alpha : \mathcal{P}(\Omega) \to \mathbb{R} \cup \{\infty\}$ is a penalty function.

Practically, the penalty term $\alpha(\cdot)$ is taken with respect to a known probability measure μ_0, in order to penalize certain extreme scenarios. One of the most popular convex risk measures is the **entropic risk measure**

$$\rho_E(X) = \sup_{\mu \in \mathcal{P}(\Omega)} \left\{\mathbb{E}_\mu[-X] - \frac{1}{\gamma} KL(\mu \| \mu_0)\right\}.$$

- The definition of the entropic risk measure reveals that the value of the risk is computed with respect to a selected level of divergence (through the choice of γ) from the prior model $\mu_0 \in \mathcal{P}(\Omega)$.

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The Issue of Model Uncertainty

However, what happens if the case where the probability measure μ_0 is **NOT** known, and in fact, there are **more** than one plausible models that could serve as priors ?

Therefore, what we consider here is a natural extention of the class of convex risk measures under the framework of *model uncertainty* or in other words, under the **multi-prior** setting.

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Wasserstein barycentric risk measures

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Robust Treatment of Prior Information

Assume that $\mathcal{M} = \{\mu_1, ..., \mu_n\}$ is a set of prior models concerning the distribution of the stochastic risk factors Z. The fact that multiple priors exist, requires the formulation of a new class of risk measure capable of **handling** multiple prior models of possibly varying validity.

First, we employ the concept of the Fréchet mean to define an aggregate prior model using the barycentric sense, in order to :

(a) **condense** the multiple information, and
(b) **robustly** represent it by a *single* model through an appropriate definition of the mean.

Since we are interested in measuring the distance of a probability measure μ from $\mathcal{M}$ is defined in terms of the Fréchet function $\sum_{i=1}^{n} w_i \mathsf{d}^2(\mu, \mu_i)$, and the element of mininal distance from $\mathcal{M}$

$$\mu_B := arg \min_{\mu \in \mathcal{P}(\Omega)} \sum_{i=1}^{n} w_i \mathsf{d}^2(\mu, \mu_i)$$

will be called the *Fréchet mean* (or barycenter) of $\mathcal{M}$, where d is an appropriate metric and $w \in \Delta^{n-1}$ denotes a weight vector.

This definition takes into account that the fact that the space of probability measures $\mathcal{P}(\Omega)$ is **not** a vector space and μ_B is the “average model” (aggregate model) compatible with $\mathcal{M}$.

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The class of Fréchet risk measures

Dispersion in the space of probability measures $\mathcal{P}(\Omega)$ can be quantified using **Fréchet variance** which is defined as

$$V_{\mathcal{M}} := \min_{\mu \in \mathcal{P}(\Omega)} \mathbb{F}_{\mathcal{M}}(\mu) = \min_{\mu \in \mathcal{P}(\Omega)} \sum_{i=1}^{n} w_i \mathsf{d}^2(\mu, \mu_i),$$

i.e., the Fréchet function value obtained by its minimizer, the Fréchet mean.

Fréchet function $\mathbb{F}_{\mathcal{M}}(\mu)$ can be used to formulate an appropriate penalty function $\alpha(\cdot)$ according to the robust representation of convex risk measures, in order to penalize deviance from the prior set $\mathcal{M}$.

In that sense, penalizing deviance from $\mathcal{M}$ is somewhat equivalent to penalizing deviance from the Fréchet barycenter of $\mathcal{M}$. Therefore, since the Fréchet barycenter μ_B treats robustly the information provided by $\mathcal{M}$ we can use this sense of variance in order to define an extension of the convex class of risk measures to the multi-prior setting.

Fréchet risk measures: Definition & Properties

Definition (Fréchet risk measure)

- $\alpha : \mathbb{R} \to \mathbb{R}_+ \cup \{\infty\}$ an increasing function and $\alpha(0) = 0$
- $\Phi_0 : \mathbb{R}^d \to \mathbb{R}_+$ a risk mapping w.r.t. the stochastic factors Z

The ***Fréchet risk measure*** is defined for any $\gamma \in (0, \infty)$ as

$$\rho_F(X) := \sup_{\mu \in \mathcal{P}(\mathbb{R}^d)} \left\{ \mathbb{E}_\mu[-X] - \frac{1}{2\gamma} \alpha(\mathbb{F}_{\mathcal{M}}(\mu)) \right\},$$

where $\mathcal{M}$ is the set of priors for Z, $-X = \Phi_0(Z)$ and $\mathbb{F}_{\mathcal{M}}$ the Fréchet function.

Proposition (Properties of Fréchet risk measures)

(a) *Fréchet risk measures belong to the convex class.*

(b) *For any* $0 \leq \gamma_1 \leq \gamma_2$ *it holds that* $\rho_F(X; \gamma_1) \leq \rho_F(X; \gamma_2)$.

(c) $\lim_{\gamma \to 0^+} \rho_F(X; \gamma) = \mathbb{E}_{\mu_B}[-X] \leq \rho_F(X; \gamma)$ *where* μ_B *is the Fréchet barycenter of* $\mathcal{M}$.

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Wasserstein risk measures: Definition

A very natural choice for the metrization of $\mathcal{P}(\Omega)$, is the class of the **Wasserstein distances**. The p-Wasserstein distance is defined as

$$\mathcal{W}_p(\mu_1, \mu_2) := \left\{ \inf_{\nu} \left(\int_{\Omega \times \Omega} |x-y|^p d\nu(x,y) \ : \ \nu \in \Pi(\mu_1, \mu_2) \right) \right\}^{\frac{1}{p}}$$

where the set of transport plans $\Pi(\mu_1, \mu_2)$ denotes all probability measures on $\Omega \times \Omega$ with marginals μ_1 and μ_2. This distance is a true metric in the space of probability measures therefore it is a very appropriate choice.

The **Wasserstein barycentric risk measure** ρ_W for a set of priors $\mathcal{M}$, a set of weights $w = (w_1, \ldots, w_n) \in \Delta^{n-1}$, and a multiplier $\gamma > 0$, is defined as

$$\rho_W(X) := \sup_{\mu \in \mathcal{P}(\Omega)} \left\{ \mathbb{E}_\mu[-X] - \frac{1}{2\gamma} F_{\mathcal{M}}(\mu) \right\},$$

where

$$F_{\mathcal{M}}(\mu) := \sum_{i=1}^{n} w_i W_2^2(\mu, \mu_i) - V_{\mathcal{M}}$$

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Proposition (1-d case)

Assume that there is only one stochastic factor Z that affects the position X. Denote by μ_B the Wasserstein barycenter, represented by the quantile function $g_B(s) := \sum_{i=1}^{n} w_i g_i(s)$ where $g_i(s)$ is the quantile function of each measure $\mu_i \in \mathcal{M}$. Then, given that Φ_0 is a sufficiently **smooth** *risk mapping*

$$\rho_W(X) = \int_0^1 \left(\Phi_0(\tilde{g}(s)) - \frac{1}{2\gamma}(\tilde{g}(s) - g_B(s))^2 \right) ds$$

where $\tilde{g}(s)$ a quantile function (increasing & monotone mapping) obtained through the solution of the equation: $\Lambda(g) := g(s) - \gamma\Phi_0'(g(s)) - g_B(s) = 0$.

For some special cases, for $\kappa = b\gamma$ and $\lambda = 1 - c\gamma$ we get:

(a) $\Phi(z) = a + bz$ (**affine risk mapping**): $\boxed{\rho_W(X) = \mathbb{E}_{\mu_B}[-X] + \frac{\gamma b^2}{2}}$

(b) $\Phi(z) = a + bz + cz^2$ (**quadratic risk mapping**):

$$\boxed{\rho_W(X) = \int_0^1 \left[\Phi_0\left(\frac{g_B(s)+\kappa}{\lambda}\right) - \frac{1}{2\gamma}\left(\frac{g_B(s)+\kappa}{\lambda} - g_B(s)\right)^2 \right] ds}$$

- On the limit $\gamma \to 0^+$ it holds that $\rho_W(X) \to \mathbb{E}_{\mu_B}[-X]$.

The case of a single risk factor

Proposition (**1-d case**)

Assume that there is only one stochastic factor Z that affects the position X. Denote by μ_B the Wasserstein barycenter, represented by the quantile function $g_B(s) := \sum_{i=1}^{n} w_i g_i(s)$ where $g_i(s)$ is the quantile function of each measure $\mu_i \in \mathcal{M}$. Then, given that Φ_0 is a sufficiently **smooth** *risk mapping*

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The case of elliptically distributed risk factors

Consider that $d > 1$ random factors affect the position X through the risk mapping $-X = \Phi_0(Z)$.

Assume that the risk factors $Z = (Z_1, ..., Z_d)'$ follow a location-scatter family, i.e. $Z = m + SZ_0$ where:

- $m \in \mathbb{R}^d$ (**location** vector)
- $S \in \mathbb{P}(d)$ (**dispersion** matrix)
- Z_0 a central random variable on $\mathbb{R}^d$ following some **spherical** distribution $\nu \in \mathcal{P}(\mathbb{R}^d)$.

In that case, the random behaviour of the random variable Z is characterized by the probability measure $\mu = LS(m, S)$.

Consider the case where each $\mu_i \in \mathcal{M}$ for $i = 1, ..., n$ can be identified with some $LS(m_i, S_i)$. In that case, the (Fréchet) Wasserstein function can be decomposed in two parts as

$$\mathbb{F}_{\mathcal{M}}(m, S) := \overline{\mathbb{F}}_{\mathcal{M}}(m) + \widetilde{\mathbb{F}}_{\mathcal{M}}(S)$$

where
$\overline{\mathbb{F}}_{\mathcal{M}}(m) := \sum_{i=1}^{n} w_i \|m - m_i\|^2$ and
$\widetilde{\mathbb{F}}_{\mathcal{M}}(S) := \sum_{i=1}^{n} w_i Tr(S + S_i - 2(S_i^{1/2} S S_i^{1/2})^{1/2}$.

The case of elliptically distributed risk factors

Proposition (L-S family)

Denote by $\Phi(m,S) := \int_{\mathbb{R}^d} \Phi_0(m+Sz)d\nu(z)$*. Assuming that* Φ_0 *is sufficiently* **smooth***, the Wasserstein risk measure takes the form:*

$$\rho_W(X) = \Phi(m,S) - \frac{1}{2\gamma}(\overline{\mathbb{F}}_{\mathcal{M}}(m) + \widetilde{\mathbb{F}}_{\mathcal{M}}(S))$$

where m, S *are the solution of the matrix system:*

$$\begin{aligned} \gamma D_m \Phi(m,S) - (m - \sum_{i=1}^{n} w_i m_i) = 0 \\ 2\gamma S^{1/2} D_S \Phi(m,S) S^{1/2} - (S - \sum_{i=1}^{n} w_i (S^{1/2} S_i S^{1/2})^{1/2}) = 0 \end{aligned} \tag{2}$$

which can be solved numerically.

- For the special case where $\gamma \to 0^+$ the solution to the above problem converges to the Wasserstein barycenter, i.e. $\mu_B = LS(m_B, S_B)$ where $m_B = \sum_{i=1}^n w_i m_i$ and S_B satisfying the equation $S_B = \sum_{i=1}^n w_i (S_B^{1/2} S_i S_B^{1/2})^{1/2}$.

The case of elliptically distributed risk factors

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Explicit formulae for the elliptical case

- **Linear risk mappings** $\Phi_0(Z) = (\beta, Z)$, $\beta \in \mathbb{R}^d$. In that case, the risk measure is computed as

$$\rho_W(X) = (\beta, m_B) + \frac{\gamma}{2}\|\beta\|_2^2.$$

- **Portfolios** Given a total position of the form $X := \sum_{k=1}^{K} \theta_k X_k$ for proportions $\theta = (\theta_1, ..., \theta_K)'$ for the risk mapping $\Phi_0(Z) = (\sum_{k=1}^{K} \theta_k \beta_k, Z)$, the occurring risk measure value is

$$\rho_W(X) = \sum_{k=1}^{K} \theta_k(\beta_k, m_B) + \frac{\gamma}{2}\|\sum_{k=1}^{K} \theta_k \beta_k\|_2^2.$$

- **Quadratic risk mappings** $\Phi_0(Z) = q^T Z + Z^T Q Z$ where $q \in \mathbb{R}^d$ and $Q \in \mathbb{R}^{d\times d}$ and symmetric. The occurring risk measure value is obtained as

$$\rho_W(X) = (q, m_B) + m_B^T Q m_B + Tr(S_B Q S_B) + \gamma C$$

where C is an appropriate constant which can be computed numerically applying Proposition for L-S family.

The class of Fréchet risk measures

Wasserstein barycentric risk measures

Risk Premia Calculation under Uncertainty

Application in Natural Resources Management

A standard problem in insurance is the estimation of the total claim amount that an insurance firm is obliged to cover at a specific time horizon $T > 0$.

A Standard Model:

- N : # of claim events within $[0, T]$, $N \sim Poi(\lambda_N)$
- C_j: claim size of the j-th claim event $(j = 1, 2, ..., N)$, where $\{C_j\}_j$ are assumed to be independent and identically distributed according to a probability distribution F
- The total claim amount up to time T is represented by the compound mixed Poisson process

$$-X := \sum_{j=1}^{N} C_j$$

Uncertainty on λ_N and F:

- Let $Z^{(1)} = (Z_1^{(1)}, ..., Z_{d_1}^{(1)})'$ the set of stochastic factors that affect the value of λ_N through the risk mapping $\Phi_0^{(1)} : \mathbb{R}^{d_1} \to \mathbb{R}$.
- $Z^{(2)} = (Z_1^{(2)}, ..., Z_{d_2}^{(2)})'$ the set of stochastic factors that affect the value of C_j through the risk mapping $\Phi_0^{(2)} : \mathbb{R}^{d_2} \to \mathbb{R}$.

- We consider different sets of stochastic factors in order to keep N and C_j independent.

Risk Premia Estimation under Model Uncertainty

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• We consider different sets of stochastic factors in order to keep N and C_j independent.

Risk Premia Estimation under Model Uncertainty

Given the independence between N and C_j we get that

$$\mathbb{E}[-X] = \mathbb{E}\left[\sum_{j=1}^{N} C_j\right] = \mathbb{E}[\lambda_N]\mathbb{E}[C_j] = \mathbb{E}[\Phi_0^{(1)}(Z^{(1)})]\mathbb{E}[\Phi_0^{(2)}(Z^{(2)})].$$

The latter expression allows for the construction of a risk mapping depending on all the stochastic factors $Z := (Z^{(1)}, Z^{(2)})'$.

Consider the set of priors $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$, where $\mathcal{M}_1 = \{\mu_1^{(1)}, ..., \mu_n^{(1)}\}$ and $\mathcal{M}_2 = \{\mu_1^{(2)}, ..., \mu_n^{(2)}\}$ are prior sets for the stochastic factors $Z^{(1)}$ and $Z^{(2)}$ respectively.

Assume that each set consists of members of the location-scatter family, i.e. for each $i = 1, 2, ..., n$ we have that

$$\mu_i^{(1)} = LS(m_i^{(1)}, S_i^{(1)}) \text{ and } \mu_i^{(2)} = LS(m_i^{(2)}, S_i^{(2)}).$$

Letting $Z_0^{(1)} \sim \nu^{(1)}$ and $Z_0^{(2)} \sim \nu^{(2)}$ be some central and spherical distributed random variables. Then, we obtain the mapping

$$\Phi(m^{(1)}, S^{(1)}, m^{(2)}, S^{(2)}) = \Phi_1(m^{(1)}, S^{(1)})\Phi_2(m^{(2)}, S^{(2)})$$

where

$$\Phi_j(m^{(j)}, S^{(j)}) = \int_{\mathbb{R}^{d_j}} \Phi_0^{(j)}(m^{(j)} + S^{(j)} z) d\nu^{(j)}(z) \text{ for } j = 1, 2.$$

Risk Premia Estimation under Model Uncertainty

In that case, the risk premium can be estimated according to the Wasserstein risk measure as

$$\rho_W(X) = \max_{(m^{(1)},S^{(1)},m^{(2)},S^{(2)})} \Big\{ \Phi(m^{(1)}, S^{(1)}, m^{(2)}, S^{(2)}) - \frac{1}{2\gamma}(F_{\mathcal{M}_1}(m^{(1)}, S^{(1)}) + F_{\mathcal{M}_2}(m^{(2)}, S^{(2)})) \Big\}$$

where $F_{\mathcal{M}_1}(\cdot,\cdot)$ and $F_{\mathcal{M}_2}(\cdot,\cdot)$ are the Fréchet functions with respect to $\mathcal{M}_1$ and $\mathcal{M}_2$. Applying Proposition for L-S family we obtain a matrix system of equations which can be solved numerically.

For the sake of example, consider the case where $d_1 = d_2 = 1$, assuming linear risk mappings of the form $\Phi_0^{(j)}(Z^{(j)}) = \alpha^{(j)} Z^{(j)}$. Then, the risk can be obtained as above for the parameters:

$$m_1 = \left(\frac{1+\gamma^2 a_1^2 a_2^2}{1-\gamma^2 a_1^2 a_2^2}\right) m_{1,B} + \left(\frac{\gamma a_1 a_2}{1-\gamma^2 a_1^2 a_2^2}\right) m_{2,B},$$

$$m_2 = \frac{1}{1-\gamma^2 a_1^2 a_2^2} m_{2,B} + \left(\frac{\gamma a_1 a_2}{1-\gamma^2 a_1^2 a_2^2}\right) m_{1,B}$$

$$\sigma_j = \left(\sum_{i=1}^{n} w_i \sigma_{j,i}^{1/2}\right)^2, \quad j = 1, 2,$$

where $m_{j,B} = \sum_{i=1}^{n} w_i m_{j,i}, \quad j = 1, 2.$

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where $m_{j,B} = \sum_{i=1}^{n} w_i m_{j,i}, \;\; j = 1,2.$

A Numerical Case Study: Gaussian Noise

Consider the case where $\lambda_N := Z^{(1)} \sim N(m_1, s_1)$ and $C_j := Z^{(2)} \sim N(m_2, s_2)$.

The true values of the parameters are not known to the decision maker, however, he is provided with a prior set by n experts with their estimations which are assumed to be contaminated with noise as:

$$\begin{aligned} m_1^{(j)} &= m_1 + u, \qquad u \sim N(0,\zeta) \\ m_2^{(j)} &= m_2 + v, \qquad v \sim N(0,\xi) \\ s_1^{(j)} &= s_1 + w, \qquad w \sim N(0,\theta) \\ s_2^{(j)} &= s_2 + z, \qquad z \sim N(0,\tau) \end{aligned}$$

for $j = 1, 2, ..., n$ and $\Theta = (\zeta, \xi, \theta, \tau)'$ the vector with the noises' scaling factors.

We simulate three scenarios for different choices of the scale vector:

(hh) **high homogeneity scenario** small perturbations around the true parameters

(mh) **medium homogeneity scenario** medium perturbations around the true parameters

(lh) **low homogeneity scenario** large perturbations around the true parameters

Then, in order to check the performance of the proposed risk measures we simulate $B = 1000$ cases in each scenario, and evaluate the deviance from the true risk values by averaging appropriate statistical indices for different number of experts, $n = 5, 10, 30$.

(hh) High Homogeneity Case

γ	$\mathbb{E}[\rho_\gamma(X)]$		$\mathbb{E}\left(\frac{\lvert\rho_\gamma(X)-\rho_0(X)\rvert}{\rho_0(X)}\right)$		$Var^{1/2}\left(\frac{\lvert\rho_\gamma(X)-\rho_0(X)\rvert}{\rho_0(X)}\right)$	
	Average	***Wasserstein***	***Average***	***Wasserstein***	***Average***	***Wasserstein***
i. n=5						
0.10	5659.20	5679.70	0.132	0.136	0.016	0.014
0.05	5658.90	5257.20	0.132	0.051	0.016	0.013
0.01	5660.10	5126.90	0.132	0.025	0.016	0.012
0.00	5660.30	5001.80	0.132	0.010	0.016	0.007
ii. n=10						
0.10	5659.20	5682.10	0.124	0.136	0.013	0.010
0.05	5619.90	5259.00	0.123	0.052	0.012	0.009
0.01	5615.20	5127.40	0.123	0.025	0.013	0.009
0.00	5617.10	5001.10	0.123	0.007	0.013	0.005
iii. n=30						
0.10	5560.10	5681.30	0.112	0.136	0.009	0.006
0.05	5558.20	5257.80	0.112	0.052	0.009	0.005
0.01	5559.50	5126.10	0.112	0.025	0.009	0.005
0.00	5557.70	5001.50	0.112	0.004	0.009	0.003

(mh) Medium Homogeneity Case

γ	$\mathbb{E}[\rho_\gamma(X)]$		$\mathbb{E}\left(\frac{\lvert\rho_\gamma(X)-\rho_0(X)\rvert}{\rho_0(X)}\right)$		$Var^{1/2}\left(\frac{\lvert\rho_\gamma(X)-\rho_0(X)\rvert}{\rho_0(X)}\right)$	
	Average	***Wasserstein***	***Average***	***Wasserstein***	***Average***	***Wasserstein***
i. n=5						
0.10	5559.90	5684.60	0.1120	0.1369	0.0315	0.0286
0.05	5554.10	5259.70	0.1108	0.0523	0.0317	0.0253
0.01	5557.80	5132.10	0.1116	0.0299	0.0317	0.0208
0.00	5547.50	5004.60	0.1095	0.0192	0.0307	0.0148
ii. n=10						
0.10	5476.70	5682.20	0.0953	0.1363	0.0095	0.1364
0.05	5478.40	5258.80	0.0957	0.0516	0.0092	0.0518
0.01	5476.10	5126.50	0.0952	0.0252	0.0091	0.0267
0.00	5471.50	5001.90	0.0943	0.0042	0.0092	0.0144
iii. n=30						
0.10	5388.40	5678.90	0.0777	0.1358	0.0156	0.0117
0.05	5388.00	5256.30	0.0776	0.0513	0.0152	0.0108
0.01	5388.90	5126.60	0.0778	0.0254	0.0149	0.0101
0.00	5386.80	5001.30	0.0774	0.0086	0.0153	0.0063

(lh) Low Homogeneity Case

γ	$\mathbb{E}[\rho_\gamma(X)]$		$\mathbb{E}\left(\frac{\lvert\rho_\gamma(X)-\rho_0(X)\rvert}{\rho_0(X)}\right)$		$Var^{1/2}\left(\frac{\lvert\rho_\gamma(X)-\rho_0(X)\rvert}{\rho_0(X)}\right)$	
	Average	***Wasserstein***	***Average***	***Wasserstein***	***Average***	***Wasserstein***
i. n=5						
0.10	5392.30	5675.40	0.084	0.136	0.056	0.064
0.05	5394.90	5263.30	0.086	0.065	0.055	0.048
0.01	5389.20	5120.50	0.084	0.050	0.051	0.037
0.00	5390.00	4998.00	0.084	0.046	0.055	0.034
ii. n=10						
0.10	5285.40	5676.40	0.061	0.135	0.039	0.045
0.05	5278.50	5245.40	0.060	0.054	0.039	0.036
0.01	5292.60	5126.00	0.062	0.039	0.039	0.029
0.00	5302.00	5009.30	0.063	0.032	0.039	0.024
iii. n=30						
0.10	5195.20	5682.40	0.040	0.136	0.023	0.026
0.05	5193.00	5260.50	0.040	0.052	0.023	0.024
0.01	5185.40	5124.70	0.038	0.028	0.023	0.019
0.00	5189.70	4998.00	0.039	0.018	0.023	0.013

The class of Fréchet risk measures

Wasserstein barycentric risk measures

Risk Premia Calculation under Uncertainty

Application in Natural Resources Management

Application: Optimal harvesting under model uncertainty

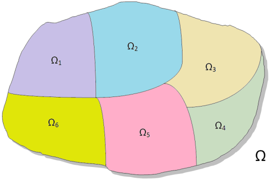
A firm needs to optimally harvest a physical capital (e.g. water resources, fishery, etc.) which evolves according to the dynamics:

$$\frac{\partial u}{\partial t}(t,x) = \mathcal{L}u(t,x) + Au(t,x) - Bc(t,x), \;\; u(0,x) = u_0 \qquad (3)$$

where

- $u(t,x)$: the physical capital density at (t,x) (**state** variable)
- $c(t,x)$: the harvest rate at (t,x) (**control** variable)
- $\mathcal{L}$: the Laplacian operator
- A, B : diagonal operators

The area of interest Ω for harvesting purposes is divided into K smaller domains Ω_i.



We consider the evolution of the average density over each region Ω_i, i.e., $u_i(t) = \frac{1}{|\Omega_i|}\int_{\Omega_i} u(t,x)dx$ for $i = 1, 2, ..., K$ and the corresponding average harvest rate $c_i(t) = \frac{1}{|\Omega_i|}\int_{\Omega_i} c(t,x)dx$ which leads to a discretized version of (3) with $\mathcal{L}$ replaced by an appropriate graph Laplacian matrix.

The Optimal Control Problem

The optimal control problem for a utility function of the total harvest of the form $U(c) = \frac{c^{1-\beta}}{1-\beta}$ for $\beta > 0$, a discount factor $r > 0$ and coefficients D_i, E_i for $i = 1, ..., K$ is expressed as

$$V(u_0) = \sup_{c_1(\cdot),\dots,c_K(\cdot)} \int_0^T e^{-rt} \frac{(\sum_{i=1}^K D_i c_i(t))^{1-\beta}}{1-\beta} dt + e^{-rT} \frac{(\sum_{i=1}^K E_i u_i(T))^{1-\beta}}{1-\beta}$$

subject to

$$\frac{\partial u_i}{\partial t}(t) = (\mathcal{L}u)_i(t) + A_i u_i(t) - B_i c_i(t), \;\; u_i(0) = u_{0,i}, \;\; i = 1, 2, ..., K.$$

Given the initial condition $u_0 = (u_{0,1}, ..., u_{0,K})' \in \mathbb{R}^K$ and for a specified time horizon $T > 0$ (large enough) we get the **optimal controls** (harvest rates):

$$\boxed{c_i = -\Phi_{0,i}(u_0) := G_i \langle \alpha, u_0 \rangle}$$

where $G_i := C(\alpha, B_i, D_i)$ a constant depending on the eigenvector corresponding to the lowest eigenvalue of $(\mathcal{L} + A)$.

The optimal turnout of the firm will be

$$X := -\Phi_0(u_0) := -\sum_{i=1}^K \Phi_{0,i}(u_0) = G\langle \alpha, u_0 \rangle = \sum_{i=1}^K c_i,$$

where $G := \sum_{i=1}^K G_i$.

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Uncertainty on initial states of biomass density

In practice u_0 is **NOT** known precisely and maybe subject to stochastic fluctuations, leading to stochastic fluctuations of the turnout X, and in this respect Φ_0 can be considered as the related risk mapping, which is **linear**.

In the above formulation, the role of risk factor Z is played by the initial state u_0 which is assumed to be uncertain and distributed according to a **location-scatter** family $u_0 \sim \mu = LS(m, S)$, with ambiguity on m, S, which is expressed in terms of a prior set of probability models $\mathcal{M} = \{\mu_1, ..., \mu_N\}$ for the true distribution of u_0.

Ambiguity follows from the inability of exact modeling or measurement of the various exogenous factors (e.g. environmental conditions) affecting the distribution of u_0.

The decision maker would like to evaluate possible losses from such fluctuations taking into account the model uncertainty and the proposed class of risk measures can offer such an evaluation.

Risk allocation in optimal harvesting

According to the **Euler's principle** for the risk allocation, one could in this case quantify not only the risk of the total position of the firm, but also the amount that each different domain contributes to the total risk.

In particular, if $X = \sum_{j=1}^{n} X_k = -\sum_j \Phi_{0,j}(u_0)$ is the **total position** of the firm then the total risk $\rho(X)$ can be computed by using an appropriate risk measure $\rho(\cdot)$. Then, the **individual** risk contribution of the j-th component can be calculated according to Euler allocation principle as

$$\rho(X_j|X) = \frac{d\rho}{dh}(X + hX_j)|_{h=0}.$$

Applying the general results for the computation of the risk, we obtain:

- The **total harvest risk** is quantified by the Wasserstein risk measure as

$$\boxed{\rho_W(X) = G\langle \alpha, m_B\rangle + \frac{\gamma}{2} G^2 \|\alpha\|_2^2}, \text{ and}$$

- the **harvest risk allocation to the j-th component** is estimated as

$$\boxed{\rho_W(X_j|X) = G_j\langle \alpha, m_B\rangle + \gamma G G_j \|\alpha\|_2^2}.$$

Concluding Remarks

- We proposed a class of risk measures which **extents** the framework of the convex class to the case of model uncertainty.
- We employed the Wasserstein barycenter in order to define an appropriate concept of mean in the space of probability measures and at the same time, to counter robustly uncertainty.
- In the case of elliptical family of distributions, the Wasserstein risk measure admits **analytic** or **semi-analytic** formulae.
- This more general concept of risk measures can be employed successfully as a decision making tool in a great variety of applications (actuarial, financial, optimal harvesting issues, energy management, etc.).

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QUESTIONS ???