Dynamic Asset Allocation under Disappointment Aversion preferences

Vasileios E. Kontosakos[1], Soosung Hwang[2], Vasileios Kallinterakis[3]
Athanasios A. Pantelous[1]

15th Summer School in Stochastic Finance, AUEB, Athens Greece

July 11, 2018

[1]Department of Econometrics and Business Statistics, Monash University. AUS

[2]School of Economics, Sungkyunkwan University (SKKU). S. Korea

[3]Management School, University of Liverpool. UK

Structure

1. From expected utility to prospect theory
2. From prospect theory (loss aversion) to disappointment aversion
3. The portfolio choice problem in disappointment aversion
4. The disappointment aversion framework
5. DA participation/non–participation
6. Non-participation in disappointment aversion
7. Dynamic asset allocation – optimization
8. Numerical examples
9. Concluding remarks

Expected Utility

- assume the following gamble :

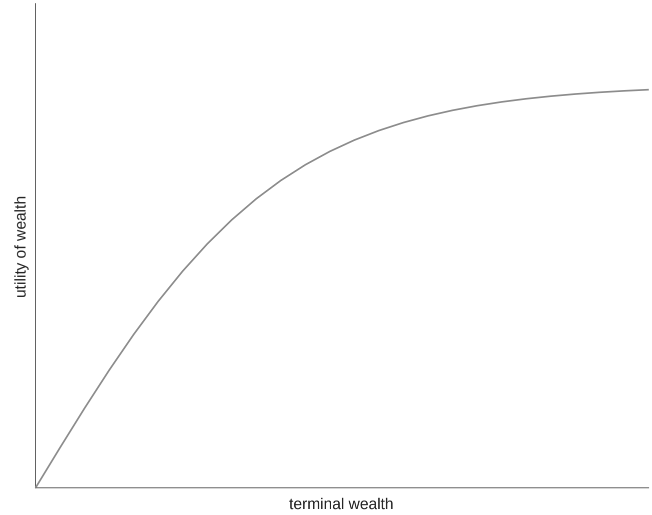
$$q = (x_1, p_1; x_2, p_2; \ldots; x_n, p_n);$$

  which reads as : "gain x_m with probability p_m", where $1 \leq m \leq n$;
- under expected utility the value of gamble q is given by

$$V = \sum_{i=1}^{n} p_i U(W(x_i)),$$

  where $U(\cdot)$ is a monotone increasing, concave function and W is the end-of-period wealth dependent on outcome x_m;
- the definition of $U(\cdot)$ implies that
  - people prefer more wealth to less;
  - an additional dollar at higher wealth levels is not as desirable as at lower ones;
  - the concavity of $U(\cdot)$ implies risk–aversion;
- under expected utility, utility is generated by *absolute levels* of wealth;

Expected utility value function $U(\cdot)$



Departing from expected utility

Utility over outcomes

- assume now the same gamble is evaluated as follows:

$$V = \sum_{i=1}^{n} \pi_i V(x_i),$$

  where π_m is a 'decision weight' and $V(\cdot)$ is a value function evaluated over each of the outcomes x_m (*prospects*);
- utility is now generated from *gains and losses* compared against a *reference point* rather than absolute levels of wealth;
- this definition implies the following properties:
  1. reference dependence;
  2. loss aversion;
  3. diminishing sensitivity;
  4. probability weighting;
- we are led to *Prospect Theory* (Kahneman and Tversky, 1979);

A famous expected utility violation: Allais' paradox

but what do we need the second formulation for?

- in practice, we have evidence that expected utility is violated;
- one of the most famous violations of expected utility was discovered by Allais (1953);
- participants in an experiment were asked to choose between the following gambles;
- A: receive: $1 M with probability 1;
  B: receive: $ 5 M with probability 0.10; $ 1 M with probability 0.89; $ 0 with probability 0.01;
- C: receive: $ 1 M with probability 0.11; $ 0 with probability 0.89;
  D: receive: $ 5 M with probability 0.10; $ 0 with probability 0.90;
- most popular response was A over B and D over C;

Why is this a paradox?

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Allais' paradox explained

- the expected value of A is \$ 1 million, while the expected value of B is \$ 1.39 million;
- by choosing A over B, people maximize *expected utility* not *expected value*;
- by $A \succ B$, we have the following expected utility relationship:

$$u(1) > 0.1u(5) + 0.89u(1) + 0.01u(0) \Leftrightarrow$$
$$0.11u(1) > 0.1u(5) + 0.01u(0);$$

- adding $0.89u(0)$ to each side, we get:

$$0.11u(1) + 0.89u(0) > 0.1u(5) + 0.90u(0),$$

- an expected utility maximizer should go for C over D!;
- but the experimental evidence of D over C creates a paradox;
- we have a violation of the *independence axiom* of expected utility;

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Prospect theory in detail

Remember the four properties of prospect theory:

- reference dependence;
- loss aversion;
- diminishing sensitivity;
- probability weighting;

Let's peruse them one by one;

Departing from expected utility theory

Reference dependence

- in prospect theory, individuals derive utility from gains and losses rather than absolute wealth levels;
- these are measured on a comparative basis against a *reference point*;
- interestingly, individuals respond to attributes other than wealth (such as temperature) based on past or present experience;
- individuals adapt their behaviour relative to that rather than the absolute current level of the attribute (Kahneman and Tversky, 1979);

Departing from expected utility theory

Loss aversion

- the value function $V(\cdot)$ captures *loss aversion*;
- loss aversion is the explanation to the observation that people are way more sensitive to losses than to gains;
- under loss aversion, the pain generated by a certain loss is a much stronger feeling compared to the satisfaction by a gain of the same magnitude;
- Kahneman and Tversky in their experiments found that the following gamble

  $$(-\$100, 0.5; \$110, 0.5)$$

  was most frequently turned down;
- indeed, assuming loss aversion the pain by a loss of $\$100$ is far stronger than the pleasure by a gain of $\$110$;
- in order for the value function $V(\cdot)$ to capture this effect, we construct it steeper in the domain of losses;

Departing from expected utility theory

Diminishing sensitivity

- $V(\cdot)$ is concave in the domain of gains and convex in the domain of losses;
- diminishing sensitivity captures the following effect: while a $\$2,000$ gain (or loss) instead of a $\$1,000$ gain (or loss) has a significant impact on utility, a $\$10,000$ gain (or loss) instead of a $\$9,000$ gain (or loss) has a much smaller impact;
- the concave shape of the utility function in the gain region shows thar people are risk-averse;
- they prefer a certain gain of $\$100$ to an uncertain gain of $\$200$;
- the opposite for losses: risk-seeking behaviour; people prefer a loss of $\$200$ with probability 0.5 compared to a certain loss of $\$100$;
- in prospect theory, the value function $V(\cdot)$ has the following form:

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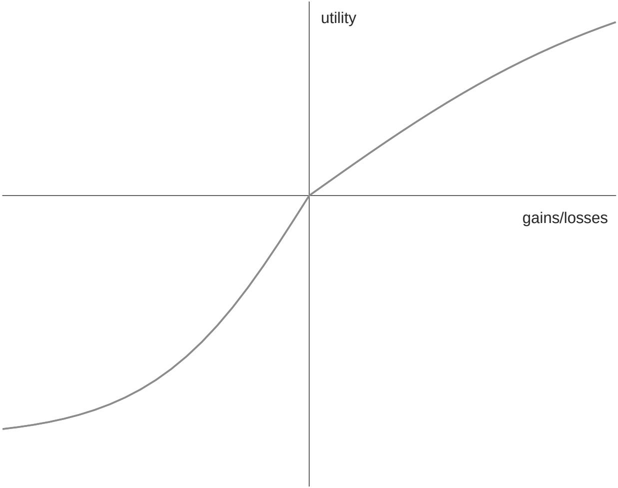
Departing from Expected Utility Theory

Asymmetric S-shaped utility function

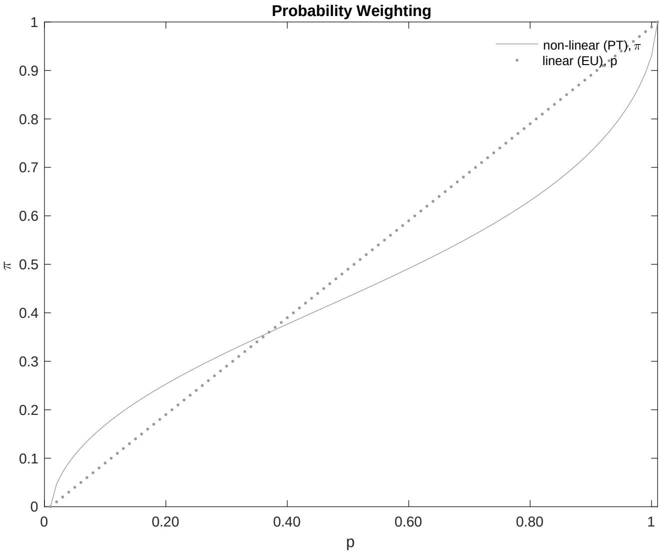


Departing from expected utility theory

Probability weighting

- in prospect theory, people do not use the *objective* probabilities p_i;
- instead, they perform a probability weighting using π_i which are non-linear transformed weights of p_i;
- the probability weighting overweights *tail-events*;
- it makes outcomes which seem to be unlikely when p_i are used, a bit more likely under π_i;
- people will opt for a certain small loss over a very large loss with an extremely small probability to happen (they like to be insured);
- people will opt for the very unlikely event to win a lottery over a certain very small gain (they like to gamble);
- prospect theory captures these effects, while expected utility doesn't seem capable of being able to explain them;

Probability weighting: transforming probabilities into decision weights



From loss aversion to disappointment aversion

- reminder: the main idea of prospect theory is that people are more interested in changes of wealth relative to some reference point, rather than absolute levels of wealth;
- but how are these reference points defined and updated?
- Kahneman and Tversky (1979) do not provide us with a clear answer to that; reference points are generally set *exogenously* equal to current wealth level;
- however, Gul (1991) comes up with *disappointment aversion* (DA), a derivative of loss aversion;
- DA theory:
  1. captures the same behavioural effects as loss aversion;
  2. maintains prospect theory's axiomatic definition;
  3. but more importantly, provides us with a purely tractable endogenous way as to how reference points are chosen and updated;

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Introduction to the asset allocation problem

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- individuals are interested not only in whether the future return of an investment is positive or not but also on whether it meets their initial expectations;
- then, a below expectation performance can generate disappointment introducing disappointment aversion tendencies in investor's trading behaviour;

Literature

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- several theoretical frameworks depart from the axioms of EU transforming probabilities into decision weights non-linearly (Handa, 1977; Chew and MacCrimmon, 1979; Fishburn 1983; Tversky and Kahneman, 1992);

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- several theoretical frameworks depart from the axioms of EU transforming probabilities into decision weights non-linearly (Handa, 1977; Chew and MacCrimmon, 1979; Fishburn 1983; Tversky and Kahneman, 1992);
- in portfolio choice, PT is by far the most widely used framework (Berkelaar et al., 2004; Gomes, 2005; Barberis and Huang, 2008; Dimmock and Kouwenberg, 2010; Bernard and Ghossoub, 2010);

Prospect theory in asset allocation decisions

How do the attributes of prospect theory carry over to an asset allocation problem?

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- on the other side, they grow more risk-averse in the domain of gains, selling the winner stocks to realize the profits while they still exist;
- according to short-term momentum investors should keep their winners and sell their losers;

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- however, the use of DA theory is rather limited; Abdellaoui and Bleichrodt (2007) attribute this to its lack of providing a way to formally extract the DA coefficient;
- it has been used for asset pricing (Routledge and Zin, 2010; Bonomo et al. 2011) and recently in asset allocation (Dalquist et al., 2017) but with the objective to derive expressions for risk measures;

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Contributions

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- second, we revisit and extend the study of the portfolio choice problem for a long-term buy-and-hold investor under return predictability and parameter uncertainty;
- third, we demonstrate how the incorporation of predictability and parameter uncertainty in asset returns affects portfolio weights at different horizons for a dynamic investor and how this can give rise to *horizon effects*.

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Quick recap and continuation

So far we saw:

- expected utility Vs prospect theory (and disappointment aversion theory);
- departures from expected utility by changing the way we measure the impact of an outcome on our utility function;
- disappointment aversion in asset allocation decisions.

The rest of this presentation focuses on:

- the formulation of the asset allocation problem under disappointment aversion preferences;
- a proposed solution;
- numerical results and conclusion;

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The disappointment aversion framework

DA utility

- the DA utility function is defined as in Ang et al. (2005) as follows

$$U(\mu_W) = \frac{1}{K}\left(\int_{-\infty}^{\mu_W} U(W)dF(W) + A \int_{\mu_W}^{\infty} U(W)dF(W) \right); \quad (1)$$

- $K = P(W \leq \mu_W) + AP(W > \mu_W)$, $U(W)$ is the utility function, $F(W)$ is the cumulative density function. The coefficient of DA, $0 < A \leq 1$ downweighs outcomes above the certainty equivalent μ_W;
- investor's objective is to

$$\max_{\alpha} U(\mu_W); \quad (2)$$

- in a static context we face the decision making problem of allocating optimally between a risky (stock index) and a riskless (bond) asset;

DA participation/non participation

Critical level of the DA coefficient (A^*)

- investors' utility function and their expectations over risky asset's drive in part their decisions;
- *expected utility* theory always predicts positive portfolio weights to risky assets when the expected equity premium is positive (i.e. $\mathbb{E}(X) > 0$);
- in DA theory, there are cases where it is optimal to hold no risky assets despite the positive risk premium. This generates the so-called *non-participation* regions;
- we prove that below a certain level of the A^*, for a DA investor is optimal to hold zero units of the risky asset in the following theorem:

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DA participation/non participation

Theorem

Let $\mu = \mu_W(A,\alpha)$*, with*

- $\mu(A,:) \in \mathbf{C}^1, \forall A \in [0,1]$
- $\frac{\partial \mu(A,0)}{\partial \alpha} = \xi(A) \le 0, \forall A \in [0,1]$
- $\mathbb{E}(X) > 0$ *and* $\mathbb{E}(X\mathbf{1}_{W \ge \xi(A)}) > 0$*, where* $X = e^y - e^r$ *is the excess return of the equity over the bond.*

Then, setting

$$A^* = \frac{\mathbb{E}(X\mathbf{1}_{W \ge \xi(A)})}{\mathbb{E}(X\mathbf{1}_{W < \xi(A)})} \tag{3}$$

we have the following:

1. *For every* $A \le A^*$*,* $\alpha^* = 0$*,*
2. *For every* $A > A^*$*,* $\alpha^* > 0$*,*

where α^* *is the portfolio allocation which maximizes* $\mu(A,\alpha)$ *for a given* A*.* A^* *is independent of the risk aversion parameter.*

Interpretation of the result in Theorem 1

- Intuitively this theorem can be presented in the following way: focusing on the disappointment aversion coefficient A, we find that, as it decreases, investors allocate less wealth to the risky asset regardless of their level of risk aversion;
- then there should be a level of A, let A^*, at which the optimal portfolio allocation to the risky asset, α^* equals zero;
- recalling the condition $\partial\mu(A,0)/\partial\alpha \leq 0$, a further decrease in the risky asset weight α^* (e.g. due to short-selling the risky asset) will result in a higher certainty equivalent since the following relationship will prevail:

$$W = \alpha^* X + r > r,$$

  for $\alpha^* < 0$ and negative states $(X < 0)$ of the excess equity return;
- therefore, the optimal allocation for this critical level of the disappointment aversion coefficient, A^* is zero and $\alpha = \alpha^* = 0$.

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DA participation/non–participation

Static case - Ang et al. (2005) revisit. 1–year horizon

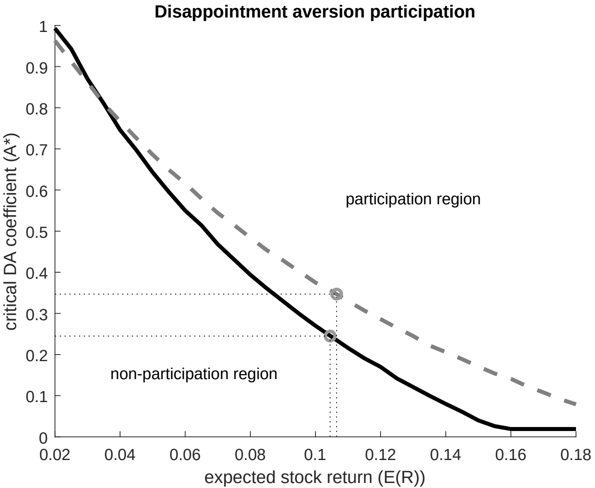


Figure: Stock market participation/non-participation regions with DA preferences.

Dynamic asset allocation: Utility maximization

Utility of wealth U(W)

- We first formulate the optimization problem for a utility function, $U(W)$ and then we extend its definition to accommodate DA preferences;
- instead of a single portfolio weight we now need to find a series of optimal portfolio weights;
- we use *Dynamic Programming* by solving first and storing the solution of the problem at T-1; we proceed recursively by using this solution to solve the problem at T-2 and so on;
- our objective is to find the optimal policy $\alpha = \{\alpha_t\}_{t=0}^{T-1}$ in order to:

$$\max_{\alpha_0, \alpha_1, \ldots, \alpha_{T-1}} \mathbb{E}_0[U(W_T)], \tag{4}$$

- where $\alpha_0, \alpha_1, \ldots, \alpha_{T-1}$ are the portfolio weights to the risky asset, $U(W) = W^{1-\gamma}/1-\gamma$, wealth $W_{t+1} = W_t R_{t+1}(\alpha_t)$ with $R_{t+1}(\alpha_t)$ being the total portfolio return over the period t to $t+1$;

Dynamic asset allocation: Optimization problem

Problem formulation

- at time t the optimization problem becomes

$$\max_{\alpha_t, \alpha_{t+1}, \ldots, \alpha_{T-1}} \mathbb{E}_t[U(W_{t+1} Q^*_{t+1,T})], \tag{5}$$

  where $Q^*_{t+1,T} = R_T(\alpha^*_{T-1}) R_{T-1}(\alpha^*_{T-2}) \cdots R_{t+2}(\alpha^*_{t+1})$ represents the aggregate return from time $t+1$ to T that maximizes the investor's expected utility;
- by plugging in the power utility function, we have the following:

$$\max_{\alpha_t} \mathbb{E}_t\left[\frac{W_{t+1}^{1-\gamma}}{1-\gamma} (Q^*_{t+1,T})^{1-\gamma} \right];$$

- and the optimal investment proportions of the risky asset at every horizon is given by:

$$\alpha^*_t = \arg\max_{\alpha_t} \mathbb{E}_t\left[W_{t+1}^{1-\gamma} (Q^*_{t+1,T})^{1-\gamma} \right];$$

Proposition (DA utility and FOC for the dynamic problem)

- *For given* $Q^*_{t+1,T} = R_T(\alpha^*_{T-1})R_{T-1}(\alpha^*_{T-2})\cdots R_{t+2}(\alpha^*_{t+1})$, *the DA utility function for the dynamic asset allocation problem is given by*

$$U(\mu_t) = \frac{1}{K_t}\Bigg[\mathbb{E}_t\Big(U(W_{t+1}Q^*_{t+1,T})\mathbf{1}_{W_{t+1}Q^*_{t+1,T}\le\mu_t}\Big) + A\mathbb{E}_t\Big(U(W_{t+1}Q^*_{t+1,T})\mathbf{1}_{W_{t+1}Q^*_{t+1,T}>\mu_t}\Big)\Bigg],$$

where $W_{t+1}Q^*_{t+1,T} = W_T$.

- *The FOC for the optimization of the utility of the certainty equivalent return is given by*

$$\mathbb{E}_t\Big(\frac{dU(W_T)}{dlW}Q^*_{t+1,T}R_{t+1}(\alpha_t)W_tX_{t+1}\mathbf{1}_{W_T\le\mu_t}\Big) + A\mathbb{E}_t\Big(\frac{dU(W_T)}{dW}Q^*_{t+1,T}R_{t+1}(\alpha_t)W_tX_{t+1}\mathbf{1}_{W_T>\mu_t}\Big) = 0;$$

$X_{t+1} = e^{y_{t+1}} - e^{r_t}$ *is the excess return of the risky asset over the riskless.*

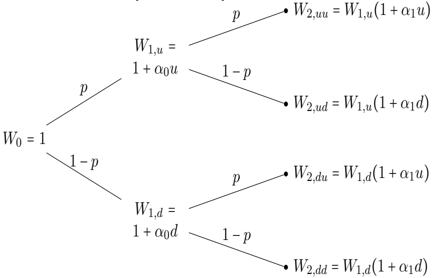
Dynamic asset allocation: Optimization problem

Computational issue

- when the product

$$Q^*_{t+1,T} = R_T(\alpha^*_{T-1})R_{T-1}(\alpha^*_{T-2})\cdots R_{t+2}(\alpha^*_{t+1})$$

  is used, the state space increases exponentially with time;
- this makes the problem intractable and very expensive to solve computationally;
- in a two-period problem, with only two states for risky asset return:



  - we need to consider $2^2 = 4$ states, namely $\{uu, ud, du, dd\}$;
  - in a multi-period framework with N time steps and s possible risky asset returns we track s^N: exponentially increases with time;

Dynamic asset allocation: Optimization problem

Remedy: Problem reduction

- we adopt a *reduction technique* proposed in Epstein and Zin (1989) and Ang et al. (2005) making the assumption that *future uncertainty on asset returns is captured in the certainty equivalent*;
- $R_{t+1}(\alpha_t^*)$ is now substituted with μ_t^* (certainty equivalent return for the corresponding period), optimal by definition (*optimality principal*);
- although μ_t^* is in general not exactly equal to $R_{t+1}(\alpha_t^*)$, it allows us to tackle the problem computationally and reach a stable solution;
- the discretized system of the adjusted utility function and its corresponding FOC can be solved by a binary search algorithm for μ_W and recursively for the portfolio weights at each horizon t;

Dimensionality reduction for the optimization problem

Proposition (DA utility function and FOC, reduced problem)

- *The utility of the certainty equivalent return is as follows:*

$$U(\mu_t) = \frac{U(\prod_{i=t+1}^{T-1} \mu_i^* W_t)}{K_t}\Bigg[\mathbb{E}_t(U(R_{t+1}(\alpha_t))\mathbf{1}_{\{R_{t+1}(\alpha_t)\leq\xi_t\}}) + A\mathbb{E}_t(U(R_{t+1}(\alpha_t))\mathbf{1}_{\{R_{t+1}(\alpha_t)>\xi_t\}})\Bigg]$$

- *and the FOC for the optimization of the utility of certainty equivalent return is given by*

$$\mathbb{E}_t\left(\frac{dU(R_{t+1}(\alpha_t))}{d\alpha_t}X_{t+1}\mathbf{1}_{\{R_{t+1}(\alpha_t)\leq\xi_t\}}\right)+A\mathbb{E}_t\left(\frac{dU(R_{t+1}(\alpha_t))}{d\alpha_t}X_{t+1}\mathbf{1}_{\{R_{t+1}(\alpha$$

where $\xi_t = \frac{\mu_t}{\mu_{T-1}^* \cdots \mu_{t+1}^* W_t}$, *with* μ^* *'s the optimal certainty equivalents between* $t+1$ *and* $T-1$.

Computational benefit

- in the reduced utility function of certainty equivalent return $\mu^*_{T-1}\cdots\mu^*_{t+1}W_t$ represents the optimal decision making between $t+1$ and $T-1$;
- it is taken outside the expectation terms;
- at every time horizon we need to keep track of only the candidate states for $R_{t+1}(\alpha_t)$, next period's return;
- the DA investor uses next period's optimal return (captured in the certainty equivalent) to calculate the utility of the current period maintaining the endogeneity in the updating of the reference point;
- this way, we keep the dimension of the problem constant to the total number of states for the risky asset return for next period only;
- plugging in the power utility function, the FOC in takes the following form:

$$\mathbb{E}_t\Big(R^{-\gamma}_{t+1}(\alpha_t)X_{t+1}\mathbf{1}_{R_{t+1(\alpha_t)\leq\xi_t}}\Big)+A\mathbb{E}_t\Big(R^{-\gamma}_{t+1}(\alpha_t)X_{t+1}\mathbf{1}_{R_{t+1(\alpha_t)>\xi_t}}\Big)=0;$$

- the advantage of using the certainty equivalent is clear by comparing the two expressions for the DA utility;

Model estimation - first-order vector autoregression (VAR)

- Under i.i.d. returns, the excess equity return is represented by

$$x_t = (\mu - r) + \epsilon_t, \tag{6}$$

where x_t is the continuously compounded annual excess return of the S&P 500 index in period t and ϵ_t are i.i.d. disturbance terms distributed as $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$; $\mu = 0.1045$, $r = 0.0344$ and $\sigma = 0.1635$,
- in this case, investor's opportunity set remains constant over time;
- when predictability is incorporated, we estimate the following VAR

$$x_t = c_1 + b_{11} x_{t-1} + b_{12} (d/p)_{t-1} + \epsilon_{1,t} \tag{7}$$

$$(d/p)_t = c_2 + b_{21} x_{t-1} + b_{22} (d/p)_{t-1} + \epsilon_{2,t} \tag{8}$$

where $y_t - r_{t-1} = x_t$ is the excess equity return, r_t is the risk–free rate and $(d/p)_{t-1}$ is the dividend–price ratio. The AR matrix B equals

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 0.0259 & 0.0220 \\ (0.1176) & (0.1354) \\ -0.7068 & 0.9978 \\ (0.0807) & (0.0929) \end{pmatrix}; \tag{9}$$

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VAR estimation

Parameter	With predictability	Without predictability
c_1	0.1222	0.0128
	(0.0173)	(0.0178)
c_2	-0.0004	-0.0317
	(0.0119)	(0.0150)
b_{11}	0.0259	0.0
	0.1176	–
b_{12}	0.0220	0.0
	(0.1354)	–
b_{21}	-0.7068	0.0
	(0.0807)	–
b_{22}	0.9978	0.9932
	(0.0929)	(0.0912)
σ_{11}	0.0850	0.0856
	(0.0037)	(0.0042)
σ_{22}	0.0408	0.0752
	(0.0017)	(0.0029)
ρ	-0.5216	-0.2980
	(0.0021)	(0.0028)

Table: VAR estimation and corresponding standard errors in parentheses.

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Parameter uncertainty

Return distribution

- One of the main decisions investors have to make is what *distribution* they will use to estimate risky asset's return and volatility;
- under *parameter uncertainty* we incorporate no prior information about the real values of model parameters and we treat them as unknown;
- relevant literature: Kandel and Stambaugh (1996); Barberis (2000); Kacperczyk and Damien (2011); Branger et al. (2013); Hoevernars et al. (2014); De Miguel et al. (2015) among others;
- Two approaches: When parameter uncertainty is ignored:

$$\max_{\alpha} \mathbb{E}_t[U(W_n)] = \max_{\alpha} \Bigg[\int_{W_{t+n} \leq \mu_W} U(W_{t+n}) p(r_{t+n}|Y,\theta) dr_{t+n} + A \int_{W_{t+n} > \mu_W} U(W_{t+n}) p(r_{t+n}|Y,\theta) dr_{t+n} \Bigg] \quad (10)$$

where $U(\cdot)$ is the utility of wealth, $p(r_{t+n}|Y,\theta)$ is the density function of the expected returns conditional on the data Y and θ;

Dynamic portfolio allocation

Construct the posterior predictive distribution

- the uncertainty in this problem revolves around parameter set θ, since their values change as we incorporate newly created data in the model;
- integrating out θ in the prior distribution $p(r_{t+n}|Y,\theta)$, we end up with the posterior predictive distribution;
- now, investors maximize the expression:

$$\int_{W_{t+n}\leq\mu_W} U(W_{t+n})p(r_{t+n}|Y)dr_{t+n} + A\int_{W_{t+n}>\mu_W} U(W_{t+n})p(r_{t+n}|Y)dr_{t+n}, \quad (11)$$

where the distribution of the returns is conditional only on the observed data and not on the parameter set θ;

Parameter uncertainty - posterior predictive

i.i.d. returns

- starting from the following uninformative prior:

$$p(\mu,\sigma)d\mu d\sigma \propto \frac{1}{\sigma}d\mu d\sigma,$$

- we construct the joint posterior of the mean return μ and volatility σ as

$$p(\mu,\sigma|Y) \propto p(\mu,\sigma) \times L(\mu,\sigma|Y),$$

where L is the likelihood function;

- then, the posterior distribution $p(\mu|\sigma,Y)$ is given by:

$$\sigma^2|Y \sim Inv-Gamma\left(\frac{N}{2},\frac{1}{2}\sum_{i=1}^{N+1}(y_i-\overline{\mu})^2\right)$$

$$\mu|\sigma,Y \sim \mathcal{N}\left(\overline{\mu},\frac{\sigma}{\sqrt{N}}\right),$$

where Y is the observed asset return data, N is the sample size and $\overline{\mu}$ is the sample mean;

Parameter uncertainty - posterior predictive

Return predictability

- a suitable uninformative prior for predictable returns is the Jeffreys prior given by:

$$\begin{aligned} p(B,\Sigma) &= p(B)p(\Sigma) \\ &\propto |\Sigma|^{-(m+1)/2}, \end{aligned}$$

  where $m = 2$ is the total number of regressors on VAR specification, $p(B)$ is constant and B is independent of Σ.
- then, posterior density $p(vec(B)|\Sigma, X)$ for the coefficient matrix, B and the variance-covariance matrix, Σ is given by:

$$\Sigma|Y \sim \mathcal{W}^{-1}((Y - Z\hat{B})'((Y - Z\hat{B}), T - n - 1)$$
$$vec(B)|\Sigma, Y \sim \mathcal{N}(vec(\hat{B}), \Sigma^{-1} Z'Z).$$

  where $\mathcal{W}^{-1}$ Wishart distribution, Z is a $(3 \times T)$ matrix with lagged excess return and dividend yield data, T is the number of observations in our sample and n is the number of predictor variables.

Non participation under DA utility function

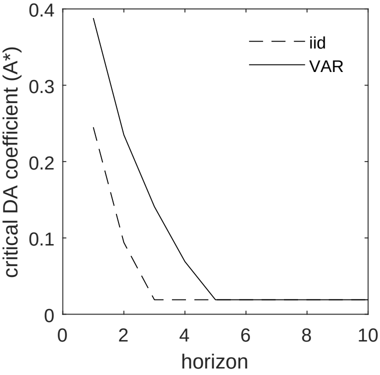

Figure: Critical DA level (A^*) that induces non–participation in the stock market for a buy–and–hold investor (left graph) and a dynamic investor (right graph). Investors would invest in the stock market when their DA coefficient lies in the area above the lines.

Non-participation in disappointment aversion

For an investor who follows a buy-and-hold strategy:

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- the choice of the underlying Data Generating Process (DGP) (i.i.d. returns or VAR) is not critical;

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- the choice of the DGP is critical;
- when returns are i.i.d., A^* is invariable to changes in investment horizon as at every horizon, the investor uses the exact same distribution to generate her expectations;
- when returns are believed to be forecastable, the longer the horizon, the more disappointment averse should an investor be in order to refrain from holding the risky asset;

Numerical examples: Buy–and–hold strategies (1/2)

i.i.d. returns

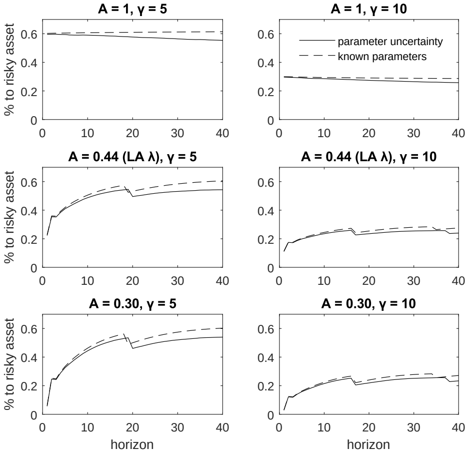

Figure: Optimal portfolio allocation to the risky asset for an investor who follows a buy-and-hold investment strategy, uses the i.i.d. return generator and either incorporates (solid line) or ignores (dashed line) uncertainty in model parameters.

Numerical examples: Buy–and–hold strategies(2/2)

VAR

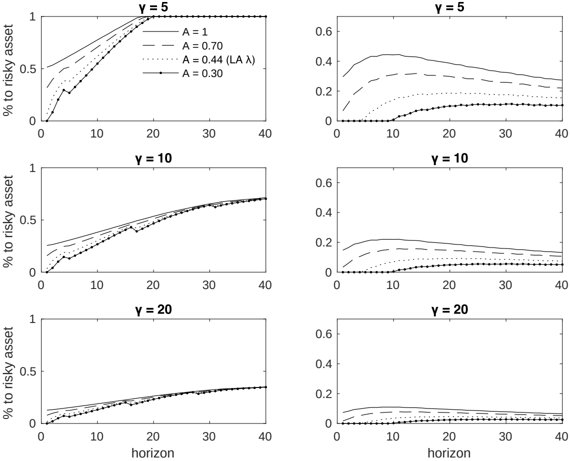

Figure: Optimal portfolio composition for different horizons when the VAR is used. Investor follows a buy–and–hold strategy with the one on the left column ignoring parameter uncertainty while the one on the right accounts for this.

Buy-and-hold strategies

when returns are i.i.d.:

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- the longer the horizon, the higher the allocation the risky asset as a result of the slower increase of the total volatility over the investment horizon (next slide's graph);

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- the impact of DA is more profound at shorter horizons as for longer ones, allocation lines converge regardless of the level of A;
- the longer the horizon, the higher the allocation the risky asset as a result of the slower increase of the total volatility over the investment horizon (next slide's graph);
- when parameter uncertainty is considered, allocation to the risky asset drops significantly, but in general it increases with investment horizon;

Volatility evolution

- When we model returns as i.i.d., the two-period variance equals:

$$var_{r_1,r_2} = var_{r_1} + var_{r_2} \Leftrightarrow \sigma_{1,2} = \sqrt{var_{r_1} + var_{r_2}};$$

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$$var_{r_1,r_2} = var_{r_1} + var_{r_2} + 2\rho\sqrt{var_{r_1} var_{r_2}};$$

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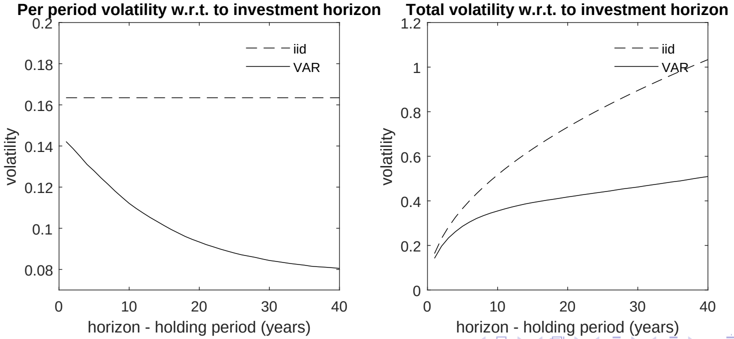

Numerical examples – Dynamic strategies (i.i.d. returns)

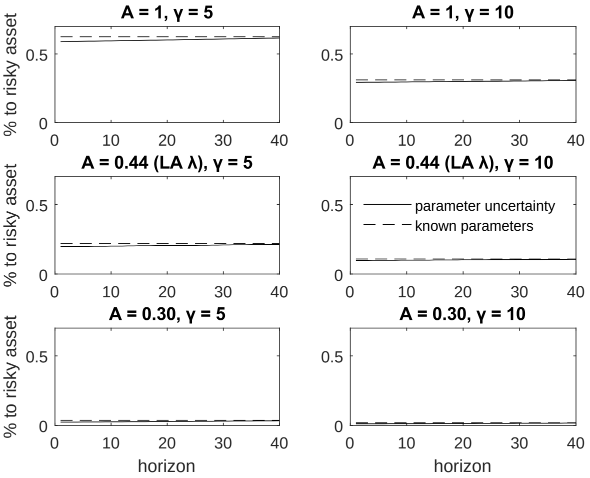

Figure: Dynamic portfolio allocation between the risky and the riskless asset for an investor who uses the i.i.d. return generator for the risky asset. The graph shows how portfolio allocation to the risky asset changes for an investor who acknowledges parameter uncertainty (solid line) compared to one who ignores it

Numerical examples – Dynamic strategies

Return predictability

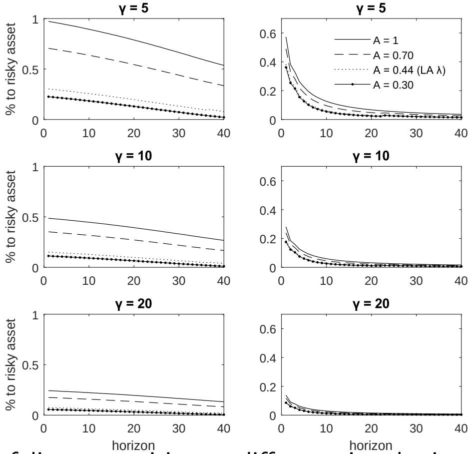

Figure: Optimal portfolio composition at different time horizons for an investor who follows a dynamic reallocation using the VAR to forecast returns. The left columns reports results when parameter uncertainty is ignored while the one on the right takes parameter uncertainty into account.

Dynamic strategies

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when returns are predictable:

- allocation to the risky asset at longer horizons is significantly higher to that at shorter ones for all levels of the disappointment aversion coefficient;
- when parameter uncertainty is incorporated, allocation to the risky asset is significantly lower to that in the case where parameter uncertainty is ignored;

Concluding remarks

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- for an investor who follows a dynamic strategy, the incorporation of *parameter uncertainty* with predictability derives lower weights to equity as a result of investors' doubts on the predictability of equity return;

Concluding remarks - cont.

- disappointment aversion tends to prompt investors to reduce their exposure to equity;

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- DA can finally help explain the relative reluctance of investors to hold equities (thus contributing to the debate on the equity premium puzzle) and re-enter the market if they have exited it previously at a loss;
- overall, parameter uncertainty is beneficial to be examined in a portfolio optimization problem as it could lead to allocations closer to investor's risk profile and prevent overallocation to the risky asset.

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Thank you for your attention.
Questions?