Sequential capacity expansion options

15th Summer School in Stochastic Finance

Athens, Greece

Alain Bensoussan[1,2] **Benoît Chevalier-Roignant**[3]

July 10, 2018

[1]University of Texas at Dallas, [2]City University of Hong Kong, [3]King's College London

INTRODUCTION

We examine **sequential capacity expansion options** for which the firm decides on

- the investment times (stopping times)
- the size of the capacity increase

Contributions to the extant literature:

- Models of irreversible investment ("capital accumulation") ignore fixed costs; the impact of fixed costs on timing is known from real options analysis (e.g., McDonald and Siegel 1986)
- Introducing fixed costs explain observed "investment bursts," e.g., in real estate

Economic literature

	OPTION TO EXPAND	CAPITAL ACCUMULATION	OUR CONTRIBUTION
References	Trigeorgis 1996	Abel and Eberly 1994	Our paper
Criticisms	No investment lump (Hubbard 1994)	Incremental (Pindyck 1988)	n.a.
Applications	Microeconomics	Macroeconomics	Microeconomics

Mathematical characterization

	Option to expand	Capital accumulation	Our contribution
Cost	Fixed	Linear	Affine
Problem	Optimal stopping	Singular control	Impulse control
DP equation[1]	VI[2]	Degenerate QVI[3]	QVI[4]

[1] DP = Dynamic programming
[2] See Bensoussan and Lions 1982 on variational inequalities (VI)
[3] See, in the context of capacity investment, the formulations in Kobila 1993; Øksendal 2000; Riedel and Su 2011; Ferrari 2015 for the irreversible case and Merhi and Zervos 2007 for the reversible case.
[4] See Bensoussan and Lions 1984 on quasi-variational inequalities (QVI)

Model setup

1. The **commodity price** process follows a geometric Brownian motion (GBM)[1]

$$dY_t = \mu Y_t \, dt + \sigma Y_t \, dW_t, \quad Y_0 = y \ (> 0) \text{ a.s.}$$

   where W is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$

[1] It is not affected by the firm's decisions on output/capacity
[2] The τ_n are stopping times with respect to the filtration generated by W; $\xi_n > 0$ are $\mathscr{F}_{\tau_n}$-measurable
[3] Capacity scrapping or depreciation is not permitted here

Two state equations

1. The **commodity price** process follows a geometric Brownian motion (GBM)[1]

$$\mathrm{d}Y_t = \mu Y_t \,\mathrm{d}t + \sigma Y_t \,\mathrm{d}W_t, \quad Y_0 = y \ (> 0) \text{ a.s.}$$

   where W is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$

2. The firm influences the **capacity stock** development

$$X_t^\nu = x + \sum_n \underbrace{\xi_n 1_{\{\tau_n \le t\}}}_{\text{Capacity additions}},$$

   via its choice of **impulse control** policy $\nu = \{\tau_n, \xi_n\}_n$.[2,3]

[1] It is not affected by the firm's decisions on output/capacity

[2] The τ_n are stopping times with respect to the filtration generated by W; $\xi_n > 0$ are $\mathscr{F}_{\tau_n}$-measurable

[3] Capacity scrapping or depreciation is not permitted here

- The *production function* $\delta(\cdot)$ is C^1, monotone increasing and concave in capacity x[1]
- The **value function** is

$$v(y,x) = \sup_{\nu} \mathbb{E}_{y,x}\Big[\int_0^\infty \underbrace{Y_t \delta(X_t^\nu)}_{\text{Firm profit}} e^{-rt}\,\mathrm{d}t - \sum_n \underbrace{(k + c\xi_n)}_{\substack{\text{Affine cost}\\ k,c>0}} e^{-r\tau_n} \Big]$$

- We assume that $r > \mu$ and focus on *admissible* controls ν such that

$$\mathbb{E}_{y,x}\Big[\sum_n (k + c\xi_n) e^{-r\tau_n} \Big] < \infty.$$

[1] In addition, $\delta(0) = 0$, and $\delta'(0) = \infty$, $\delta'(\infty) = 0$

Dynamic programming equation

We introduce

$$\mathcal{L}\varphi(y) := r\varphi(y) - y\mu\varphi'(y) - \frac{1}{2}y^2\sigma^2\varphi''(y) \qquad \text{(infinitesimal generator)}$$

$$\mathcal{M}v(y,x) := \sup_{z\geq x}\{v(y,z) - c\,[z-x]\} - k \qquad \text{(intervention operator)}$$

$$v_L(y,x) := \frac{y\delta(x)}{r-\mu} \qquad \text{(perpetuity value of capacity)}$$

We are looking for a solution v of the **QVI**[1]

$$\min\left\{\mathcal{L}v(y,x) - y\delta(x); v(y,x) - \mathcal{M}v(y,x)\right\} = 0, \tag{1a}$$

$$v(0,x) = 0, \tag{1b}$$

$$\liminf_{y\to\infty}\{v(y,x) - v_L(y,x)\} \geq 0, \tag{1c}$$

with the **regularity**

$$v \in C^1(\mathbb{R}^2_+) \quad \text{and} \quad v_{yy}(\cdot,x) \text{ locally integrable}$$

[1] Equation (1a) to be interpreted as

$$\mathcal{L}v(y,x) \geq y\delta(x), \qquad \text{Waiting is a lower bound}$$
$$v(y,x) \geq \mathcal{M}v(y,x) \qquad \text{Investing is a lower bound}$$
$$[\mathcal{L}v(y,x) - y\delta(x)][v(x,y) - \mathcal{M}v(x,y)] = 0, \qquad \text{Either one is optimal}$$

The *continuation region* $\mathcal{C} = \{(y, x) \mid v(y, x) > \mathcal{M}v(y, x)\}$ is conjectured to be

$$\mathcal{C} = \{(y, x) \mid y < \bar{y}(x)\},$$

where the threshold $\bar{y}(\cdot)$ is monotone increasing. We define $\bar{x} = \bar{y}^{-1}$ and denote by $\mathcal{S} = \mathbb{R}^2_+ \setminus \mathcal{C}$ the *stopping region*

[1]Note that (i) value function, (ii) QVI's solution and (iii) FBP's solution do not necessarily coincide

The *continuation region* $\mathcal{C} = \{(y,x) \mid v(y,x) > \mathcal{M}v(y,x)\}$ is conjectured to be

$$\mathcal{C} = \{(y,x) \mid y < \bar{y}(x)\},$$

where the threshold $\bar{y}(\cdot)$ is monotone increasing. We define $\bar{x} = \bar{y}^{-1}$ and denote by $\mathcal{S} = \mathbb{R}^2_+ \setminus \mathcal{C}$ the *stopping region*

Given this conjecture, we look for a regular solution v of the **free-boundary problem** (FBP):[1]

$$\begin{aligned} \mathcal{L}v(y,x) &= y\delta(x), && \text{a.e. } y < \bar{y}(x), \\ v(y,x) &= \mathcal{M}v(y,x), && y \geq \bar{y}(x), \\ v(0,x) &= 0, && \\ \liminf_{y\to\infty}\{v(y,x) - v_L(y,x)\} &\geq 0. && \end{aligned}$$

[1]Note that (i) value function, (ii) QVI's solution and (iii) FBP's solution do not necessarily coincide

In a nutshell, we are able in the *general case* to

- prove that the “minimum” QVI’s solution coincides with the value function (provided regularity)
- express conditions on the (s, S) boundaries to ensure existence of a regular FBP’s solution

In a *particular case*,

1. we construct (s, S) boundary *curves* corresponding to a regular FBP’s solution
2. we prove that the FBP’s solution solves the QVI
3. we establish that the QVI’s solution coincides with the value function (verification problem)

Benchmark models

The value function v is bounded below by the **perpetuity value of capacity**:

$$v(y,x) \geq v_L(x,x) := \frac{y\delta(x)}{r-\mu}$$

The value function v is bounded below by the **perpetuity value of capacity**:

$$v(y,x) \geq v_L(x,x) := \frac{y\delta(x)}{r-\mu}$$

In the (perfectly) **reversible case**, the firm should raise capacity if

$$(y,x) \in \mathcal{S}_L := \left\{(y,x) \mid y \geq \mathsf{y}_\mathsf{L}(\mathsf{x}) := \frac{\mathsf{c}[\mathsf{r}-\mu]}{\delta'(\mathsf{x})}\right\}$$

We have $\frac{\partial v_L}{\partial x}(\bar{y}_L(x), x) = c$ and define $\bar{x}_L := \bar{y}_L^{-1}$

The value function v is bounded above by the value function of the **singular control** problem ($k = 0$):

$$v(y,x) \le v_U(y,x) := \sup_{\xi(\cdot)} \mathbb{E}_{y,x}\left[\int_0^\infty Y_t \delta(X_t) e^{-rt} \, \mathrm{d}t - c \int_0^\infty e^{-rt} \, \mathrm{d}\xi_t \right]$$

The value function v is bounded above by the value function of the **singular control** problem ($k = 0$):

$$v(y,x) \le v_U(y,x) := \sup_{\xi(\cdot)} \mathbb{E}_{y,x}\left[\int_0^\infty Y_t\delta(X_t)e^{-rt}\,\mathrm{d}t - c\int_0^\infty e^{-rt}\,\mathrm{d}\xi_t\right]$$

The dynamic programming equation for v_U is a degenerate QVI

$$\begin{aligned}\min\left\{\mathcal{L}v_U(y,x) - y\delta(x); c - \frac{\partial v_U}{\partial x}(y,x)\right\} &= 0,\\ v_U(0,x) &= 0,\\ \liminf_{y\to\infty}\{v_U(y,x) - v_L(y,x)\} &\ge 0\end{aligned}$$

The singular control's QVI is easier to solve than the impulse control's QVI

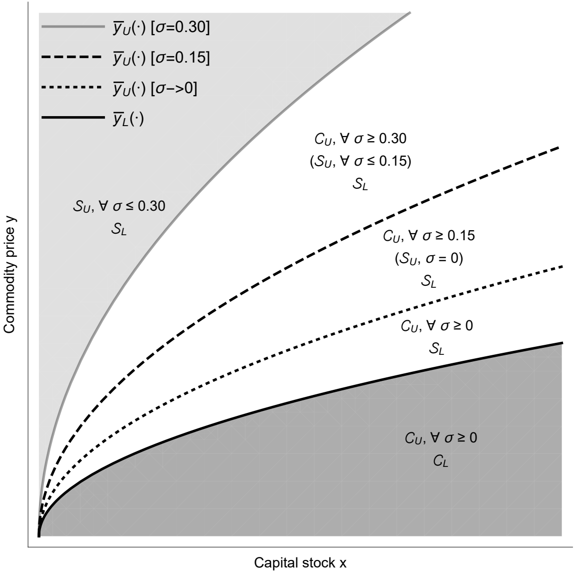
It admits a **regular solution** given by

$$v_U(y,x) = \begin{cases} v_L(y,x) + A_U(x) y^\beta, & y \le \bar{y}_U(x), \\ v_L(y, \bar{x}_U(y)) - c[\bar{x}_U(y) - x] + A_U(\bar{x}_U(y)) y^\beta, & y \ge \bar{y}_U(x), \end{cases}$$

$$S_U := \left\{ (y,x) \mid y \ge \mathrm{y_U(x)} := \frac{\beta}{\beta - 1} \frac{\mathrm{c}[\mathrm{r} - \mu]}{\delta'(\mathrm{x})} = \bar{x}_U^{-1}(x) \right\},$$

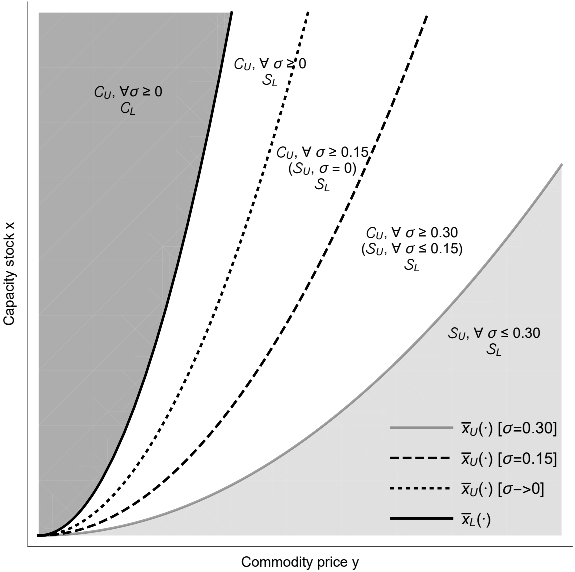
$$A_U(x) := \left(\frac{\beta c}{\beta - 1} \right)^{\beta - 1} \frac{1}{\beta} \left(\frac{1}{r - \mu} \right)^\beta \int_x^\infty \delta'(\xi)^\beta \, \mathrm{d}\xi$$

$$\beta := -\frac{\mu - \sigma^2/2}{\sigma^2} + \sqrt{\left(\frac{\mu - \sigma^2/2}{\sigma^2} \right)^2 + \frac{2r}{\sigma^2}} \in (1, r/\mu)$$



[1] Graphs generated assuming $\delta(x) = \sqrt{x}$, $r = 0.05$, $\mu = 0.014$, $c = 10$

Reversible investment vs. singular control: $\bar{x}_U(\cdot)$ vs. $\bar{x}_L(\cdot)$[1]



[1]Graphs generated assuming $\delta(x) = \sqrt{x}$, $r = 0.05$, $\mu = 0.014$, $c = 10$

General case

Statement[1]

The QVI has solutions in $[\underline{v}, \overline{v}] \subset [v_L, v_U]$. The sequence $\{v_n\}_n$ defined by $v_0 \equiv v_L$ and the recurrence relation

$$\min\left\{\mathcal{L}v_{n+1}(y,x) - y\delta(x); v_{n+1}(y,x) - \mathcal{M}v_n(y,x)\right\} = 0$$

$$v_{n+1}(0,x) = 0,$$

$$\liminf_{y\to\infty}[v_{n+1}(y,x) - v_L(y,x)] \geq 0,$$

converges to the **minimum solution** $\underline{v}$.[2]

---

[1] To establish this "statement" would require to prove regularity. We assume v_n are regular. Regularity is proved in the particular case

[2] $\overline{v}$ is the "maximum solution" (limit of a sequence starting at $v_U > v_L$)

[3] A QVI can be viewed as a fixed-point equation with an "implicit obstacle" $\mathcal{M}v$. The **min** term here is a VI (not a QVI) with an "explicit obstacle" $\mathcal{M}v_n$. Formally, this sequence is an approach to solve the fixed-point equation

Verification theorem

Assume

i) we can find a (s, S) policy:
   - "s": $\bar{x} = \bar{y}^{-1}$
   - "S": $z(y) = \bar{x}(y) + \arg\max_{\xi>0} \{v(y, \bar{x}(y) + \xi) - c\xi\} \quad [> \bar{x}(y)]$

ii) the impulse control $\hat{\nu}$ is admissible[1]

iii) the transversality condition $\mathbb{E}_{y,x}[v_U(Y_T, \hat{X}_T)e^{-rT}] \to 0$ is satisfied

iv) $\underline{v}$ is regular.

Then $\underline{v}$ coincides with the value function v.[2]

[1] We define $X_0 = x$, $X_{\hat{\tau}_1} = z(y \vee \bar{y}(x))$ and $X_{\hat{\tau}_n} = z(Y_{\hat{\tau}_n})$. $\hat{\nu}$ is defined by $\hat{\tau}_1 = \inf\{t \mid Y_t = y \vee \bar{y}(x)\}$, $\hat{\tau}_{n+1} = \inf\{t > \hat{\tau}_n \mid Y_t = \bar{y}(X_{\hat{\tau}_n})\}$, $\hat{\xi}_n = X_{\hat{\tau}_n} - X_{\hat{\tau}_n}$

[2] Proof sketch: The function $\underline{v}$ (i) is the smallest upper solution, (ii) majorizes v, and (iii) coincides with v (generalized Itô's lemma + assumption iii)

General case — Free boundary problem

FBP's regular solution

If we can find (s, S) boundaries $\bar{x}(\cdot)$ and $z(\cdot)$ satisfying certain conditions (see Appendix A), then the function v given by

$$v(y,x) = \begin{cases} v_L(y,x) + A(x)y^\beta, & y \le \bar{y}(x), \\ v_L(y,z(y)) - c[z(y)-x] - k + A(z(y))y^\beta, & y \ge \bar{y}(x), \end{cases}$$

$$A(x)y^\beta = \underbrace{\int_x^\infty \underbrace{\left[\frac{\partial v_L}{\partial x}(\bar{y}(\xi),\xi) - c\right]}_{\substack{\text{Marginal net perpetuity} \\ \text{gain at } \bar{y}(\xi)}} \underbrace{\left(\frac{y}{\bar{y}(\xi)}\right)^\beta}_{\substack{\text{Expected discount} \\ \text{factor for } \bar{y}(\xi) > y}} \mathrm{d}\xi}_{\substack{\text{Value of supramarginal capacity units} \\ \text{(“capacity expansion options”)}}},$$

is a regular solution of the FBP

The boundaries $x(\cdot)$ and $z(\cdot)$ are defined *implicitly* as solutions of[1]

$$v_L(y, z(y)) - v_L(y, \bar{x}(y)) = \frac{\beta c}{\beta - 1}[z(y) - \bar{x}(y)] + \frac{\beta k}{\beta - 1}$$

$$\frac{\partial v_L}{\partial x}(y, z(y)) - c = \left[\frac{\partial v_L}{\partial x}(\bar{y}(z(y)), z(y)) - c\right]\left(\frac{y}{\bar{y}(z(y))}\right)^{\beta}$$

[1]The first condition can be stated as

$$\underbrace{\int_{\bar{x}(y)}^{z(y)} \left[\frac{\partial v_L}{\partial x}(y, \xi) - \frac{\beta c}{\beta - 1}\right] \mathrm{d}\xi}_{\text{Additional excess perpetuity value of the capacity lump}} = \underbrace{\frac{\beta}{\beta - 1} k}_{\text{Fixed opportunity cost}}$$

If we introduce

$$F(y,x) := (\beta - 1)v_L(y,x) - \beta cx,$$
$$G(y,x) := -\left[\frac{\partial v_L}{\partial x}(y,x) - c\right]y^{-\beta},$$

then the boundaries $\bar{x}(\cdot)$ and $z(\cdot)$ can be expressed as solutions of

$$F(y,z(y)) = F(y,\bar{x}(y)) + \beta k,$$
$$G(y,z(y)) = G\big(\bar{y}(z(y)), z(y)\big).$$

Technical challenge

Solving this system is nontrivial because the unknown $z(\cdot)$ is an argument of the unknown $\bar{y}(\cdot)$

We introduce the functions[1]

- Y: $Y(y, x) \in [\bar{y}_L(x), \infty) \setminus \{y\}$ is the unique solution of

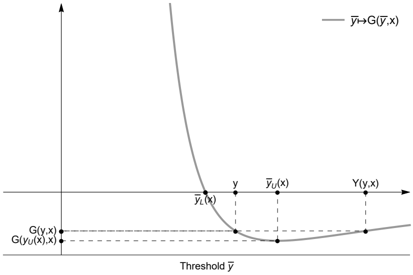
$$G(y, x) = G(Y(y, x), x)$$

- X: $X(y, x) \in [0, \bar{x}_U(y)]$ is the unique solution of

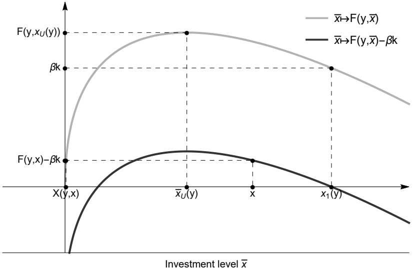
$$F(y, X(x, y)) = F(y, x) - \beta k$$

(s, S) BOUNDARIES IN THE GENERAL CASE

Function Y



Function X



We thus look for functions $\bar{x}(\cdot)$ and $z(\cdot)$ expressed explicitly as[2,3]

$$0 \leq \bar{x}(y) \leq X(y, \bar{x}_U(y)) \leq \bar{x}_U(y) \leq z(y) \leq \min\{\bar{x}_L(y), x_1(y)\},$$
$$z(y) = \bar{x}\big(Y(y, z(y))\big),$$
$$\bar{x}(y) = X(y, z(y))$$

[1]See details on the domains of definition in the working paper

[2]The singular control's boundary $\bar{x}_U$ corresponds to a degenerate case $[k = 0 \implies \bar{x} \equiv \bar{x}_U \equiv z]$

[3]$z(\cdot)$ solves the fixed-point equation $z(y) = X\big(Y(y, z(y)), z\big(Y(y, z(y))\big)\big)$

Particular case

Particular case: Assumptions

To fully solve the problem, we consider a restrictive case, namely

Particular case

$$\delta(x) = \sqrt{x} \quad \text{and} \quad 3\sigma^2 = r - 3\mu > 0.$$

Here, we are able to:

1. construct the (s, S) boundary *curves*
2. prove that the regular FBP's solution (expressed in closed form) solves the QVI
3. establish the verification problem and thus prove the optimality of the (s, S) policy

If we consider the

Particular case

$$\delta(x) = \sqrt{x} \quad \text{and} \quad 3\sigma^2 = r - 3\mu > 0,$$

then

- $Y(y,x)$ is a root of a polynomial of power 3 (y is another root):

$$Y(y,x) = \frac{y}{2}\left[-1 + \sqrt{\frac{y + 6c(r-\mu)\sqrt{x}}{y - 2c(r-\mu)\sqrt{x}}}\right]$$

- $X(y,x)$ also has an explicit expression, namely

$$X(y,x)^{1/2} = \frac{y - \sqrt{[3c(r-\mu)\sqrt{x} - y]^2 + 9kc(r-\mu)^2}}{3c(r-\mu)}$$

We introduce

$$\psi(y) := \sqrt{\frac{z(y)}{\bar{x}_U(y)}} - 1, \quad \varphi(y) := \frac{Y(y, z(y))}{y}, \quad \rho(y) := \frac{3(r-\mu)\sqrt{kc}}{y},$$

After a change of variables and re-arrangement, ψ obtains to be the **fixed point** of the map $\mathcal{T} : \theta(\cdot) \to \zeta(\cdot)$ where $\zeta(\cdot)$ is the solution of[1]

$$\sqrt{\frac{1+2\zeta(y)/3}{1-2\zeta(y)}} - 1 - \frac{2}{3}\zeta(y) = \frac{2}{3}\sqrt{\rho^2(y) + \chi^2(y)\theta^2(y\chi(y))}$$

$$\chi(y) = -\frac{1}{2} + \frac{3}{2}\sqrt{\frac{1+2\zeta(y)/3}{1-2\zeta(y)}}$$

[1] We here provide a sketch; more details, e.g., on bounds and monotonicity available in the working paper

To obtain ψ, we

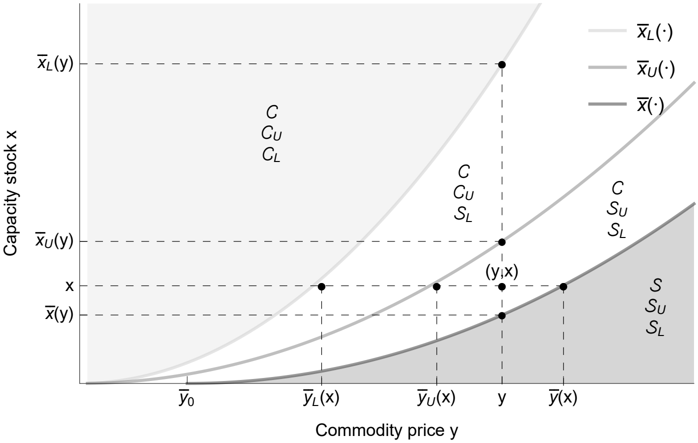
- construct a monotone sequence $\{\zeta^k(\cdot)\}_k$ that converges to $\zeta(\cdot)$
- show that the map $\mathcal{T}$ is monotone increasing
- prove that $\{\zeta_n(y)\}_n$ defined by $\zeta_1(y) = 0$ and $\zeta_{n+1}(y) = \mathcal{T}(\zeta_n)(y)$ increases and converges to the minimum solution $\psi(y)$

This allows us to obtain the boundaries $z(\cdot)$ and $\bar{x}(\cdot)$ explicitly from

$$\psi(y) = \sqrt{\frac{z(y)}{\bar{x}_U(y)}} - 1 \quad \text{and} \quad \sqrt{\frac{\bar{x}(y)}{\bar{x}_U(y)}} = 1 - \sqrt{\psi^2(y) + \rho^2(y)}$$

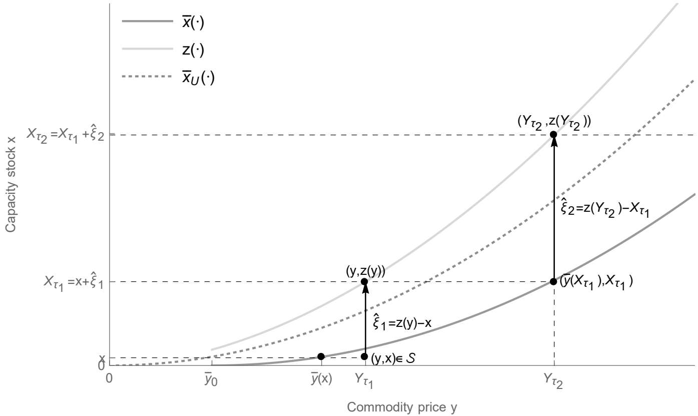
We also obtain that $\bar{x}(\cdot)$ and $z(\cdot)$ are monotone increasing and asymptotically equivalent to $\bar{x}_U(\cdot)$[1]

[1] Formally, the fixed cost k becomes negligible when $y \to \infty$



Hysteresis (white region) because of:

- Effect of irreversibility ($\mathcal{C}_U \cap \mathcal{S}_L \neq \emptyset$)
- *Effect of fixed cost* $k > 0$ ($\mathcal{C} \cap \mathcal{S}_U \neq \emptyset$)



"If there is a stock fixed cost …the optimal policy will allow the capital stock to jump in discrete steps at isolated moments"

(Dixit and Pindyck 1994)

Concluding remarks

Contributions

- we consider staged capacity investments under uncertainty (optimal timing & optimal capacity choices)
- we derive and solve a QVI with two states
- we compare with benchmarks in micro- and macroeconomics and investigate another source of hysteresis

Concluding remarks

Contributions

- we consider staged capacity investments under uncertainty (optimal timing & optimal capacity choices)
- we derive and solve a QVI with two states
- we compare with benchmarks in micro- and macroeconomics and investigate another source of hysteresis

Research outlook:

- Larger classes of profit functions ($y\sqrt{x}$ is somewhat restrictive)
- Impact of output $\delta(x)$ on market-clearing price y
- Broader class of Lévy processes
- Depreciation and resale of capacity
- Strategic interactions between investing firms

QUESTIONS?

Abel, A. B., and J. C. Eberly. 1994. “A Unified Model of Investment Under Uncertainty”. *American Economic Review* 84 (5): 1369–1384.

Bensoussan, A., and J.-L. Lions. 1982. *Applications of Variational Inequalities in Stochastic Control*. Vol. 13. Studies in Mathematics and Its Applications. New York, NY: North-Holland.

– . 1984. *Impulse Control and Quasi-Variational Inequalities*. Paris, France: Gauthiers-Villars.

Dixit, A. K., and Pindyck. 1994. *Investment under Uncertainty*. Princeton, NJ: Princeton University Press.

Ferrari, G. 2015. “On an integral equation for the free-boundary of stochastic, irreversible investment problems”. *Annals of Applied Probability* 25 (1): 150–176.

- Hubbard, R. G. 1994. “Investment Under Uncertainty: Keeping One’s Options Open”. *Journal of Economic Literature* 32 (4): 1816–1831.
- Kobila, T. 1993. “A class of solvable stochastic investment problems involving singular controls”. *Stochastics and Stochastic Reports* 43 (1-2): 29–63.
- McDonald, R., and D. Siegel. 1986. “The Value of Waiting to Invest” [inlangEnglish]. *Quarterly Journal of Economics* 101 (4): 707–728.
- Merhi, A., and M. Zervos. 2007. “A Model for Reversible Investment Capacity Expansion”. *SIAM Journal of Control and Optimization* 46 (3): 839–876.
- Øksendal, A. 2000. “Irreversible investment problems”. *Finance and Stochastics* 4 (2): 223–250.

- Pindyck. 1988. “Irreversible Investment, Capacity Choice, and the Value of the Firm”. *American Economic Review* 78 (5): 969–985.
- Riedel, F., and X. Su. 2011. “On irreversible investment”. *Finance and Stochastics* 15 (4): 607–633.
- Trigeorgis, L. 1996. *Real options: Managerial flexibility and strategy in resource allocation*. MIT press.