Summer School in Stochastic Finance

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Objectives

A key focus in the theory of investment under uncertainy is on the timing of decisions

We will consider today several cases whereby the payoff receives at the exercise of the real options is nontrivial, which leads sometimes to new insights and request the use of more advanced techniques

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In the first session, we will cover

1. Introduction to real options
2. Time and scale flexibility
3. Capacity expansion and performance-sensitive debt

We will then consider an extension of the model in “2.” to account for

4. Sequential capacity expansion options

and finally consider

5. Capacity and output choices in oligopoly

Intro to real options

What is value?

Consider

- a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$
- Brownian motion $Z = (Z_t; t \geq 0)$
- an "underlying source" of uncertainty: $X^x : \Omega \times [0, T] \mapsto \mathcal{X} \subset \mathbb{R}$ follows a diffusion

$$X_0 = x \text{ a.s.}, \tag{1}$$

$$\mathrm{d}X_t = \nu(X_t, t)\,\mathrm{d}t + \sigma(X_t, t)\,\mathrm{d}Z_t, \quad t \in \mathbb{R}_+ \tag{2}$$

- a payout/cashflow function f on $\mathcal{X} \times [0, T]$
- a "stochastic discount factor" (SDF) $\delta : \ \Omega \times \mathcal{T} \to (0, 1]$

The present value is the discounted expected sum of the cashflows given by

$$\text{PV} := \mathbb{E}^{\mathbb{P}}\left[\int_0^T \delta_t f(X_t^x, t)\,\mathrm{d}t\right]$$

What is value?

Critical steps in appraising this value are

1. characterize the cashflow stream $f \circ X^x$, e.g., linear versus nonlinear w.r.t. the state $x \in \mathcal{X}$
2. determine the appropriate SDF δ
3. ensure consistency between the probability measure $\mathbb{P}$, the process X^x and the SDF δ

The NPV paradigm and ROA gives different interpretations

Implicit assumptions in NPV:

1. the cashflow $x \mapsto f(\cdot, t)$ is linear
2. the discount fact $\delta_t = e^{-\int_0^t \mu(s)\mathrm{d}s}$ surmises risk-adjusted discount rate $\mu(\cdot)$ given by, e.g., CAPM. Discount rate often constant
3. physical probability measure $\mathbb{P}$.

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Drawbacks of NPV paradigm:

- Fails to capture managerial flexibility/nonlinearities
- Discount rate $\mu(\cdot)$ may depend on states ω or firms' decisions $\alpha : \Omega \times [0, T] \to \mathrm{A}$

Assumptions borrowed from nonlinear derivative pricing:

1. firms have managerial flexibility: firm strategy $\alpha : \Omega \times [0, T] \to A$ impacts payout $f(x, t, \alpha_t(\omega))$, which is nonlinear in x
2. use of riskfree rate $r > 0$ with (S)DF $\delta_t = e^{-rt}$
3. simplification on SDF allowed under "martingale equivalent probability" measure $\mathbb{Q}$

This leads to the present value expression

$$\text{PV} = \sup_{\alpha} \mathbb{E}^{\mathbb{Q}} \left[\int_0^T e^{-rt} f(X_t^x, t, \alpha_t) \, \mathrm{d}t \right]$$

Reference books are: Dixit and Pindyck (1994) and Trigeorgis (1996)

Real options analysis (ROA)

Firm strategy α needs be specified:

Nonexhaustive list of real options

Type	Optionality
Option to defer	Delay start of project
Option to expand	Raise capacity, possibly in stages
Option to contract	Reduce capacity, possible in stages
Options to exit	Leave the market

The option to defer is a canonical example we discuss next

Brief model description:

- Investment cost K is not recoupable, i.e., investment is irreversible
- infinite planning horizon ($T = \infty$)
- Management decides on the “exercise time” τ

Model features:

- Risk-neutral firm or risk-neutral pricing (if no arbitrage and complete it holds). Riskfree rate r
- Analogy with the perpetual American call option (McDonald and Siegel 1986)

Option to defer

- Project value X^x evolves as a GBM, i.e., it solves the SDE

$$X_0 = x \text{ a.s.}, \tag{3}$$

$$\mathrm{d}X = \mu X \,\mathrm{d}t + \sigma X \,\mathrm{d}W_t, \quad t \in \mathbb{R}_+ \tag{4}$$

- The **value function**,

$$F(x) = \sup_{\tau} \; \mathbb{E} e^{-r\tau} \underbrace{\left(X_\tau^x - K \right)}_{\text{“Obstacle”}},$$

relates to an optimal stopping problem

Dynamic programming (DP) consists in simplifying dynamic optimization problem into pointwise optimization problems

The DP equation for optimal stopping is a **variational inequality** (VI) [Bensoussan and Lions 1982]

[1] Obtained from Dynkin formula as $t \downarrow 0$, whereby $\mathcal{L}$ is a second-order differential operator $\mathcal{L} := r - \mu x \, \frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{2}\sigma^2 x^2 \frac{\mathrm{d}^2}{\mathrm{d}x}$.

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Here, the VI derives from the following heuristics:

- Given optionality, investing straight away is suboptimal:

$$F(x) \geq x - K$$

- Deferring investment for sure is also suboptimal:

$$F(x) \geq \mathbb{E}\left[e^{-rt}F(X_t^x)\right] \implies \mathcal{L}F(x) \geq 0 \text{ a.e.}^1$$

- Either one is optimal ("complementarity slackness")

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In summary

Variational inequality (see Bensoussan and Lions 1982)

If the value function F satisfies

$$F(\cdot) \in C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+) \text{ a.e. (“regularity”)}$$

then it solves[1]

$$\min\left\{F(x) - x + K, \mathcal{L}F(x)\right\} = 0$$

We set the boundary condition:

$$F(0) = 0$$

[1] General verification theorem to prove the reverse implication

- The continuation set $\mathcal{C} := \{x \in \mathbb{R}_+ \mid F(x) > x - K\}$ is conjectured to be of the form $(0, \bar{x}) \subset \mathbb{R}_+$
- If this holds, the regular VI solution $F(\cdot)$ then solves a FBP[1]

$$\begin{aligned} \mathcal{L}F(x) &= 0, && x \in (0, \bar{x}), \\ F(x) &= x - K, && x \in [\bar{x}, \infty), \\ F(0) &= 0 \end{aligned}$$

- A solution to the FBP does not necessarily solves the VI

[1] $F(\cdot)$ is conjectured to be C^1; hence, we determine the free boundary $\bar{x}$ by smooth fit

- ODE solution conjectured of the form $x \mapsto x^\gamma$, which is true if γ is a root of

$$\mathcal{Q}(\gamma) = r - \mu\gamma - \frac{1}{2}\sigma^2\gamma(\gamma - 1)$$

- If $r > \mu$, this quadratic function has two roots: $\gamma_B < 0 < 1 < \gamma_A$
- Because $F(0) = 0$, the ODE has solution of the firm $F(x) = Ax^{\gamma_A}$
- From smooth fit, the function

$$F(x) = \begin{cases} \frac{K}{\gamma_A - 1}\left(\frac{x}{\bar{x}}\right)^{\gamma_A}, & x < \bar{x} := \frac{\gamma_A}{\gamma_A - 1}K > K, \\ x - K, & x \geq \bar{x} \end{cases}$$

  solves the FBP

To prove optimality, it remains to verify the VI, i.e., prove[1]

- For any $\underline{x < \bar{X}}$,

$$f(x) := x - \frac{K}{\gamma_A - 1}\left(\frac{x}{\bar{x}}\right)^{\gamma_A} \leq K,$$

  which obtains because $f'(x) > 0$ [recall that $\gamma_A > 1$] and $f(\bar{x}) = K$.

- For any $\underline{x \geq \bar{x}}$, that

$$\mathcal{L}(x - K) \geq 0,$$

  which is immediate because $\frac{\gamma_A}{\gamma_A - \gamma_A} > \frac{r}{r - \mu}$

The optimal stopping time is $\hat{\tau}_x = \inf\{t \geq 0 \mid X_t^x \geq \bar{x}\}$

[1]See Bensoussan and Lions 1982; Øksendal 2000 for verification theorem

Optimal investment rule:

- The optimal stopping rule invalidates the (binary) NPV investment rule $\bar{x} \geq K$
- Two effects are accounted for ("hysteresis") in $\frac{\gamma_A}{\gamma_A - 1} > 1$
  - Irreversibility: $\frac{\mu}{r-\mu} > 1$
  - Further delay because of uncertainty: $\frac{\gamma_A}{\gamma_A - 1} > \frac{\mu}{r-\mu}$ for $\sigma > 0$

Further economic interpretations:

- NPV is a lower bound on value function: $F(x) \geq x - K$
- $(x/\bar{x})^{\gamma_A}$ is the "state price" of receiving \$ 1 at stochastic time $\hat{\tau}_x$

Time and scale flexibility

Extension of "option to defer"

Joint work with Alain Bensoussan [Bensoussan and Chevalier-Roignant 2013]

Next, the firm decides on the investment size *at* the time of investment, e.g.,

- a retailer decides on the extent of the premises
- a market entrant devises a marketing strategy
- a refurbished luxury hotel commits to a number of rooms

STATES & PROFIT

- **States**: commodity price $x > 0$ and the firm's capital stock $\delta \geq 0$[1]
- We assume **firm profit** $\pi(x, \delta)$ satisfying

$\pi_x > 0$	profit increases with price $[\pi(0, \delta) \equiv 0]$
$\pi_\delta > 0$	profit increasing with capacity $[\pi(x, 0) \equiv 0]$
$\pi_{\delta\delta} 0$	diminishing marginal returns
$\pi_{x\delta} > 0$	

- **Firm strategy** ν consisting of investment time τ and investment lump ξ[2] involves investment costs of $K + c\xi$, $K, c > 0$
- **State equations**:
  - Commodity price X^x follows a GBM
  - Capacity at time t is

$$\Delta_t^\nu = \delta \mathbf{1}_{t<\tau} + (\delta + \xi)\mathbf{1}_{t \geq \tau}$$

[1]Alternative interpretation as labor or capital more generally
[2]Assumed $\mathcal{F}_\tau$ measurable

Perpetuity value of capacity[1]

$$V(x,\delta) := \mathbb{E}\left[\int_0^\infty e^{-rt}\pi\left(X_t^x,\delta\right)\mathrm{d}t\right]$$

Explicit expression for V

$$V(x,\delta) = \frac{2}{\left(\gamma_A - \gamma_B\right)\sigma^2}\left[x^{\gamma_B}\int_0^x \frac{\pi(\eta,\delta)}{\eta^{\gamma_B+1}}\,\mathrm{d}\eta + x^{\gamma_A}\int_x^\infty \frac{\pi(\eta,\delta)}{\eta^{\gamma_A+1}}\,\mathrm{d}\eta\right]$$

Perpetuity value V inherits curvature of π[2]

[1] V above is the probabilistic interpretation of $\mathcal{L}V = \pi$

[2] Namely, perpetuity value increases with capacity ($V_\delta > 0$), returns to scale diminish ($V_{\delta\delta} < 0$), and shadow price of capacity increases with price ($V_{x\delta} > 0$)

Value function with capacity expansion

The **value function**,

$$F(x, \delta) = \sup_{\nu} \mathbb{E}\left[\int_0^{\infty} e^{-rt} \pi\left(X_t^x, \delta_t^{\nu}\right) \mathrm{d}t - (K + c\xi) e^{-r\tau} \right]$$

Value function with capacity expansion

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can be expressed as an **optimal stopping problem**

$$F(x,\delta) = \sup_{\tau} \mathbb{E}\left[\int_0^\tau e^{-rt}\pi(X_t^x,\delta)\mathrm{d}t + e^{-r\tau} \underbrace{\left\{\Phi\left(X_\tau^x,\delta\right) - K\right\}}_{\text{“Obstacle”}} \right]$$

where

$$\Phi(x,\delta) = \sup_{\xi \geq 0} \left\{ V(x,\delta+\xi) - c\xi \right\}$$

embeds the optimal choice of capacity

The solution of the capacity optimization problem is $\delta \vee z(x)$ where $z(\cdot)$ is given by

a if $V_\delta(x,0) \not\equiv \infty$, then there exists a unique $x_0 > 0$ s.t. $V_\delta(x_0, 0) = c$. and

$$z(x) = \begin{cases} 0, & x < x_0, \\ \{z > 0 \mid V_\delta(x,z) = c\}, & x > x_0 \end{cases}$$

b if $V_\delta(x,0) \equiv \infty$, then

$$V_\delta(x, z(x)) = c$$

Economic interpretation

When a firm invests, it does so up to the point where marginal value equals marginal cost, c

Optimal capacity choice

Define

$$\phi(x,\delta) := V\big(x, z(x)\big) - c\big[z(x) - \delta\big]$$

The obstacle,

$$\Phi(x,\delta) = \begin{cases} \phi(x,\delta), & \delta < z(x), \\ V(x,\delta), & \delta \geq z(x). \end{cases}$$

obtains to be regular

As $z(\cdot)$ is monotone, $z^{-1}(\cdot)$ is well defined on $[\delta^{\star}, \infty)$. Above the price threshold $z^{-1}(\delta)$, the marginal capacity unit δ's value exceeds (linear) cost c

Variational Inequality (VI) formulation

If the value function F is regular, then it satisfies the VI

$$\begin{aligned} \min\{F - \Phi + K, \mathcal{L}F - \pi\} &= 0, \\ \phi(0, x) &= 0. \end{aligned}$$

We conjecture a structure for the continuation set, with a **threshold** $\bar{x}(\delta)$ s.t.

$$\begin{aligned} \mathcal{L}F(x,\delta) &= \pi(x,\delta), & x &< \bar{x}(\delta) \\ F(x,\delta) &= \Phi(x,\delta) - K, & x &\geq \bar{x}(\delta) \end{aligned}$$

Threshold solution

From smooth fit, $\bar{x}(\delta) > z^{-1}(\delta)$ is conjectured the unique solution of $G(x,\delta) = K$ in x, where

$$G(x,\delta) := V(x,z(x)) - V(x,\delta) - \frac{x}{\gamma_A}\Big[V_\delta(x,z(x)) - V_\delta(x,\delta)\Big] - c\,[z(x) - \delta]$$

We make two assumptions:

- **Assumption 1**: Function $x \mapsto G_x(x,\delta)$ increases once positive
- **Assumption 2**: (a) given assumption 1, $\bar{x}(\delta)$ solves $G(x,\delta) = K$ uniquely on $(z^{-1}(\delta), \infty)$ and (b) the function

$$x \mapsto \frac{1}{x^{\gamma_A - 2}}\left[V_{xx}(x,z(x)) - V_{xx}(x,\delta) - \frac{V_{x\delta}(x,z(x))^2}{V_{\delta\delta}(x,z(x))}\right]$$

decreases on $\left[z^{-1}(\delta), \bar{x}(\delta)\right]$.

Value function

Under assumptions 1 and 2,

$$x \mapsto F(x,\delta) = \begin{cases} V(x,\delta) + \left[(\phi - V)(\bar{x}(\delta),\delta) - K\right]\left(\frac{x}{\bar{x}(\delta)}\right)^{\gamma_A}, & x < \bar{x}(\delta), \\ \phi(x,\delta) - K, & x \geq \bar{x}(\delta), \end{cases}$$

solves the VI and coincides with the value function

Main **economic insights**

- Investment at time $\hat{\tau}(x,\delta) = \inf\left\{t \geq 0 \mid X_t^x \geq \bar{x}(\delta) > z^{-1}(\delta)\right\}$
- Hence, further delay due to hysteresis, contradicting the ”NPV rule” for the marginal unit
- Hysteresis also arises if $K = 0$

a. **Linear revenues** of the form

$$\pi(x,\delta) = yk(\delta), \quad k'(\cdot) > 0,\ k''(\cdot) < 0$$

b. Revenues involving **Cobb-Douglas** production function[2]

$$\pi(x,\delta) = \psi\, x^{\gamma}\, \delta^{\alpha}$$

c. Revenue function involving **bounded production**:

$$\pi(x,\delta) = x(1 - e^{-\alpha\delta}), \quad \alpha > 0$$

[2] Denote labor cost by w, elasticity of capital by κ and elasticity of labor by $1-\epsilon$. Management maximizes $x\boldsymbol{\Delta}(l,\delta) - wl$ w.r.t. labor supply l. By setting $\alpha = \kappa/\epsilon$, $\gamma = 1/\epsilon$ and $\psi = (w/[1-\epsilon])^{(\epsilon-1)/\epsilon}\,\epsilon$, we obtain π [concave in δ if $\alpha \in (0,1)$, convex in x if $\epsilon > 0$]

[3] In this case, $k(\infty) = 1 < \infty$.

The perpetuity value of capacity is

$$V(y,x) = y\frac{\delta(x)}{r-\mu}.$$

Investment intensity is given by

$$z(x) = \begin{cases} 0, & x < x_0, \\ (k')^{-1}\left(\frac{c(r-\mu)}{x}\right), & x > x_0 \end{cases}$$

Free boundary of FBP $\hat{x}(\delta)$ solves

$$\underbrace{\bar{x}(\delta)\frac{k(z(\bar{x}))-k(\delta)}{r-\mu}}_{\text{Perpetuity value}} = \underbrace{\frac{\gamma_A}{\gamma_A-1}}_{>1}\underbrace{[K+c\,[z(\bar{x})-\delta]]}_{\text{Investment cost}}.$$

Benchmark cases:

- "**Canonical option to expand**" by fixed size ξ (Trigeorgis, 1996):

$$\bar{x}(\delta)\frac{k(\delta+\xi)-k(\delta)}{r-\mu} = \frac{\gamma_A}{\gamma_A-1}[K+c\xi]$$

- **Irreversible investment** à la Abel and Eberly (singular control)

$$\tilde{x}(\delta)\frac{k'(\delta)}{r-\mu} = \frac{\gamma_A}{\gamma_A-1}c$$

In contrast,

$$\bar{x}(\delta) > \tilde{x}(\delta) > z^{-1}(\delta), \quad K \geq 0.$$

(Specialized) optimality conditions

For these three cases, we specialize **assumption 1** as

a. $g(y) := (\gamma_A - 1)k(y) + k'(x)^2/k''(y)$ increases with either
   i. $k(\infty) < \infty$ and $k'(\delta)^2/k''(\delta) \to 0$ as $\delta \to \infty$ or
   (ii) $k(\infty) = \infty$ and $g(\infty) = \infty$;

b. $(1 - \alpha)\gamma_A \geq 1$;

c. $\emptyset$

and **assumption 2** as

a. $\delta \mapsto -k'(\delta)^2/k''(\delta)$ decreases.

b. $\gamma \geq 1$;

c. $\emptyset$

Economic interpretations

In case b ("Cobb-Douglas"), the firm cannot perform too well in production (large α)

- Contributions:
  - extend the real options approach to include a scale decision at the time of investment;
  - prove existence, unicity, and optimality of a trigger policy under restrictive conditions.
- Managerial insights:
  - capital-accumulation rule reminiscent of marginal Tobin's q theory
  - main insight from the option to defer carry over;
  - restrictions may impose "mediocrity" on the firm.
- Possible extensions:
  - allow both capital purchase and resale;
  - allow repeated capital adjustments at a fixed cost.

Capacity and output choices in oligopoly

Motivation

Joint work with Christoph Flath, Peter Kort and Lenos Trigeorgis

We study oligopoly firms' capacity investment decisions given novel assumptions of cost heterogeneity and output flexibility within capacity constraints under demand uncertainty

Firms have a narrow "window of opportunity" to enter an emerging technology market. At market inception, they choose capacity, facing a tradeoff between

- the risk of carrying expensive idle capacity in case of low demand
- the risk of unsatisfied demand in case of demand upsurge

What is the impact of

- capacity cost heterogeneity,
- output flexibility,
- capacity constraints,
- demand uncertainty

on capacity decisions?

Key novel insights

- Firm invests in more capacity when uncertainty is larger (vs depressing effect of uncertainty on investment)
- Lower-cost firm invests in more capacity while a less efficient rival reduces capacity (strategic substituability).
- Larger capacity also helps a large firm attain market power (and convexity). Uncertainty induces more upfront capacity investment for the more cost-efficient firm
- In oligopoly, greater uncertainty leads to more dispersion of equilibrium capacities and potentially a welfare loss

Related articles	Key model features	Differences
Pindyck 1988; Abel and Eberly 1996	Monopoly, repeated capital extensions (and reductions)	Output flexibility, more firms, uncertainty increases investment (rather than decreases)
Gabszewicz and Poddar 1997	Two stages, duopoly, output flexibility	More firms, more stages, cost heterogeneity, social optimum
Huisman and Kort 2015	Duopoly, capacity timing, preemption, entry deterrence vs. accommodation	Output flexibility, more firms, no capacity timing, cost heterogeneity

Firm $i = 1, \ldots, k$ decides on its capacity $\bar{q}_i$ and subsequent output strategy $q_i(\cdot)$, subject to a **capacity constraint**:

$$0 \leq q_i(t) \leq \bar{q}_i.$$

Q_t is the aggregate output at time t, while $\bar{Q}$ is the capacity vector.

We consider a **stochastic intercept** $(X_t, t \geq 0)$ that follows the geometric Brownian motion in the linear inverse demand function:

$$p(X_t, Q_t) = X_t - b\ Q_t, \quad b > 0.$$

Firm i maximizes its payoff

$$J_i\big(x_0, Q(\cdot), \bar{Q}\big) = \mathbb{E}\left[\int_0^\infty e^{-rt}\pi_i(X_t, Q_t)\,\mathrm{d}t\right] - C_i(\bar{q}_i).$$

where

$$\begin{aligned} \pi_i(X_t; Q_t) &= p(X_t, Q_t)q_i(t) - cq_i(t) && \{\text{profit}\} \\ C_i(\bar{q}_i) &= c_i(\chi)\,\bar{q}_i, && \{\text{capital cost}\} \end{aligned}$$

with $c_i(\chi) \leq c_j(\chi)$ if $i > j$ and parameter χ drives the differential.

We look for Markov perfect equilibrium such that
$\bar{q}_1 \leq \cdots \leq \bar{q}_m \leq \cdots \leq \bar{q}_k$.

Cournot output

Firm *i*'s **Cournot output** is adjusted for capacity constraints:

$$q_i^C(x,\bar{Q}) = \begin{cases} 0, & x \in (0,c) \quad \{\text{idle/flat}\} \\ \frac{x-\Sigma(x,\bar{Q})}{b[1+K(x,\bar{Q})]}, & x \in [c,\bar{x}_i) \quad \{\text{unconstrained/linear}\} \\ \bar{q}_i, & x \in [\bar{x}_i,\infty) \quad \{\text{constrained/flat}\}, \end{cases}$$

where

$$\bar{x}_i := c + b[\sum_{j=0,\ldots,i-1} \bar{q}_j + (k-i+2)\,\bar{q}_i], \qquad \{\text{Demand thresholds}\}$$

$$\Sigma(x,\bar{Q}) := c + b \sum_{m=0,\ldots,k} \bar{q}_m \mathbf{1}_{\{x \geq \bar{x}_m\}}, \qquad \{\text{cost of competing}\}$$

$$K(x,\bar{Q}) := \sum_{m=0}^{k} \mathbf{1}_{\{x \leq \bar{x}_m\}}, \qquad \{\text{\# of unconstrained firms}\}$$

It readily obtains firm i's **Cournot profit** as:

$$\pi_i^C(x,\bar{Q}) = \begin{cases} 0 & \text{if } x \in (0,c) \quad \{\text{idle/nil}\} \\ \frac{1}{b}\left[\frac{x-\Sigma(x,\bar{Q})}{1+K(x,\bar{Q})}\right]^2 & \text{if } X_t \in [c,\bar{x}_i) \quad \{\text{unconstrained/convex}\} \\ \bar{q}_i\left[\frac{x-\Sigma(x,\bar{Q})}{1+K(x,\bar{Q})}\right] & \text{if } x \in [\bar{x}_i,\infty) \quad \{\text{constrained/linear}\}. \end{cases}$$

At each threshold $\bar{x}_m$, a new capacity constraint gets binding or relaxed. This generates kinks in the profit functions.

Firm value V_i has a closed-form expression:[1]

$$V_i\left(x,\bar{Q}\right) = \underbrace{v_i\left(x,\bar{Q}\right)}_{\text{Perpetuity value}} + \underbrace{A_i\left(x,\bar{Q}\right)x^{\gamma_A}}_{\text{Output expansion benefits}} + \underbrace{B_i\left(x,\bar{Q}\right)x^{\gamma_B}}_{\text{Loss-reduction benefits}}, \quad (5)$$

where v_i, A_i and B_i reflect changes in the industry structure with some firms constrained, while other exert market power.

Function V_i is

- convex increasing in x because demand increases lead to price inflation and output expansion (if unconstrained)
- concave increasing in $\bar{q}_i$ because raising capacity helps relax the constraint but large demand states become less likely (ceteris paribus)

[1] $\gamma_{A/B}$ are the roots of the "fundamental quadratic."

The **perpetuity value** is given by

$$v_i\left(x,\bar{Q}\right):=\begin{cases}0, & x\in(0,c),\\ \frac{1}{b\left[1+K(x,\bar{Q})\right]^2}\left[\frac{x^2}{r-2\mu-\sigma^2}-\frac{2x\Sigma(x,\bar{Q})}{r-\mu}+\frac{\Sigma(x,\bar{Q})^2}{r}\right], & x\in[c,\bar{x}_i),\\ \frac{\bar{q}_i}{1+K(x,\bar{Q})}\left[\frac{x}{r-\mu}-\frac{\Sigma(x,\bar{Q})}{r}\right], & x\in[\bar{x}_i,\infty)\end{cases}$$

while the "**truncated** perpetuity values" are

$$\nu_i^{A/B}\left(x,\bar{Q}\right):=\begin{cases}0, & x\in\\ \frac{1}{b[1+K_0]^2}\left\{\frac{2-\gamma_{B/A}}{\gamma_A-\gamma_B}\frac{x^2}{r-2\mu-\sigma^2}-\frac{1-\gamma_{B/A}}{\gamma_A-\gamma_B}\frac{2x\Sigma(X,\bar{Q})}{r-\mu}-\frac{\gamma_{B/A}}{\gamma_A-\gamma_B}\frac{\Sigma(X,\bar{Q})^2}{r}\right\}, & x\in\\ \frac{\bar{q}_i}{1+K(X,\bar{Q})}\left\{\frac{1-\gamma_{B/A}}{\gamma_A-\gamma_B}\frac{x}{r-\mu}+\frac{\gamma_{B/A}}{\gamma_A-\gamma_B}\frac{\Sigma(X,\bar{Q})}{r}\right\}, & x\in\end{cases}$$

The upside adjustment is given by

$$A_i(x, \bar{Q})\, x^{\gamma_A} := \sum_{m=0}^{k} 1_{\{x \leq \bar{x}_m\}} \underbrace{\left(\nu_i^A(\bar{x}_m+, \bar{Q}) - \nu_i^A(\bar{x}_m-, \bar{Q})\right)}_{\text{Rival constraints become binding}} \left(\frac{x}{\bar{x}_m}\right)^{\gamma_A},$$

while the downside adjustment is

$$B_i(x, \bar{Q})\, x^{\gamma_B} := \sum_{m=0}^{k} 1_{\{x \geq \bar{x}_m\}} \underbrace{\left(\nu_i^B(\bar{x}_m+, \bar{Q}) - \nu_i^B(\bar{x}_m-, \bar{Q})\right)}_{\text{Constraints get relaxed}} \left(\frac{x}{\bar{x}_m}\right)^{\gamma_B}$$

Upfront capacity choice in monopoly (I)

We solve the monopoly firm's capacity choice problem explicitly:[1]

$$\bar{q}_1^C(x_0) = \hat{q}(x_0)\mathbf{1}_{\{x_0 \geq \bar{x}_\star\}}.$$

where

$$\bar{x}_\star = \begin{cases} \left(\frac{2C_1'(0)}{c(\gamma_A - 2)\kappa_A}\right)^{1/\gamma_A} c, & 0 \leq f(0) \\ \{x \mid \frac{x}{r-\mu} - \frac{c}{r} - (2-\gamma_B)\frac{c^{1-\gamma_B}}{2}\kappa_B x^{\gamma_B} = C_1'(0)\}, & 0 > f(0) \end{cases}$$

$$\hat{q}(x) = \begin{cases} \left\{\bar{q}_1 \mid \frac{\kappa_A}{2}(\gamma_A - 2)x^{\gamma_A}(c + 2b\bar{q}_1)^{1-\gamma_A} = C_1'(\bar{q}_1)\right\}, & f\left(\frac{x-c}{2b}\right. \\ \{\bar{q}_1 \mid \frac{x}{r-\mu} - \frac{c+2b\bar{q}_1}{r} + \frac{\kappa_B}{2}(\gamma_B - 2)x^{\gamma_B}(c + 2b\bar{q}_1)^{1-\gamma_B} = C_1'(\bar{q}_1)\}, & f\left(\frac{x-c}{2b}\right. \end{cases}$$

$$f(\bar{q}_1) := \frac{\kappa_A}{2}(\gamma_A - 2)(c + 2b\bar{q}_1) - C_1'(\bar{q}_1)$$

$$\kappa_{A/B} := \frac{1}{\gamma_A - \gamma_B}\left[\frac{2-\gamma_{B/A}}{r - 2\mu - \sigma^2} - 2\frac{1-\gamma_{B/A}}{r-\mu} - \frac{\gamma_{B/A}}{r}\right]$$

[1] Assuming $\varphi(r,\mu,\sigma) := \frac{2}{r-\mu}\frac{1}{(2-\gamma_B)\gamma_B\kappa_B} > 1$

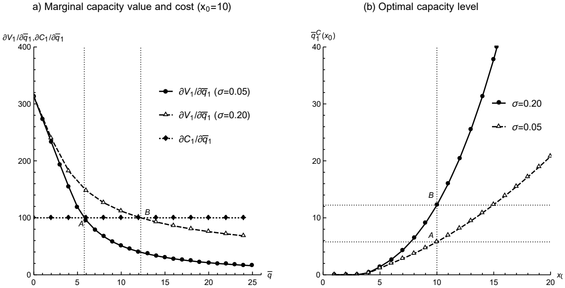

Figure 1: Marginal value ($\partial V_1/\partial \bar{q}_1$) and optimal capacity choice $\bar{q}_1^C(x_0)$ at varying volatility levels ($\sigma = 0.05$ or 0.20) in monopoly ($b = 1, c = 1, \mu = 0.02, r = 0.05, C_1(\bar{q}_1) = 100 \times \bar{q}_1$)

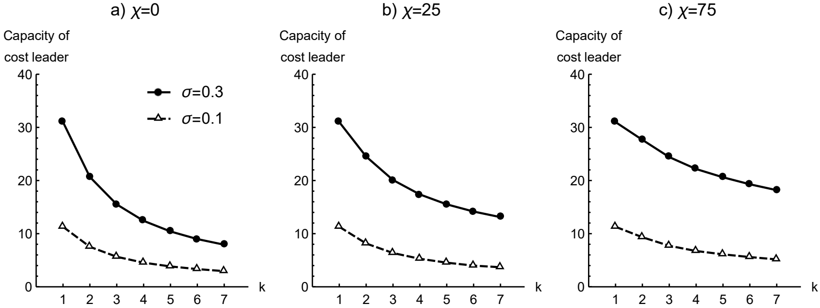

Figure 2: Effect of cost heterogeneity χ and demand volatility σ on firm k's capacity choice for different industry structures.

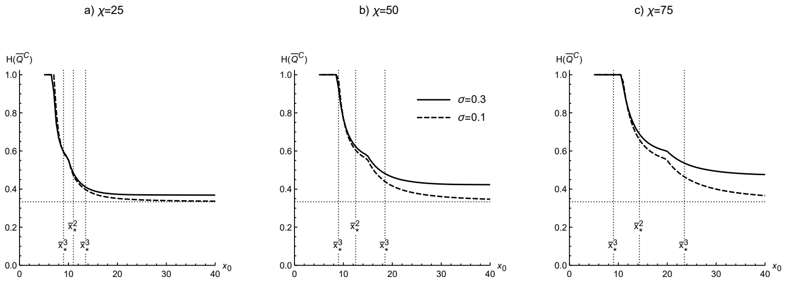

Figure 3: HHI in triopoly for different cost heterogeneity χ and demand volatility levels σ

Demand uncertainty exacerbates capacity heterogeneity because (i) low-cost firms invest more to benefit from larger value convexity and (ii) capacities are strategic substitutes.

The **social surplus**,

$$\mathcal{W} := \mathcal{PS} + \mathcal{CS},$$

consists of the producer and consumer surpluses:

$$\mathcal{PS}\left(x, Q(\cdot)\right) = \mathbb{E}\left[\int_0^\infty e^{-rt}\left[p\left(X_t, Q(t)\right) - c\right] Q(t)\,\mathrm{d}t\right],$$

$$\mathcal{CS}(x, Q(\cdot)) = \mathbb{E}\left[\int_0^\infty e^{-rt}\left\{\int_{X_t - bQ_t}^{X_t} D(X_t, Q_t)\mathrm{d}p\right\}\,\mathrm{d}t\right].$$

For a given capacity vector $\bar{Q}$, we derive in closed form

- the social surplus in constrained Cournot oligopoly;
- the social optimum.

We can compare the capacity choices and social surplus in constrained Cournot oligopoly with the socially optimum.

Capacity choices in Cournot vs. social optimum

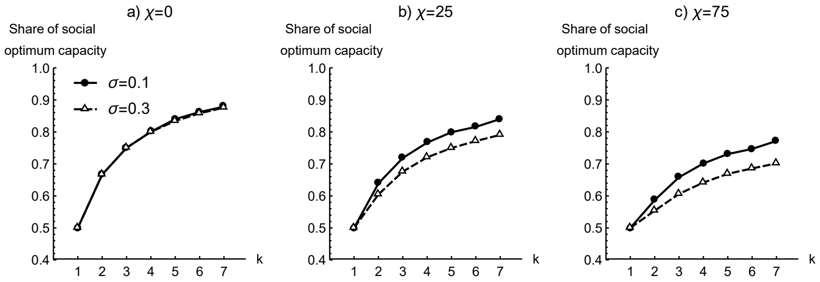

Figure 4: Effect of heterogeneity χ, volatility σ and industry structures k on total industry capacity ($x_0 = x = 10$, $b = 1$, $c = 1$, $\mu = 0.02$, $r = 0.05$)

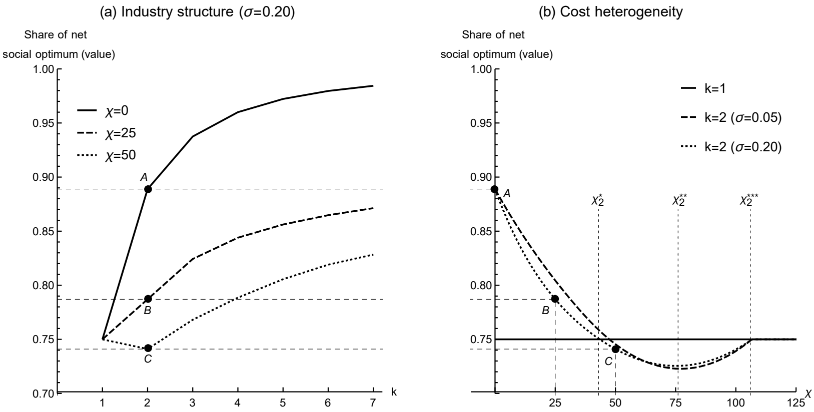

Figure 5: Effects of industry structure k and cost heterogeneity χ on net welfare ($x_0 = x = 10, b = 1, c = 1, \mu = 0.02, r = 0.05$)

Conclusions

- Optimal output policy is demand-contingent. Cournot profits is convex when a firm wields market power and linear if the capacity constraint binds
- The value of a larger firm is more convex because it can wield market power, while smaller rivals are constrained. Demand volatility increases convexity even more
- A cost-advantaged firm invests more to benefit from convexity, especially when demand volatility is large. By contrast, less efficient rivals invest less. Hence, the industry gets more concentrated
- Demand volatility leads to gross welfare loss under cost asymmetry. Encouraging more competition reduces net welfare when firms face highly heterogeneous costs in uncertain environments

Opportunities for future research

- Sequential (Stackelberg) game: greater dispersion of production capacities
- Broader notion of cost heterogeneity (e.g., marginal production costs, economies of scale, learning experience or network effects)
- Staged capacity investments/depreciation
- Costs incurred to adjust output (e.g., hiring and firing costs)

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