Robust Portfolio Decisions for Financial Institutions

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15th Summer School in Stochastic Finance

July 9, 2018

It's all about robustness

Robust grass endures mighty winds;
loyal ministers emerge through ordeal.

Li Shimin, 568-649 A.D.
Tang Dynasty of China

The model: Financial Market

Suppose that we have a financial market on the fixed time horizon $[0, T]$ with $T > 0$ and two investment possibilities :

- A risk free asset (bond or bank account) with unit price $S_0(t)$ at time t and dynamics described by the ordinary differential equation

$$\boxed{\mathrm{d}S_0(t) = rS_0(t)dt, \quad S_0(0) = 1} \tag{1}$$

- A risky asset (stock or index) with unit price $S_1(t)$ at time t which evolves according to the stochastic differential equation

$$\boxed{\mathrm{d}S_1(t) = \mu S_1(t)dt + \sigma S_1(t)dW(t), \quad S_1(0) > 0} \tag{2}$$

Here, $r > 0$, μ (with $\mu > r$) and $\sigma > 0$, are given constants.

The model: Cash Flow

- In collective risk theory, a sound mathematical model for describing the surplus of a large portfolio of claims is the Cramer-Lundberg model:

$$Y(t) = Y(0) + P(t) - L(t) \tag{3}$$

- In some situation, it is easier to work with its diffusion approximation.
- Here, the cumulative claims process is modeled by

$$\boxed{\mathrm{dL(t)} = \alpha dt - \beta dB(t)} \tag{4}$$

where α and β are positive constants.

- The drift term can be interpreted as the mean claims up to time t.
- The stochastic term can be interpreted as the fluctuations around the mean claims.

The setting

- We consider a financial firm, who, at time $t = 0$, starts with some initial wealth $x_0 > 0$.
- The risk manager of the firm decides the proportion $\pi(t)$ of its wealth $X(t)$ to be invested in the risky asset (2).
- The remaining proportion $(1 - \pi(t))X(t)$ is invested in the risk-less asset (1).
- The firm is designed to offer some very specific services to its clients (e.g., financial investments consultancy, pension fund management, insurance, etc) by entering a contract.
- In exchange for its services, the firm collects compensation (continuously) at the constant rate $c_0\alpha$, where $c_0 \geqslant 1$.

The setting

- However, such a contract also generates a stochastic cash flow of liabilities (e.g., long term payments, operating costs, etc) that evolves according to (4).
- As a means of reducing this additional exposure, the risk manager of the firm has the ability to transfer a proportion of its liabilities to another party (e.g. external investor, financial fund, reinsurance firm, e.t.c).
- The risk manager decides the proportion $q(t)$ of its claims process to be covered, by entering a contract with the third party.
- In exchange for this coverage, the third party collects an income continuously at the constant rate $c_1 \alpha q(t)$, where $c_1 \geqslant c_0$.

Stochastic Differential Equations of firm's wealth

- To sum up, the wealth process corresponding to the strategy $\eta_1 = (\pi(t), q(t))$, is denoted as $X^{\eta_1}(t)$ and is defined as the solution of the following linear stochastic differential equation

$$dX^{\eta_1}(t) = \pi(t)X^{\eta_1}(t)\frac{dS_1(t)}{S_1(t)} + (1-\pi(t))\,X^{\eta_1}(t)\frac{dS_0(t)}{S_0 t)} + dR(t),$$

where

$$\begin{aligned} dR(t) &= (c_0 - c_1)dt - dL(t) + q(t)dL(t) \\ &= \alpha(\theta - \eta)q(t)dt + \beta(1 - q(t))dB(t). \end{aligned}$$

- Therefore, in view of (1-4)

$$\begin{aligned} dX^{\eta_1}(t) = &\Big[X^{\eta_1}(t)(r + (\mu - r)\pi(t)) + \alpha(\theta - \eta q(t))\Big]dt \\ &+ \beta(1 - q(t))dB(t) + \sigma\pi(t)X^{\eta_1}(t)dW(t), \end{aligned}$$

with initial condition $X^{\eta_1}(0) = x_0 > 0$.

The original problem

- The risk manager aims to choose the control process so as to maximize some certain goal, e.g., the expected utility from her terminal wealth:

$$\sup_{\pi, q \in \mathcal{A}^{\mathbb{F}}} \mathbb{E}\Big[U(X^{\eta_1}(T))\Big],$$

subject to the state process

$$dX^{\eta_1}(t) = \Big[X^{\eta_1}(t)(r + (\mu - r)\pi(t)) + \alpha(\theta - \eta q(t))\Big]dt + \beta(1 - q(t))dB(t) + \sigma\pi(t)X^{\eta_1}(t)dW(t).$$

- A standard way to proceed is by employing the techniques of stochastic optimal control.

Model uncertainty aspects

- Stochastic optimal control theory is an indispensable part of mathematical economics and modern financial management.
- Great importance and wide range of applicability !

Assumption

The decision maker has complete faith in her model !

L.P.Hansen (Nobel Prize in Economics, 2013) and T.Sargent (Nobel Prize in Economics, 2011):

- Questioning the validity of your model is the first step towards realistic modeling: Model uncertainty aspects.

Solution: Robust control theory!

Stochastic Control vs Robust Control

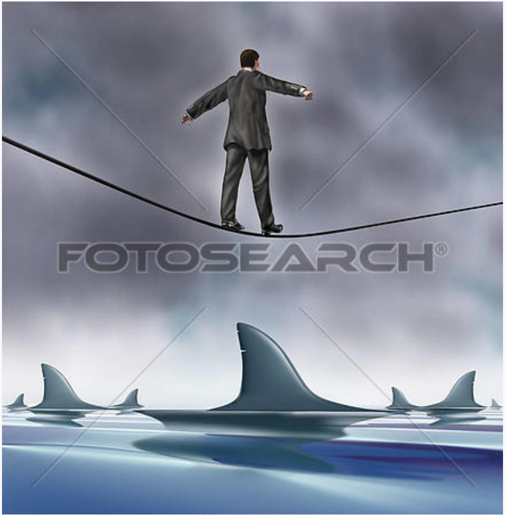


Figure: Stochastic Optimal Control Theory

Stochastic Control vs Robust Control

Figure: Robust Optimal Control Theory

Introduction to Robust Control Theory

Robust control theory is a mixture of two things:

- Stochastic control theory.
- Model selection techniques.

Main Philosophy:

Solve an optimal control problem under the worst possible scenario.
$\Longrightarrow$ Using the model that may provide the worst case for the problem at hand.

In Mathematical terms:

Model$\sim$ Probability Measure

Model uncertainty aspects

- We assume that the risk manager is uncertain as to the true nature of the stochastic processes W and B in the sense that the exact law of W and B is not known.
- There exists a "true" probability measure related to the true law of the processes W and B, the risk manager is unaware of, and a probability measure Q, which is her idea of what the exact law of W and B looks like.
- The manager in uncertain about the validity of Q:

$$\inf_{Q \in \mathcal{Q}} \mathbb{E}_Q \Big[U(X^{\eta_1}(T)) \Big],$$

- As a result, the manager faces the robust control problem

$$\sup_{\pi, q \in \mathcal{A}^{\mathbb{F}}} \inf_{Q \in \mathcal{Q}} \mathbb{E}_Q \Big[U(X^{\eta_1}(T)) \Big],$$

The class of measures $\mathcal{Q}$

Definition (The set $\mathcal{Q}$)

The set of acceptable probability measures $\mathcal{Q}$ for the agent is a set enjoying the following two properties:

(i) Considering the stochastic process W under the reference probability measure $\mathbb{P}$ and under the probability measure $\mathbb{Q}$ results to a change of drift to the Brownian motion W.

(ii) There is a maximum allowed deviation of the managers measure $\mathbb{Q}$ from the reference measure $\mathbb{P}$. In other words, the manager is not allowed to freely choose between various probability models as every departure will be penalized by an appropriately defined penalty function, a special case of which is the Kullback-Leibler relative entropy $\mathcal{H}(\mathbb{P}|\mathbb{Q})$.

Change of measure - Girsanov

Theorem

Assume that $y_1, y_2 \in \mathcal{Y} \subset \mathbb{R}^2$ *satisfy the condition*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T y_1^2(s) + y_2^2 ds\right)\right] < \infty.$$

Then, the stochastic processes $\widetilde{W}$ *and* $\widetilde{B}$ *with decomposition given by*

$$\widetilde{W}(t) = W(t) - \int_0^t y_1(s)ds,$$

and

$$\widetilde{B}(t) = B(t) - \int_0^t y_2(s)ds,$$

are $(\mathbb{F}, Q)$ *Brownian motions.*

The robust control problem

$$\sup_{\pi, q \in \mathcal{A}^{\mathbb{F}}} \inf_{Q \in \mathcal{Q}} J(t,x)$$
$$= \sup_{\pi, q \in \mathcal{A}^{\mathbb{F}}} \inf_{y_1, y_2 \in \mathcal{Y}} \mathbb{E}_Q \left[U(\widetilde{X}^{\eta_1,\eta_2}(T)) + \frac{1}{2\lambda} \int_t^T y_1^2(s) + y_2^2(s) ds \right], \tag{5}$$

subject to the state dynamics

$$\begin{aligned} d\widetilde{X}^{\eta_1,\eta_2}(s) = \Big[& r\widetilde{X}^{\eta_1,\eta_2}(s) + (\mu - r)\pi(s)\widetilde{X}^{\eta_1,\eta_2}(s) + \alpha(\theta - \eta q(s)) \\ & + \sigma\pi(s) y_1(s) \widetilde{X}^{\eta_1,\eta_2}(s) + \beta(1 - q(s)) y_2(s) \Big] ds \\ & + \sigma\pi(s)\widetilde{X}^{\eta_1,\eta_2}(s) d\widetilde{W}(s) + \beta(1 - q(s)) d\widetilde{B}(s), \end{aligned} \tag{6}$$

with initial condition $\widetilde{X}^{\eta_1,\eta_2}(s) = x_0 > 0$.

Solution Procedure

1. Derive the Hamilton-Jacobi-Bellman-Isaacs equation(HJBI) for the problem at hand. This is a partial differential equation (PDE) for an unknown function, e.g. V.
2. Fix an arbitrary point in time-space and solve the resulting static optimization problems (minimization $\rightarrow$ maximization).
3. From s2 we get a candidate for the optimal control laws.
4. This yields to a second order (for the problem at hand) PDE.
5. Solve the PDE of s4.
6. Verification theorem: The solution of the HJBI equation (V) is the value function of the problem at hand and the control choices we found earlier are indeed the optimal ones.

On the solvability of the HJBI

Is it possible to find a (smooth) solution to the HJBI ?

NOT IN GENERAL !!

There are three ways to proceed:

1. Guess a solution and pray !
2. Numerical Approximation.
3. Weak solutions (viscosity, mild, etc)

Robust Control Problem = Stochastic Differential Game

Figure: World Chess Championship 2016

Stochastic Differential Games

- The evolution of the underlying system is described by a Stochastic differential equation.
- The system is controlled by two (or more) players with conflicting goals.
- The controllers decide their control process so as to drive the system to a desired state.

A robust control problem is written as a SDG:

- **Player I.** Decision maker: Chooses the control process.
- **Player II.** Imaginary player (Nature): Chooses the model (the measure)

Theorem (Main Result)

Suppose that the risk manager has preference for robustness as described by the non-negative constant λ. The optimal robust strategy is to invest in the risky asset proportion of the firm's wealth equal to

$$\pi^*(t,x) = -\frac{\mu - r}{\sigma^2 x}\frac{V_x}{V_{xx} - \lambda V_x^2},$$

and purchase proportional coverage for the firm's claims, equal to

$$q^*(t,x) = 1 + \frac{\alpha\eta}{\beta^2}\frac{V_x}{V_{xx} - \lambda V_x^2}.$$

On the other hand, Nature chooses the worst-case scenario defined by

$$y_1^*(t,x) = \frac{\mu - r}{\sigma}\frac{\lambda V_x^2}{V_{xx} - \lambda V_x^2} \quad \text{and} \quad y_2^*(t,x) = \frac{\alpha\eta}{\beta}\frac{\lambda V_x^2}{V_{xx} - \lambda V_x^2}.$$

In this case, the optimal robust value function is a smooth solution of the following non-linear partial differential equation

$$V_t + [rx + \alpha(\theta - \eta)]V_x - \frac{1}{2}\left[\left(\frac{\mu - r}{\sigma}\right)^2 + \left(\frac{\alpha\eta}{\beta}\right)^2\right]\frac{V_x^2}{V_{xx} - \lambda V_x^2} = 0,$$

with boundary condition $V(T,x) = U(x)$.

Theorem (Exponential Utility)

Assume Exponential preferences $\left(u(x) = -\frac{\delta}{\gamma} e^{-\gamma x}\right)$. *The optimal robust value function admits the form:*

$$V(t,x) = -\frac{\delta}{\gamma} \exp\left[-\gamma x e^{r(T-t)} + g(t)\right], \tag{7}$$

where

$$g(t) = \alpha\gamma(\theta-\eta)\frac{1-e^{r(T-t)}}{r} - \frac{\gamma}{2(\lambda+\gamma)}\left[\left(\frac{\mu-r}{\sigma}\right)^2 + \left(\frac{\alpha\eta}{\beta}\right)^2\right](T-t). \tag{8}$$

In this case, the optimal robust strategy for the risk manager is to invest in the risky asset the constant amount

$$\pi^*(t,x) = \frac{\mu-r}{\sigma^2 x}\frac{e^{-r(T-t)}}{\lambda+\gamma}, \tag{9}$$

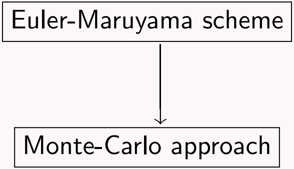
and purchase proportional coverage for the firm's claims, equal to

$$q^*(t,x) = 1 - \frac{\alpha\eta}{\beta^2}\frac{e^{-r(T-t)}}{\lambda+\gamma}. \tag{10}$$

On the other hand, Nature chooses the worst-case scenario defined by

$$y_1^*(t,x) = -\frac{\mu-r}{\sigma}\frac{\lambda}{\lambda+\gamma} \quad \text{and} \quad y_2^*(t,x) = -\frac{\alpha\eta}{\beta}\frac{\lambda}{\lambda+\gamma}. \tag{11}$$

Numerical study of the optimal investment strategy



E-M: For a time step of size $\Delta t = T/N$ with $N = 2^{11}$ points, we define the step size in the Euler-Maruyama scheme as $\delta t = \Delta t$.

M-C: Simulate a large number M of of paths of π^* and q^* in the time interval $[0, T]$ and at each time point we plot the average of M different values. We also use for each path $N = 2^{\alpha}$ number of points (here $N = 2^{11}$ and $M = 6000$ paths).

We let $M = 6000$, $T = 10$ months, $X(0) = 1.5$, $\gamma = 0.5$ and $\lambda = 0.2$. The parameters of the financial market are chosen as $\mu = 12\%$, $r = 6\%$, $\sigma = 40\%$. The parameters for the insurance market are chosen as $\alpha = 1$, $\beta = 0.2$ and $c_1 = 1.1$.

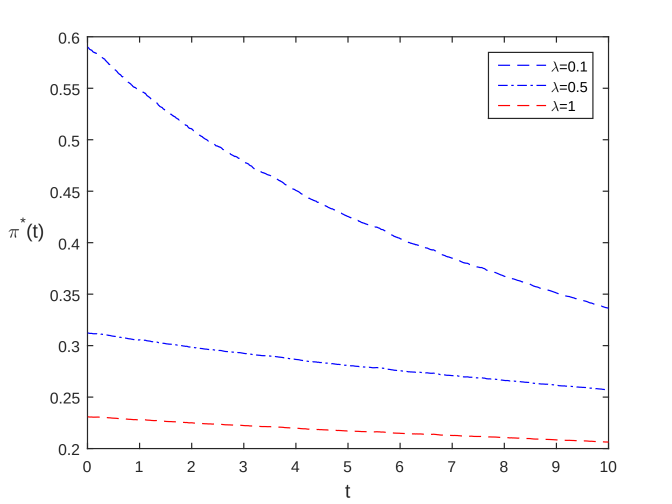


Figure: Average of 6000 optimal investment strategy paths for various levels of the preference for robustness parameter, in the case of the exponential utility function.

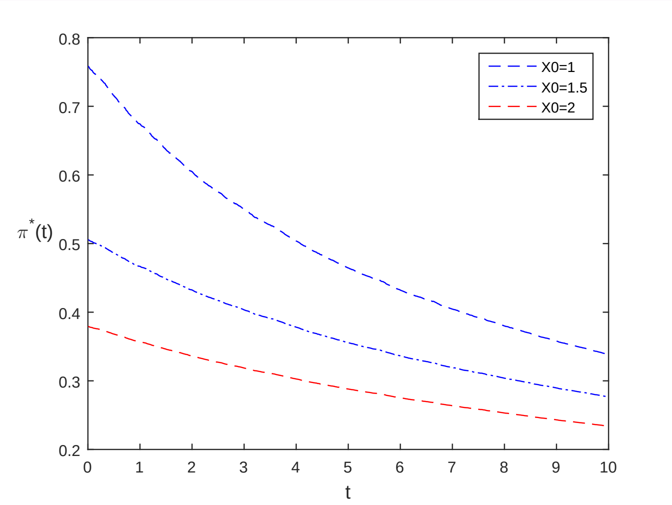


Figure: Average of 6000 optimal investment strategy paths for various levels of the initial wealth, in the case of the exponential utility function.

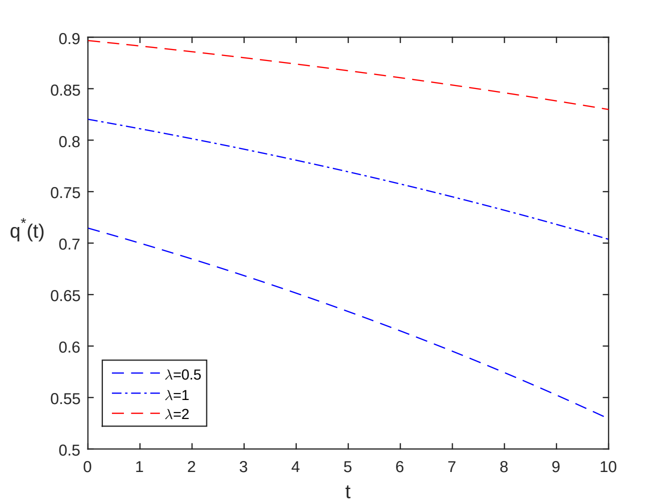


Figure: Average of 6000 optimal proportional coverage strategy paths for various levels of the preference for robustness parameter, in the case of the exponential utility function.

Limiting behavior: Cases $\lambda \to 0$ and $\lambda \to \infty$

- It is well known (see e.g. Anderson, Hansen and Sargent) that as $\lambda \to 0$ the decision maker fully trusts her model and exhibits no preference for robustness.
- As $\lambda \to +\infty$, the decision maker has no faith in the model she is offered and is willing to consider alternative models with larger relative entropy.
- The vast majority of the available works examines the limiting behavior of the optimal robust strategies, after the problem has been solved.
- Here, we are concerned with the structural behavior of the robust control problem itself in these limiting cases (well-posedness?)

Theorem (Limiting behavior as $\lambda \to 0$)

The optimal robust strategy for the risk manager is to invest in the risky asset proportion of the firm's wealth equal to

$$\pi^*(t,x) = -\frac{\mu - r}{\sigma^2 x}\frac{V_x}{V_{xx}}, \tag{12}$$

and also, to purchase proportional coverage for the firm's liabilities, equal to

$$q^*(t,x) = 1 - \frac{\alpha(1-c_1)}{\beta^2}\frac{V_x}{V_{xx}}. \tag{13}$$

On the other hand, Nature chooses the myopic worst-case scenario defined by

$$y_1^*(t,x) = y_1^*(t,x) = 0. \tag{14}$$

In this case, the optimal robust value function is a smooth solution of the following non-linear partial differential equation

$$V_t + [rx + \alpha(\theta - \eta)]V_x - \frac{1}{2}\left[\frac{(\mu - r)^2}{\sigma^2} + \frac{\alpha^2(1-c_1)^2}{\beta^2}\right]\frac{V_x^2}{V_{xx}} = 0, \tag{15}$$

with boundary condition $V(T,x) = U(x)$, assuming that such a solution exists.

Limiting behavior I: Case $\lambda \to 0$

We have some interesting findings

- The risk manager has complete faith in the model described by Equations (2) and (4).
- Operates under the probability measure $\mathbb{P}$.
- The controls (12), (13) and the PDE (15), are the optimal Markovian control laws and PDE associated with the stochastic optimal control problem:

$$\sup_{\pi, q \in \mathcal{A}^{\mathbb{F}}} \mathbb{E}_{\mathbb{P}}\Big[U(X^{\eta_1}(T))\Big],$$

  subject to the original state dynamics.
- Robust Control Problem $\to$ Optimal Control Problem.

Theorem (Limiting behavior as $\lambda \to +\infty$)

Assume that $\mathcal{Y}$ is the rectangle $\left[\underline{y_1}, \overline{y_1}\right] \times \left[\underline{y_2}, \overline{y_2}\right]$. The optimal robust strategy for the risk manager is to invest in the risky asset proportion of the firm's wealth equal to

$$\pi^*(t,x) = -\left(\frac{\mu - r}{\sigma} + \underline{y_1}\right)\frac{V_x}{\sigma x V_{xx}}, \tag{16}$$

and to purchase proportional coverage for the firm's liabilities, equal to

$$q^*(t,x) = 1 + \left(\frac{\alpha(c_1 - 1)}{\beta} + \underline{y_2}\right)\frac{V_x}{\beta V_{xx}}. \tag{17}$$

On the other hand, Nature chooses the myopic worst-case scenario defined by

$$y_1^*(t,x) = \underline{y_1}, \quad \text{and} \quad y_2^*(t,x) = \underline{y_2}. \tag{18}$$

In this case, the optimal robust value function is a smooth solution of the following non-linear partial differential equation

$$V_t + [rx + \alpha(c_0 - c_1)]V_x - \frac{1}{2}\left[\left(\frac{\mu - r}{\sigma} + \underline{y_1}\right)^2 + \left(\frac{\alpha(c_1 - 1)}{\beta} + \underline{y_2}\right)^2\right]\frac{V_x^2}{V_{xx}} = 0, \tag{19}$$

with boundary condition $V(T,x) = U(x)$, assuming that such a solution exists.

Solution break-down

- We construct a case where loss of convexity leads to break-down of the solution of the HJBI equation.
- For simplicity we assume that $c_0 = c_1$.
- The HJBI equation is restated as

$$V_t + rxV_x - A\left(\underline{y_1}, \underline{y_2}\right)\frac{V_x^2}{V_{xx}} = 0, \tag{20}$$

where

$$A\left(\underline{y_1}, \underline{y_2}\right) := \frac{1}{2}\left[\left(\frac{\mu - r}{\sigma} + \underline{y_1}\right)^2 + \left(\frac{\alpha(c_1 - 1)}{\beta} + \underline{y_2}\right)^2\right] \geqslant 0.$$

- We assume that the risk manager operates under quadratic preferences, that is a utility function of the form

$$U(x) = \kappa\frac{x^\rho}{\rho}, \tag{21}$$

for some $\kappa > 0$ and $0 < \rho < 1$.

Solution break-down

1. Assume that the PDE (20) admits a classical solution $V \in \mathcal{C}^{1,2}(\mathbb{S})$.
2. We look for a solution using the guess

$$V(t,x) = e^{-\delta t}\widetilde{V}(x),$$

   where $\widetilde{V} \in \mathcal{C}^{1,2}(\mathbb{S})$. Differentiating the above expression with respect to (t,x), yields

$$V_t = -\delta e^{-\delta t}\widetilde{V}(x)$$
$$V_x = e^{-\delta t}\widetilde{V}_x$$
$$V_{xx} = e^{-\delta t}\widetilde{V}_{xx}.$$

Solution break-down

4. Substituting these expressions back in the partial differential equation (20), results to the elliptic partial differential equation

$$\delta \widetilde{V} - rx\widetilde{V}_x + A\Big(\underline{y_1}, \underline{y_2}\Big)\frac{\widetilde{V}_x^2}{\widetilde{V}_{xx}} = 0. \tag{22}$$

5. We propose a solution to the partial differential equation of the form

$$\widetilde{V}(x) = \kappa \frac{x^\rho}{\rho}.$$

Inserting this trial solution in (22), yields to the following condition for the discounting factor

$$\delta = r\rho - A\Big(\underline{y_1}, \underline{y_2}\Big)\frac{\rho}{\rho - 1},$$

or equivalently

$$A\Big(\underline{y_1}, \underline{y_2}\Big) = \frac{1-\rho}{\rho}(\delta - r\rho).$$

Solution break-down

We distinguish the following four cases:

(i). If $A\left(\underline{y_1}, \underline{y_2}\right) = 0$ and $\delta = r\rho$, a solution exists.

(ii). If $A\left(\underline{y_1}, \underline{y_2}\right) > 0$ and $\delta = r\rho$, the solution breaks down.

(iii). If $A\left(\underline{y_1}, \underline{y_2}\right) = 0$ and $\delta > r\rho$, the solution breaks down.

(iv). If $A\left(\underline{y_1}, \underline{y_2}\right) > 0$ and $\delta - r\rho > 0$, as $\underline{y_1}$ and $\underline{y_2}$ increase in absolute value, the solution breaks down.

Thank you for your attention !