Robust Control and Applications in Finance and Insurance

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Outline of Lecture

- **Part I**
  - A motivating problem from Finance.
  - A motivating problem from Insurance.
  - Brief introduction to stochastic optimal control.
  - Brief introduction to robust control and model uncertainty.
- **Part II**
  - **Paper I.** Robust portfolio decisions for financial institutions (along with A.N Yannacopoulos and T. Xepapadeas).
  - **Paper II.** Optimal management of Defined Contribution pensions funds under the effect of inflation, mortality and uncertainty (along with A.N. Yannacopoulos, G.-W. Weber, L. Dopierala, K. Kolodziejczyk and M. Szczepański).

It's all about uncertainty

Uncertainty is an uncomfortable position.
But certainty is an absurd one.

Voltaire

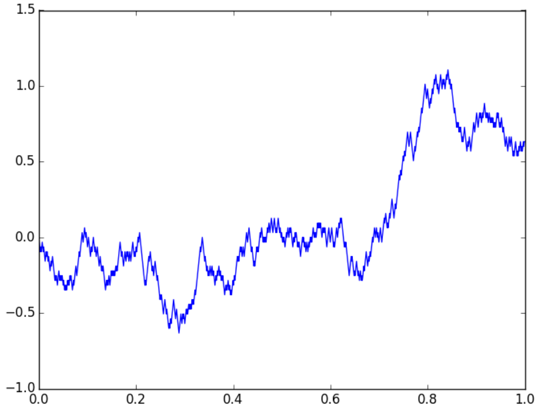
Life's most precious gift is uncertainty.

Yoshida Kenko

Uncertainty that comes from knowledge is different
from uncertainty that comes from ignorance

Isaac Asimov

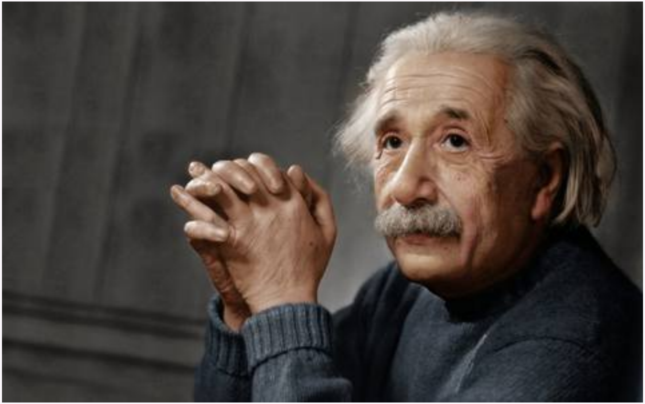
The building block: Brownian motion



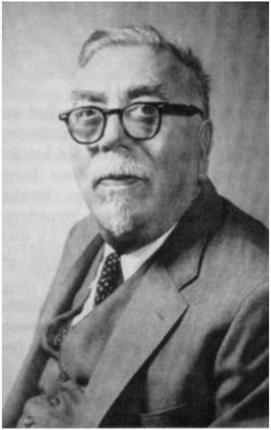
Robert Brown - 1827

Luis Bachelier - 1900

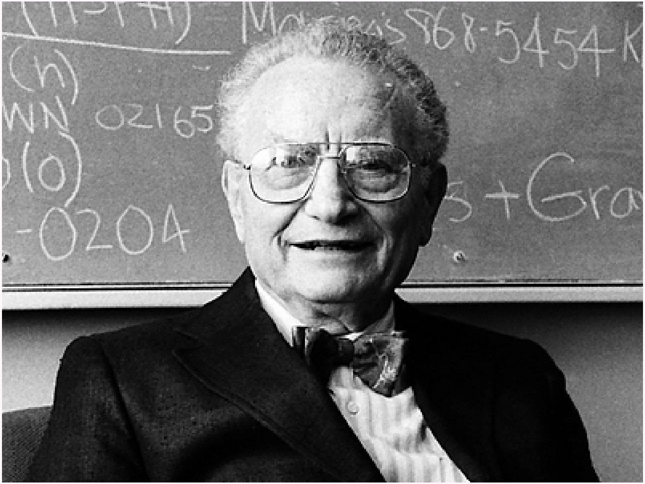
Albert Einstein - 1905



Norbert Wiener - 1923

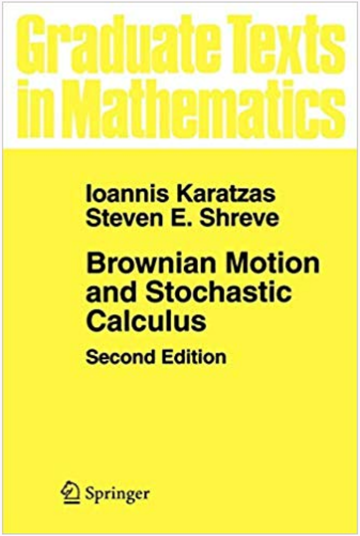


Paul Samuelson - 1973



Kiyoshi Itô - 1944

An advanced and rigorous book



Motivating Example I. A standard problem from Finance

Suppose that we have a classical Black-Scholes type financial market on the fixed time horizon $[0, T]$ with $T > 0$ and two investment possibilities :

- A risk free asset (bond or bank account) with unit price $S_0(t)$ at time t and dynamics described by the ordinary differential equation

$$dS_0(t) = rS_0(t)dt, \quad S_0(0) = 1$$

- A risky asset (stock or index) with unit price $S_1(t)$ at time t which evolves according to the stochastic differential equation

$$dS_1(t) = \mu S_1(t)dt + \sigma S_1(t)dW(t), \quad S_1(0) > 0$$

Here, the interest rate $r > 0$, the appreciation rate of the stock prices μ (with $\mu > r$) and the volatility of the stock prices $\sigma > 0$, are given constants.

Motivating Example I. A standard problem from Finance

- We consider an economic agent who at time $t = 0$ is endowed with some initial wealth $x_0 > 0$ and whose actions cannot affect the market prices.
- The portfolio process $\pi(t) = \pi(t, \omega) : [0, T] \times \Omega \to \mathbb{R}$ denotes the proportion of her wealth $X(t)$ invested in the risky asset.
- The remaining proportion $(1-\pi(t))X(t)$ is invested in the riskless asset.

the wealth process $X(t)$ corresponding to the strategy $\pi(t)$, is defined as the solution of the following stochastic differential equation (SDE)

$$
\begin{aligned}
dX(t) &= X(t)\left(r + \sigma\pi(t)\theta\right)dt + \sigma\pi(t)X(t)dW(t) \\
X(0) &= x_0 > 0
\end{aligned}
$$

where $\theta := (\mu - r)/\sigma$.

Motivating Example I. A standard problem from Finance

- The evolution of the portfolio process depends on the choice of the stochastic process $\pi(t)$, that is

$$X^{\pi}(t) = X(t, \pi(t))$$

- This process is known as a control process; The agent has complete control on it.

The Problem

Choose the control process so as to maximize the expected utility from her terminal wealth, i.e.

$$\sup_{\pi \in \Pi} \mathbb{E}\Big[\Phi(X^{\pi}(T))\Big]$$

Motivating Example II. Investment & Reinsurance

- We envision an insurance firm.
- The firm faces a (cummulative) claims process $C(t)$ with dynamics:

$$dC(t) = \alpha dt - \beta dB(t), \ \alpha > 0, \beta > 0.$$

- The firm collects premia (continuously) at the rate

$$c_0 = (1 + \theta)\alpha,$$

  where θ is the safety loading.
- The dynamics of the surplus process:

$$\begin{aligned} dR(t) &= c_0 dt - dC(t) \\ &= \alpha\theta dt + \beta dB(t) \end{aligned}$$

Motivating Example II. Investment & Reinsurance

- We assume that the insurer has the possibility to purchase proportional reinsurance (transfer risk) to reduce the underlying risk involved with its claims process.
- The insurance firm enters a reinsurance contract with a reinsurance firm.
- Reinsurance premium is paid continuously to the reinsurance firm at the rate

$$c_1 = (1 + \eta)\alpha q,$$

  where
  - $\eta \geqslant \theta$ is the safety loading of the proportional reinsurance.
  - q proportion reinsured.
- At each time $t \in [0, T]$ the insurance firm decides the strategy $(\pi(t), q(t))$.

Motivating Example II. Investment & Reinsurance

- The insurance firm faces the problem

$$\inf_{\pi, q \in \mathcal{A}^F} \mathbb{E}\left[\left(X^{\pi,q}(T) - A \right)^2 \right]$$

subject to the dynamic constraint

$$\begin{aligned} dX^{\pi,q}(t) = &\left(rX^{\pi,q}(t) + \pi(t)(\mu - r)X^{\pi,q}(t) + (\theta - \eta q(t))\alpha \right) dt \\ &+ \beta(1 - q(t))dB(t) + \sigma\pi(t)X^{\pi,q}(t)dW(t), \end{aligned}$$

with initial condition $X^{\pi,q}(0) = x_0 > 0$.

Stochastic Optimal Control

Elements of a control problem

- **Time horizon.** It can be: (i) finite, (ii) infinite, or even (iii) indefinite!
- **State process.** The state process is a stochastic process which describes the state of the physical system of interest. It is usually a stochastic differential equation that is influenced by the controller.
- **Control Process.**The control process is a stochastic process, chosen by the "controller" to influence the state of the system. We will consider only **admissible** controls, that is, controls that satisfy certain assumptions (i.e., technical, constraints).
- **Cost/reward function.** There is always some cost/reward associated with the system, which may depend on the system state itself and on the control.
- **Value function.** The value function describes the value of the maximum possible reward (or of the minimum possible cost) of the system. It is obtained, by optimizing the cost/reward over all admissible controls.

General form of a Stochastic Optimal Control Problem

We want to choose the control process u, so as to solve the problem

$$V(t,x) = \sup_{u \in \mathcal{U}} \mathbb{E}\left[\int_t^T F(s, X^u(s), u(s))dt + \Phi(X^u(T))\right]$$

subject to:

$$\begin{aligned} dX^u(s) &= \mu(s, X^u(s), u(s))ds + \sigma(s, X^u(s), u(s))dW(s) \\ X(0) &= x_0 (\in \mathbb{R}^n) \\ u(s) &\in \mathcal{U}, \ \forall \ s \in [t, T] \end{aligned}$$

Terminology:

- $X^u(s)$ = state of the system at time s (in the world scenario $\omega \in \Omega$)
- u = control variable.
- $\mathcal{U}$ = constraints for the control.

The Hamilton-Jacobi-Bellman equation

Theorem (HJB equation)

Under Standing Assumption, the following hold:

- *V satisfies the (HJB) equation:*

$$\frac{\partial V}{\partial t}(t,x) + \sup_{u \in \mathcal{U}} \left[\mathcal{L}^u V(t,x) + F(t,x,u) \right] = 0, \ \forall (t,x) \in (0,T) \times \mathbb{R}^n$$
$$V(T,x) = \Phi(x), \ \forall x \in \mathbb{R}^n$$

  where

$$\mathcal{L}^u := \mu(t,x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2}{\partial x^2}$$

  is the generator for the state process.

- *For each $(t,x) \in [0,T] \times \mathbb{R}^n$ the supremum in the HJB equation above is attained by $\hat{u}$.*

The Hamilton-Jacobi-Bellman equation

- This thoerem has the form of a necessary condition.
- If V is the optimal value function and if $\hat{u}$ is the optimal control...
  - then V satisfies the HJB equation
  - and $\hat{u}$ realizes the supremum in the equation.
- A surprising fact: the HJB equation also acts as a sufficient condition for the control problem!
- This is known as the verification theorem.

The Hamilton-Jacobi-Bellman equation

Theorem (Verification)

Suppose that we have two functions $H(t,x)$ and $g(t,x)$ such that

- *H is smooth enough solves the HJB equation.*

$$\frac{\partial H}{\partial t} + \sup_{u \in \mathcal{U}} \Big[\mathcal{L}^u H(t,x) + F(t,x,u) \Big] = 0, \ \forall (t,x) \in (0,T) \times \mathbb{R}^n$$

$$H(T,x) = \Phi(x), \ \forall x \in \mathbb{R}^n$$

- *For each fixed (t,x) the supremum in the expression*

$$\sup_{u \in \mathcal{U}} \Big[\mathcal{L}^u H(t,x) + F(t,x,u) \Big]$$

  is attained by the choice $u = g(t,x)$.

Then, $V = H$, there exists an optimal control $\hat{u}$ and $\hat{u} = g$.

Handling the HJB equation

1. Derive the Hamilton-Jacobi-Bellman equation(HJB) for the problem at hand. This is a partial differential equation (PDE) for an unknown function, e.g. V.
2. Fix an arbitrary point in (t, x) and solve the resulting static optimization problem.
3. From s2 we get a candidate for the optimal control laws.
4. This yields to a second order (for the problem at hand) PDE.
5. (Attempt to) Solve the PDE of s4.
6. Verification theorem: The solution of the HJBI equation (V) is the value function of the problem at hand and the control choices we found earlier are indeed the optimal ones.

Handling the HJB equation

- The challenging part is solving this PDE (highly nonlinear nature)!
- There are no general analytic methods available.
- There exist few known optimal control problems with an analytic solution is very small indeed.
- **Strategy:** We try to guess a solution !
- we typically make a parameterized Ansatz for V then use the PDE in order to identify the parameters.
- **Hint:** V often inherits some structural properties from the boundary function Φ as well as from the instantaneous utility function F.

Motivating Example. A standard problem from Finance

Theorem (General solution)

The optimal investment strategy for the controller is to invest in the risky asset proportion of her wealth equal to

$$\pi^*(t,x) = -\frac{\theta}{\sigma}\frac{V_x}{xV_{xx}}.$$

In this case, the optimal value function is a smooth solution of the following non-linear partial differential equation

$$V_t + rxV_x - \frac{1}{2}\theta^2\frac{V_x^2}{V_{xx}} = 0,$$

with terminal condition $V(T,x) = \Phi(x)$.

Sketch of the proof

- Consider the HJB equation

$$V_t + \sup_{\pi} \left[(r + \sigma\pi\theta) V_x + \frac{1}{2}\sigma^2\pi^2 x^2 V_{xx} \right] = 0$$

- Fix an arbitrary (t, x) and solve the static problem

$$\sup_{\pi} \left[(r + \sigma\pi\theta) V_x + \frac{1}{2}\sigma^2\pi^2 x^2 V_{xx} \right].$$

- First order conditions yield:

$$\hat{\pi}(t, x) = -\frac{\theta}{\sigma} \frac{V_x}{x V_{xx}}.$$

- Substituting the candidate control $\hat{\pi}$ to the HJB and we get the PDE
- Solve the PDE (and invoke verification theorem).

The case of the Exponential utility

Theorem

Let us assume the exponential utility function

$$\Phi(x) = -\frac{1}{\gamma} e^{-\gamma x},$$

where $\gamma > 0$ *stands for the risk aversion parameter. The optimal value function admits the form*

$$V(t, x) = -\frac{1}{\gamma} \exp \left[-\gamma x e^{r(T-t)} - \frac{1}{2} \theta^2 (T - t) \right].$$

In this case, the optimal strategy for the controller is to invest in the risky asset proportion of her wealth equal to

$$\pi^*(t, x) = \frac{\theta}{\sigma} \frac{e^{-r(T-t)}}{\gamma x}.$$

Sketch of the proof

- Let us assume the exponential utility function

$$\Phi(x) = -\frac{1}{\gamma} e^{-\gamma x},$$

- where $\gamma > 0$ stands fro the risk aversion parameter. We propose the Ansatz:

$$V(t,x) = -\frac{1}{\gamma} \exp\left[-\gamma x f(t) + g(t) \right],$$

- where f, g are appropriate functions (to be determined later) with boundary conditions $f(T) = 1$ and $g(T) = 0$.
- The boundary conditions follow from $V(T,x) = \Phi(x)$.

Sketch of the proof

$$V_t(t,x) = V(t,x)\Big[-\gamma x f^{'}(t) + g^{'}(t)\Big]$$
$$V_x(t,x) = V(t,x)\Big[-\gamma f(t)\Big]$$
$$V_{xx}(t,x) = V(t,x)\Big[-\gamma f(t)\Big]^2$$

- Substituting the above back into the HJB equation yields:

$$f^{'}(t) - rf(t) = 0$$
$$g^{'}(t) - \frac{1}{2}\theta^2 = 0$$

- solve the ODEs

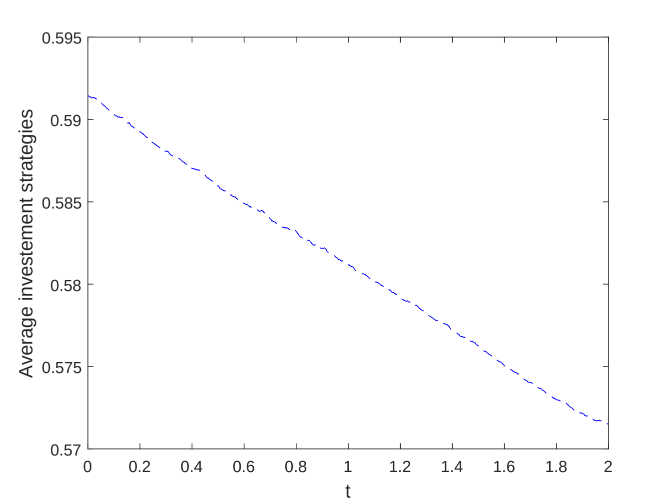


Figure: Average of 10000 optimal proportional investment strategy paths in the case of the exponential utility function.

References on Stochastic Control

The literature is vast:

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- W. H. Fleming and H. M. Soner (2006), Controlled Markov processes and viscosity solutions, Springer, New York.
- J. Yong and X. Zhou (1999), Stochastic controls, Hamiltonian systems and HJB equations, Springer.
- B. Øksendal (2003), Stochastic differential equations: An introduction with applications, Springer Verlag.
- T. Bjork (2009), Arbitrage theory in continuous time, Oxford.

Stochastic Optimal Control

Main Idea

- The underlying system is represented by a controlled stochastic process.
- The decision maker chooses the control process to drive the system to the desired state.
- Many applications in a variety of fields:
    - Mathematical Finance
    - Insurance
    - Risk Management, etc.

Main Assumption

- The decision maker blindly trusts the model he faces.
- The exact probability law of the stochastic risk factors in the underlying model, is precisely known.

Robust Control Theory

- An important part of stochastic control.
- In some sense it is the most realistic version of control theory.

Main Idea

- We wish to control a system but we do not know the exact law of evolution of the state process.
- What we have is a family of laws (scenarios), and we want to control the worst possible scenario.
- The best policy for the worst scenario is our robust control.

Stochastic Control vs Robust Control

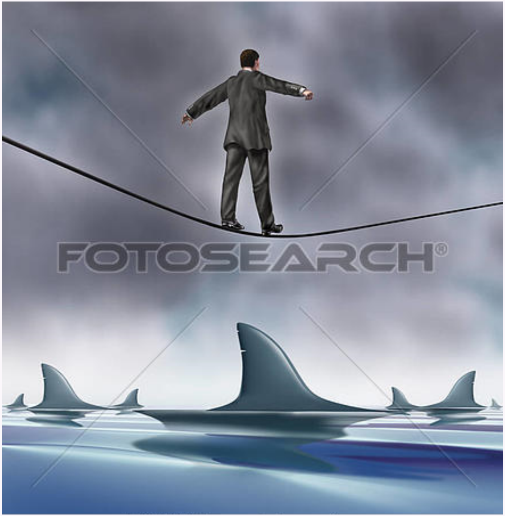

Figure: Stochastic Optimal Control Theory

Stochastic Control vs Robust Control

Figure: Robust Optimal Control Theory

Robust control theory is a mixture of two things:

- This theme has become extremely useful in economics and Finance
- T. J. Sargent (Nobel Prize in Economics; 2011) devoted most of his research in this field (along with L. Harsen).

Main Philosophy:

Solve an optimal control problem under the worst possible scenario.
$\Longrightarrow$ Using the model that may provide the worst case for the problem at hand.

In Mathematical terms:

Model$\sim$ Probability Measure

Model Uncertainty Aspects

- Uncertainty concerning the "true" statistical distribution of the state of the system.
- We assume that the controller is uncertain as to the true nature of the stochastic process W in the sense that the exact law of W is not known.
- There exists a "true" probability measure related to the true law of the process W, the controller is unaware of and a probability measure $\mathbb{Q}$, which is his/her idea of what the exact law of W looks like.
- As the controller is uncertain about the validity of $\mathbb{Q}$ as a proper description of the futures states of the world, she seeks to make her decision robust.

Model Uncertainty Aspects

- She adopts a "cautionary" approach that of seeking to maximize the worst possible scenario concerning the true description of the noise term. This is quantified as:

$$\inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\Big[U(Y(T))\Big],$$

- As a result, the manager faces the robust control problem

$$\sup_{\pi \in \mathcal{A}^{\mathbb{F}}} \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\Big[U(Y(T))\Big],$$

The Class of Measures $\mathcal{Q}$

Definition (The set $\mathcal{Q}$)

The set of acceptable probability measures $\mathcal{Q}$ for the agent is a set enjoying the following two properties:

(i) Considering the stochastic process W under the reference probability measure $\mathbb{P}$ and under the probability measure $\mathbb{Q}$ results to a change of drift.

(ii) There is a maximum allowed deviation of the controller's measure $\mathbb{Q}$ from the reference measure $\mathbb{P}$. In other words, the controller is not allowed to freely choose between various probability models as every departure will be penalized by an appropriately defined penalty function, a special case of which is the Kullback-Leibler relative entropy.

Change of measure - Girsanov

Theorem

Assume that $u \in \mathcal{Y} \subset \mathbb{R}$ *satisfies the condition*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T u^2(s)\right)ds\right] < \infty.$$

Then, the stochastic process $\widetilde{W}$ *with decomposition given by*

$$\widetilde{W}(t) = W(t) - \int_0^t u(s)ds,$$

is an $(\mathbb{F}, \mathbb{Q})$ *Brownian motion.*

The Robust Control Problem

$$\sup_{\pi \in \mathcal{A}^{\mathbb{F}}} \inf_{\mathbb{Q} \in \mathcal{Q}} J(t, y)$$

$$\sup_{\pi \in \mathcal{A}^{\mathbb{F}}} \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}} \Big[\Phi(X(T)) \Big] + \frac{1}{\beta} \mathcal{H}(\mathbb{P}|\mathbb{Q}) \tag{1}$$

$$= \sup_{\pi \in \mathcal{A}^{\mathbb{F}}} \inf_{u \in \mathcal{Y}} \mathbb{E}_{\mathbb{Q}} \left[\Phi(X(T)) + \frac{1}{2\beta} \int_t^T u^2(s) ds \right],$$

subject to the state dynamics

$$\frac{dX(s)}{X(s)} = \big[r + \sigma\theta\pi(s) + \sigma\pi(s)u(s) \big] ds + \sigma\pi(s) d\widetilde{W}(s)$$

with initial condition $X(s) = x > 0$.

The Robust Control Problem - Parameter β

- Departures from the reference probability model are penalized.
- These penalizations are weighted by the term $1/\beta$.
- $\beta > 0$ is referred to as the preference for robustness parameter, and serves as a measure to quantify the preference for robustness.

Two interesting limiting cases:

- $\beta \to 0$: In this case, the controller fully trusts the model he/she is offered and seeks no robustness.
- $\beta \to \infty$: In this case, the controller has no faith in the model he/she faces and seeks alternative models with larger entropy.

Example of a Stochastic Differential Game

Figure: World Chess Championship 2016

Stochastic Differential Games

- The evolution of the underlying system is described by a Stochastic differential equation.
- The system is controlled by two (or more) players with conflicting goals.
- The controllers decide their control process so as to drive the system to a desired state.

A robust control problem is written as a SDG:

- **Player I.** Decision maker: Chooses the control process.
- **Player II.** Imaginary player (Nature): Chooses the model (the measure)

Hamilton-Jacobi-Bellman-Isaacs equation

The generator for the state process has the form:

$$\mathcal{L}^{\pi,u} := \left[r + \sigma\pi\theta + \sigma\pi u\right] x \frac{\partial}{\partial x} + \frac{1}{2}\sigma^2\pi^2 x^2 \frac{\partial^2}{x^2}$$

Bellman-Isaacs equation

$$\frac{\partial V}{\partial t} + \sup_{\pi\in\Pi} \inf_{u\in\mathcal{U}} \left[\mathcal{L}^{\pi,u} V(t,x) + \frac{1}{\beta} F(u)\right] = 0$$

$$V(T,x) = \Phi(x).$$

On the solvability of the BI

Is it possible to find a smooth solution to the BI ?

NOT IN GENERAL !!

There are (almost) three ways to proceed.

One way of treating the BI

Figure: Praying as a means of solving the BI

Figure: Numerical approximation to the BI

Another way of treating the BI: Viscosity solutions

- Pierre-Louis Lions and Michael Crandall (1983) for first order PDEs
- The term "viscosity" refers to the vanishing viscosity method: A special approach to solve a first order PDE.
- The theory of viscosity solutions have matured at that point that is considered an inseparable part in the study of PDEs: HJB equation and HJBI equation.
- Barles, Fleming and Souganidis, Caffarelli, Cabre, Ishii, e.t.c

Important

- We do not require smoothness.
- We start from the problem itself and not the HJB or the BI equations.

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Journal of Dynamics and Games

Journal of Dynamics and Games

Robust portfolio decisions for financial institutions
(along with A.N. Yannacopoulos and A. Xepapadeas).

A joint work

The model: Financial Market

Suppose that we have a financial market on the fixed time horizon $[0, T]$ with $T > 0$ and two investment possibilities :

- A risk free asset (bond or bank account) with unit price $S_0(t)$ at time t and dynamics described by the ordinary differential equation

$$\boxed{dS_0(t) = rS_0(t)dt, \quad S_0(0) = 1} \tag{2}$$

- A risky asset (stock or index) with unit price $S_1(t)$ at time t which evolves according to the stochastic differential equation

$$\boxed{dS_1(t) = \mu S_1(t)dt + \sigma S_1(t)dW(t), \quad S_1(0) > 0} \tag{3}$$

Here, $r > 0$, μ (with $\mu > r$) and $\sigma > 0$, are given constants.

The model: Cash Flow

- In collective risk theory, a sound mathematical model for describing the surplus of a large portfolio of claims is the Cramer-Lundberg model:

$$Y(t) = Y(0) + P(t) - L(t) \tag{4}$$

- In some situations, it is easier to work with its diffusion approximation.
- Here, the cumulative claims process is modeled by

$$\boxed{\mathrm{dL(t)} = \alpha dt - \beta dB(t)} \tag{5}$$

  where α and β are positive constants.
- The drift term can be interpreted as the mean claims up to time t.
- The stochastic term can be interpreted as the fluctuations around the mean claims.

The setting

- We consider a financial firm, who, at time $t = 0$, starts with some initial wealth $x_0 > 0$.
- The risk manager of the firm decides the proportion $\pi(t)$ of its wealth $X(t)$ to be invested in the risky asset.
- The remaining proportion $(1 - \pi(t))X(t)$ is invested in the risk-less asset.
- The firm is designed to offer some very specific services to its clients (e.g., financial investments consultancy, pension fund management, insurance, etc) by entering a contract.
- In exchange for its services, the firm collects compensation (continuously) at the constant rate $c_0\alpha$, where $c_0 \geqslant 1$.

The setting

- However, such a contract also generates a stochastic cash flow of liabilities (e.g., long term payments, operating costs, etc) that evolves according to (5).
- As a means of reducing this additional exposure, the risk manager of the firm has the ability to transfer a proportion of its liabilities to another party (e.g. external investor, financial fund, reinsurance firm, e.t.c).
- The risk manager decides the proportion $q(t)$ of its claims process to be covered, by entering a contract with the third party.
- In exchange for this coverage, the third party collects an income continuously at the rate $c_1 \alpha q(t)$, where $c_1 \geqslant c_0$.

Example 1

Example 1

- Consider a bank that issues mortgages.
- All the loans that have been issued are part of the bank's loan portfolio.
- This portfolio generates both income (interest) and claims (default, liquidity).
- Describe the evolution of this process is the model (5).
- The bank, in order to reduce the risk associated with the claims generated by this portfolio, decides to sell part of it to some external investor (fund).
- The bank also has the opportunity to invest part of its assets/reserves in a financial market like the one described here.

Example 2

Example 2

- Another interesting example which falls in the above mentioned general framework, is the classical insurance/reinsurance setting.
- Let us consider an insurance firm who has the opportunity to invest part of its reserves in the financial market like the one described here.
- In addition, the insurance firm collects premia (continuously and at a constant rate) from its clients.
- In this case, the stochastic process (5) may be considered as the claims process the insurance firm faces.
- As a means of reducing the underlying risk involved with this claims process, the insurance firm faces the possibility of entering a reinsurance contract and purchase coverage.
- For this contract, the reinsurance firm collects premia.

Stochastic Differential Equations of firm's wealth

- To sum up, the wealth process corresponding to the strategy $\eta_1 = (\pi(t), q(t))$, is denoted as $X^{\eta_1}(t)$ and is defined as the solution of the following linear stochastic differential equation

$$dX^{\eta_1}(t) = \pi(t) X^{\eta_1}(t) \frac{dS_1(t)}{S_1(t)} + (1 - \pi(t))\, X^{\eta_1}(t) \frac{dS_0(t)}{S_0 t)} + dR(t),$$

where

$$\begin{aligned} dR(t) &= (c_0 - c_1)dt - dL(t) + q(t)dL(t) \\ &= \alpha(\theta - \eta) q(t) dt + \beta(1 - q(t)) dB(t). \end{aligned}$$

- Therefore, in view of (29-5)

$$\begin{aligned} dX^{\eta_1}(t) = &\Big[X^{\eta_1}(t)(r + (\mu - r)\pi(t)) + \alpha(\theta - \eta q(t)) \Big] dt \\ &+ \beta(1 - q(t)) dB(t) + \sigma \pi(t) X^{\eta_1}(t) dW(t), \end{aligned} \tag{6}$$

with initial condition $X^{\eta_1}(0) = x_0 > 0$.

The original problem

- The risk manager aims to choose the control process so as to maximize some certain goal, e.g., the expected utility from her terminal wealth:

$$\sup_{\pi, q \in \mathcal{A}^{\mathbb{F}}} \mathbb{E}\Big[U(X^{\eta_1}(T))\Big],$$

subject to the state process

$$\begin{aligned} dX^{\eta_1}(t) = \Big[X^{\eta_1}(t)(r + (\mu - r)\pi(t)) + \alpha(\theta - \eta q(t))\Big]dt \\ + \beta(1 - q(t))dB(t) + \sigma\pi(t)X^{\eta_1}(t)dW(t). \end{aligned} \tag{7}$$

- A standard way to proceed is by employing the techniques of stochastic optimal control.

Model uncertainty aspects

- We assume that the risk manager is uncertain as to the true nature of the stochastic processes W and B in the sense that the exact law of W and B is not known.
- There exists a "true" probability measure related to the true law of the processes W and B, the risk manager is unaware of, and a probability measure Q, which is her idea of what the exact law of W and B looks like.
- The manager in uncertain about the validity of Q:

$$\inf_{Q \in \mathcal{Q}} \mathbb{E}_Q \Big[U(X^{\eta_1}(T)) \Big],$$

- As a result, the manager faces the robust control problem

$$\sup_{\pi, q \in \mathcal{A}^{\mathbb{F}}} \inf_{Q \in \mathcal{Q}} \mathbb{E}_Q \Big[U(X^{\eta_1}(T)) \Big],$$

Change of measure - Girsanov

Theorem

Assume that $y_1, y_2 \in \mathcal{Y} \subset \mathbb{R}^2$ *satisfy the condition*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T y_1^2(s) + y_2^2 ds\right)\right] < \infty.$$

Then, the stochastic processes $\widetilde{W}$ *and* $\widetilde{B}$ *with decomposition given by*

$$\widetilde{W}(t) = W(t) - \int_0^t y_1(s)ds,$$

and

$$\widetilde{B}(t) = B(t) - \int_0^t y_2(s)ds,$$

are $(\mathbb{F}, Q)$ *Brownian motions.*

The robust control problem

$$
\begin{aligned}
&\sup_{\pi, q \in \mathcal{A}^{\mathbb{F}}} \inf_{Q \in \mathcal{Q}} J(t, x) \\
&= \sup_{\pi, q \in \mathcal{A}^{\mathbb{F}}} \inf_{y_1, y_2 \in \mathcal{Y}} \mathbb{E}_Q \left[U(\widetilde{X}^{\eta_1, \eta_2}(T)) + \frac{1}{2\lambda} \int_t^T y_1^2(s) + y_2^2(s) ds \right],
\end{aligned}
\tag{8}
$$

subject to the state dynamics

$$
\begin{aligned}
d\widetilde{X}^{\eta_1, \eta_2}(s) = \Big[& r\widetilde{X}^{\eta_1, \eta_2}(s) + (\mu - r)\pi(s)\widetilde{X}^{\eta_1, \eta_2}(s) + \alpha(\theta - \eta q(s)) \\
& + \sigma\pi(s) y_1(s) \widetilde{X}^{\eta_1, \eta_2}(s) + \beta(1 - q(s)) y_2(s) \Big] ds \\
& + \sigma\pi(s)\widetilde{X}^{\eta_1, \eta_2}(s) d\widetilde{W}(s) + \beta(1 - q(s)) d\widetilde{B}(s),
\end{aligned}
\tag{9}
$$

with initial condition $\widetilde{X}^{\eta_1, \eta_2}(s) = x_0 > 0$.

Theorem (Main Result)

Suppose that the risk manager has preference for robustness as described by the non-negative constant λ. The optimal robust strategy is to invest in the risky asset proportion of the firm's wealth equal to

$$\pi^*(t,x) = -\frac{\mu - r}{\sigma^2 x}\frac{V_x}{V_{xx} - \lambda V_x^2},$$

and purchase proportional coverage for the firm's claims, equal to

$$q^*(t,x) = 1 + \frac{\alpha\eta}{\beta^2}\frac{V_x}{V_{xx} - \lambda V_x^2}.$$

On the other hand, Nature chooses the worst-case scenario defined by

$$y_1^*(t,x) = \frac{\mu - r}{\sigma}\frac{\lambda V_x^2}{V_{xx} - \lambda V_x^2} \quad \text{and} \quad y_2^*(t,x) = \frac{\alpha\eta}{\beta}\frac{\lambda V_x^2}{V_{xx} - \lambda V_x^2}.$$

In this case, the optimal robust value function is a smooth solution of the following non-linear partial differential equation

$$V_t + [rx + \alpha(\theta - \eta)]V_x - \frac{1}{2}\left[\left(\frac{\mu - r}{\sigma}\right)^2 + \left(\frac{\alpha\eta}{\beta}\right)^2\right]\frac{V_x^2}{V_{xx} - \lambda V_x^2} = 0,$$

with boundary condition $V(T,x) = U(x)$.

Theorem (Exponential Utility)

Assume Exponential preferences $(u(x) = -\frac{\delta}{\gamma}e^{-\gamma x})$*. The optimal robust value function admits the form:*

$$V(t,x) = -\frac{\delta}{\gamma}\exp\left[-\gamma x e^{r(T-t)} + g(t)\right], \tag{10}$$

where

$$g(t) = \alpha\gamma(\theta-\eta)\frac{1-e^{r(T-t)}}{r} - \frac{\gamma}{2(\lambda+\gamma)}\left[\left(\frac{\mu-r}{\sigma}\right)^2 + \left(\frac{\alpha\eta}{\beta}\right)^2\right](T-t). \tag{11}$$

In this case, the optimal robust strategy for the risk manager is to invest in the risky asset the constant amount

$$\pi^*(t,x) = \frac{\mu-r}{\sigma^2 x}\frac{e^{-r(T-t)}}{\lambda+\gamma}, \tag{12}$$

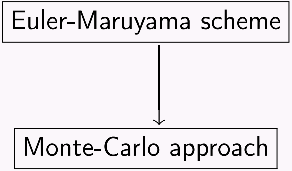
and purchase proportional coverage for the firm's claims, equal to

$$q^*(t,x) = 1 - \frac{\alpha\eta}{\beta^2}\frac{e^{-r(T-t)}}{\lambda+\gamma}. \tag{13}$$

On the other hand, Nature chooses the worst-case scenario defined by

$$y_1^*(t,x) = -\frac{\mu-r}{\sigma}\frac{\lambda}{\lambda+\gamma} \quad \text{and} \quad y_2^*(t,x) = -\frac{\alpha\eta}{\beta}\frac{\lambda}{\lambda+\gamma}. \tag{14}$$

Numerical study of the optimal investment strategy



E-M: For a time step of size $\Delta t = T/N$ with $N = 2^{11}$ points, we define the step size in the Euler-Maruyama scheme as $\delta t = \Delta t$.

M-C: Simulate a large number M of of paths of π^* and q^* in the time interval $[0, T]$ and at each time point we plot the average of M different values. We also use for each path $N = 2^{\alpha}$ number of points (here $N = 2^{11}$ and $M = 6000$ paths).

We let $M = 10000$, $T = 10$ months, $X(0) = 1.5$, $\gamma = 0.5$ and $\lambda = 0.2$. The parameters of the financial market are chosen as $\mu = 12\%$, $r = 6\%$, $\sigma = 40\%$. The parameters for the insurance market are chosen as $\alpha = 1$, $\beta = 0.2$ and $c_1 = 1.1$.

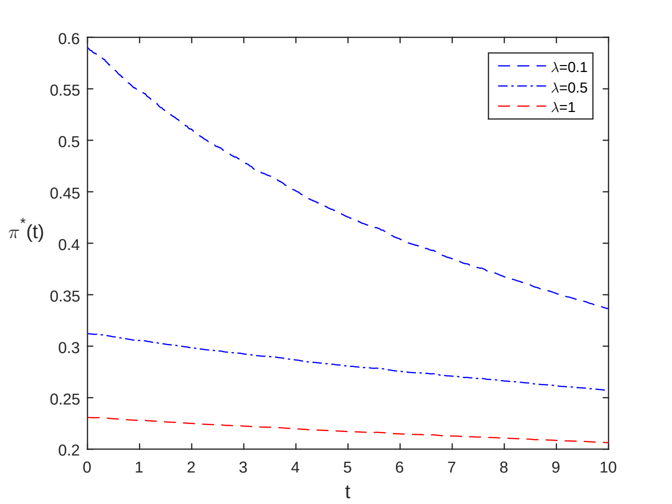


Figure: Average of 10000 optimal investment strategy paths for various levels of the preference for robustness parameter, in the case of the exponential utility function.

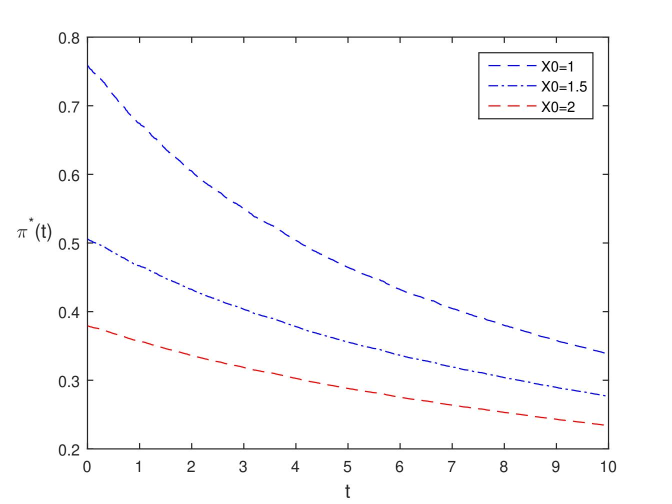


Figure: Average of 10000 optimal investment strategy paths for various levels of the initial wealth, in the case of the exponential utility function.

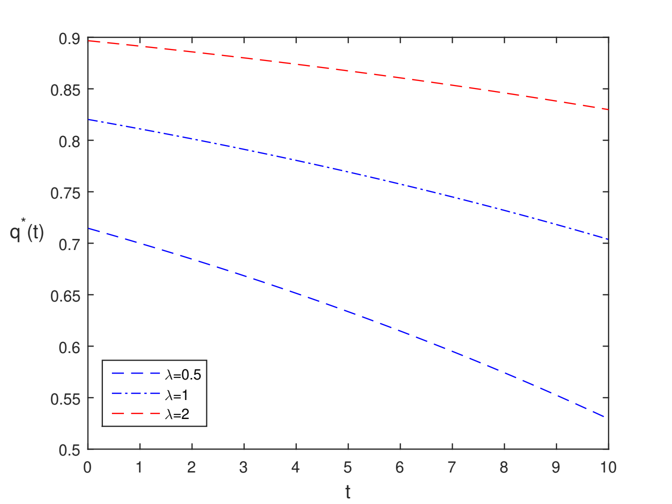


Figure: Average of 10000 optimal proportional coverage strategy paths for various levels of the preference for robustness parameter, in the case of the exponential utility function.

Limiting behavior: Cases $\lambda \to 0$ and $\lambda \to \infty$

- It is well known (see e.g. Anderson, Hansen and Sargent) that as $\lambda \to 0$ the decision maker fully trusts her model and exhibits no preference for robustness.
- As $\lambda \to +\infty$, the decision maker has no faith in the model she is offered and is willing to consider alternative models with larger relative entropy.
- The vast majority of the available works examines the limiting behavior of the optimal robust strategies, after the problem has been solved.
- Here, we are concerned with the structural behavior of the robust control problem itself in these limiting cases (well-posedness?)

Theorem (Limiting behavior as $\lambda \to 0$)

The optimal robust strategy for the risk manager is to invest in the risky asset proportion of the firm's wealth equal to

$$\pi^*(t,x) = -\frac{\mu - r}{\sigma^2 x}\frac{V_x}{V_{xx}}, \tag{15}$$

and also, to purchase proportional coverage for the firm's liabilities, equal to

$$q^*(t,x) = 1 - \frac{\alpha(1-c_1)}{\beta^2}\frac{V_x}{V_{xx}}. \tag{16}$$

On the other hand, Nature chooses the myopic worst-case scenario defined by

$$y_1^*(t,x) = y_1^*(t,x) = 0. \tag{17}$$

In this case, the optimal robust value function is a smooth solution of the following non-linear partial differential equation

$$V_t + [rx + \alpha(\theta - \eta)]V_x - \frac{1}{2}\left[\frac{(\mu-r)^2}{\sigma^2} + \frac{\alpha^2(1-c_1)^2}{\beta^2}\right]\frac{V_x^2}{V_{xx}} = 0, \tag{18}$$

with boundary condition $V(T,x) = U(x)$, assuming that such a solution exists.

Limiting behavior I: Case $\lambda \to 0$

We have some interesting findings

- The risk manager has complete faith in the model described by Equations (3) and (5).
- Operates under the probability measure $\mathbb{P}$.
- The controls (15), (16) and the PDE (18), are the optimal Markovian control laws and PDE associated with the stochastic optimal control problem:

$$\sup_{\pi, q \in \mathcal{A}^{\mathbb{F}}} \mathbb{E}_{\mathbb{P}} \Big[U(X^{\eta_1}(T)) \Big],$$

  subject to the original state dynamics.
- Robust Control Problem $\to$ Optimal Control Problem.

Theorem (Limiting behavior as $\lambda \to +\infty$)

Assume that $\mathcal{Y}$ is the rectangle $\left[\underline{y_1}, \overline{y_1}\right] \times \left[\underline{y_2}, \overline{y_2}\right]$. The optimal robust strategy for the risk manager is to invest in the risky asset proportion of the firm's wealth equal to

$$\pi^*(t,x) = -\Big(\frac{\mu - r}{\sigma} + \underline{y_1}\Big)\frac{V_x}{\sigma x V_{xx}}, \tag{19}$$

and to purchase proportional coverage for the firm's liabilities, equal to

$$q^*(t,x) = 1 + \left(\frac{\alpha(c_1 - 1)}{\beta} + \underline{y_2}\right)\frac{V_x}{\beta V_{xx}}. \tag{20}$$

On the other hand, Nature chooses the myopic worst-case scenario defined by

$$y_1^*(t,x) = \underline{y_1}, \quad \text{and} \quad y_2^*(t,x) = \underline{y_2}. \tag{21}$$

In this case, the optimal robust value function is a smooth solution of the following non-linear partial differential equation

$$V_t + [rx + \alpha(c_0 - c_1)]V_x - \frac{1}{2}\left[\left(\frac{\mu - r}{\sigma} + \underline{y_1}\right)^2 + \left(\frac{\alpha(c_1 - 1)}{\beta} + \underline{y_2}\right)^2\right]\frac{V_x^2}{V_{xx}} = 0, \tag{22}$$

with boundary condition $V(T,x) = U(x)$, assuming that such a solution exists.

Solution break-down

- We construct a case where loss of convexity leads to break-down of the solution of the HJBI equation.
- For simplicity we assume that $c_0 = c_1$.
- The HJBI equation is restated as

$$V_t + rxV_x - A\Big(\underline{y_1}, \underline{y_2}\Big)\frac{V_x^2}{V_{xx}} = 0, \tag{23}$$

where

$$A\Big(\underline{y_1}, \underline{y_2}\Big) := \frac{1}{2}\left[\left(\frac{\mu - r}{\sigma} + \underline{y_1}\right)^2 + \left(\frac{\alpha(c_1 - 1)}{\beta} + \underline{y_2}\right)^2\right] \geqslant 0.$$

- We assume that the risk manager operates under quadratic preferences, that is a utility function of the form

$$U(x) = \kappa\frac{x^\rho}{\rho}, \;\; \kappa > 0, \;\; 0 < \rho < 1. \tag{24}$$

Solution break-down

Assume that the PDE (23) admits a classical solution $V \in \mathcal{C}^{1,2}(\mathbb{S})$. We look for a solution using the guess

$$V(t,x) = e^{-\delta t}\widetilde{V}(x),$$

where $\widetilde{V} \in \mathcal{C}^{1,2}(\mathbb{S})$. Differentiating the above expression with respect to (t,x), yields

$$V_t = -\delta e^{-\delta t}\widetilde{V}(x)$$
$$V_x = e^{-\delta t}\widetilde{V}_x$$
$$V_{xx} = e^{-\delta t}\widetilde{V}_{xx}.$$

Solution break-down

Substituting these expressions back in the partial differential equation (23), results to the elliptic partial differential equation

$$\delta \widetilde{V} - rx\widetilde{V}_x + A\left(\underline{y_1}, \underline{y_2}\right)\frac{\widetilde{V}_x^2}{\widetilde{V}_{xx}} = 0. \tag{25}$$

We propose a solution to the partial differential equation of the form

$$\widetilde{V}(x) = \kappa\frac{x^\rho}{\rho}.$$

Inserting this trial solution in (25), yields to the following condition for the discounting factor

$$\delta = r\rho - A\left(\underline{y_1}, \underline{y_2}\right)\frac{\rho}{\rho - 1},$$

or equivalently

$$A\left(\underline{y_1}, \underline{y_2}\right) = \frac{1-\rho}{\rho}(\delta - r\rho).$$

Solution break-down

We distinguish the following four cases:

If $A\left(\underline{y_1}, \underline{y_2}\right) = 0$ and $\delta = r\rho$, a solution exists.

If $A\left(\underline{y_1}, \underline{y_2}\right) > 0$ and $\delta = r\rho$, the solution breaks down.

If $A\left(\underline{y_1}, \underline{y_2}\right) = 0$ and $\delta > r\rho$, the solution breaks down.

If $A\left(\underline{y_1}, \underline{y_2}\right) > 0$ and $\delta - r\rho > 0$, as $\underline{y_1}$ and $\underline{y_2}$ increase in absolute value, the solution breaks down.

European Journal of Operational Research

Optimal management of Defined Contribution pension funds under the effect of inflation, mortality and uncertainty
(along with A.N. Yannacopoulos).

A joint work with G.-W. Weber (PUT) and . . .

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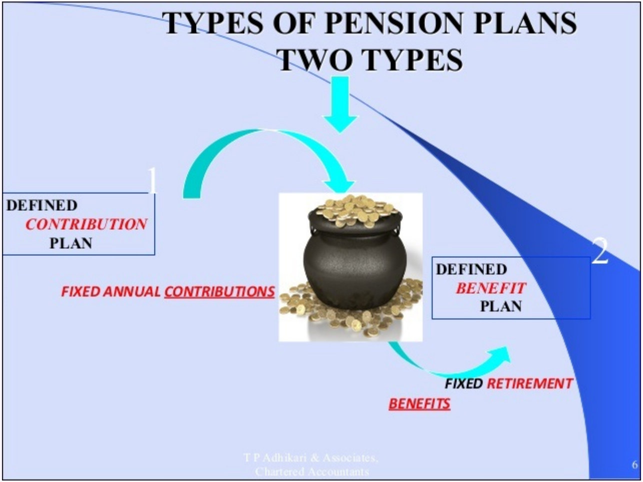
Optimal management of pension funds

Optimal management of pension funds

- During the last decade, the problem of providing supplementary pensions to the retirees has attracted a lot of attention from official bodies, as well as private financial institutions worldwide.
- In this vein, there exist various possible directions, with the most popular provided by the mechanism of pension fund schemes.
- Essentially, a pension fund scheme constitutes an independent legal entity that represents accumulated wealth stemming from pooled contributions of its members.
- This wealth is to be invested over a long period of time
- (usually from 20 to 40 years) in order to provide its members with retirement benefits (in the form of periodic pension payments or a one-off payment).

Optimal management of pension funds



The model I: Inflation index

- DC pension schemes usually last for many years (e.g., 10-20) hence **Inflation** plays a crucial role.
- We define the inflation index level as an Itô process of the following type:

$$\frac{dI(t)}{I(t)} = (r_N - r_R + \sigma_I \theta_I)dt + \sigma_I dB(t), \qquad (26)$$
$$I(0) = i_0 > 0,$$

where:

- $\theta_I \in \mathbb{R}$ stands for the market price of inflation risk.
- $r_R > 0$ stands for the real interest rate.
- $r_N > 0$ stands for the nominal interest rate.
- $\sigma_I > 0$ stands for the volatility of the inflation index.
- B is a standard Brownian motion.

The model I: Financial Market

Suppose that we have a financial market on the fixed time horizon $[0, T]$ with $T > 0$ and three investment possibilities:

Asset 1

An inflation-adjusted bond that matures at time $T > 0$. The inflation adjusted bond offers a constant rate of return r_R and its dynamics are described by the following SDE:

$$\begin{aligned} \frac{dP(t,T)}{P(t,T)} &= r_R dt + \frac{dI(t)}{I(t)} \\ &= (r_N + \sigma_I \theta_I) dt + \sigma_I dB(t), \\ P(0,T) &= p_0 > 0, \end{aligned} \tag{27}$$

where:

- $P(t, T)$ denotes the price of the inflation-indexed bond at time t with maturity at time T.

The model I: Financial Market

Asset 2

Another risky asset (e.g., a financial index or stock) which evolves according to the stochastic differential equation

$$\begin{aligned} \frac{dS(t)}{S(t)} &= \nu dt + \sigma_S dW(t) + \sigma_{SI} dB(t), \\ S(0) &= S_0 > 0, \end{aligned} \tag{28}$$

where

- $S(t)$ denotes the price of the index at time $t \in [0, T]$.
- $\nu > r_N > 0$ stands for the appreciation rate of the stock prices.
- $\sigma_S > 0$ stands for the volatility of the stock prices due to the financial market.
- $\sigma_{SI} > 0$ is another volatility source due to the exposure to the inflation risk
- W is another standard Brownian motion.

Asset 3

A risk free asset (bank account) with unit price $S_0(t)$ at time $t \in [0, T]$ and dynamics described by the ordinary differential equation

$$\begin{aligned} dS_0(t) &= r_N S_0(t) dt, \\ S_0(0) &= 1. \end{aligned} \tag{29}$$

Key points

- Market parameters are assumed to be constants for simplicity.
- Extension with time varying parameters is possible (but painful).
- Brownian motions are assumed orthogonal for algebraic simplicity.

The model II: The Pension Fund

- We consider a standard defined contribution (DC) pension fund scheme.
- Employees that become part of the pension fund pay contributions during employment period.
- Each one of them receives an accumulated amount X_F at the time of retirement.
- X_F is split into two amounts: X_A and X_B.
- The amount X_A is converted to a pension annuity.
- The amount X_B is converted to a **life-assurance contract**.

The model II: The life assurance contract

- At the time of retirement $\ell(0)$ lives enter such a contract with the same insurance firm by paying upfront the amount X_B.
- The accumulated wealth $X_0 := \ell(0)X_B$ is then collected by the fund manager of the insurance firm to a portfolio savings account and is to be invested optimally in the market (2)-(3)-(4).
- **If a member dies** at time $t \in [0, T]$ his beneficiaries immediately receive a benefit payment of $q(t) > 0$.
- **If the member is alive** at the end of the investment period, he is promised the amount $A_0 := e^{r_F T} X_B - KT > 0$, where:
  - $r_F > r_N$ (scheme more attractive that cash account),
  - $K > 0$ (fixed amount, kept by insurance firm p.a. to cover operational costs and additional expenses).

Stochastic Differential Equation of Fund's wealth

- The fund's wealth process corresponding to the strategy $(\pi(t), b(t))$, is denoted as $X(t)$ and is defined as the solution of the following linear stochastic differential equation:

$$\frac{dX(t)}{X(t)} = \pi(t)\frac{dS(t)}{S(t)} + b(t)\frac{dP(t,T)}{P(t,T)} + (1 - \pi(t) - b(t))\,\frac{dS_0(t)}{S_0(t)} - q\mu(t)\hat{\ell}(t)dt.$$

- Therefore, in view of (1)-(4) (and referring to the initial wealth as x):

$$\frac{dX(t)}{X(t)} = \left([r_N + \pi(t)(\nu - r_N) + b(t)\sigma_I\theta_I] - q\delta(t)\right)dt + \pi(t)\sigma_S dW(t) + (b(t)\sigma_I + \pi(t)\sigma_{SI})dB(t),$$

$$X(0) = x > 0,\ \ \delta(t) := \mu(t)\hat{\ell}(t)$$

The Initial Problem

- The fund manager chooses the control processes so as to maximize some certain goal, e.g., the expected utility from her terminal inflation-adjusted wealth:

$$\sup_{(\pi,b)\in\mathcal{A}^{\mathbb{F}}} \mathbb{E}_{\mathbb{P}}\left[U_{\varepsilon}\left(Y(T)-\hat{A}\right)\Big|\mathcal{F}_t\right],\ \ Y(T)=\frac{X(T)}{I(T)},$$

where $\hat{A} := (1+\theta)\hat{I}(T)A_0$, subject to the state process

$$\begin{aligned} dY(t) = \Big[&k+\pi(t)(\nu-r_N)+b(t)\sigma_I\theta_I-b(t)\sigma_I^2-\pi(t)\sigma_{SI}\sigma_I \\ &-q\delta(t)\Big]dt+\pi(t)\sigma_S dW(t)+(b(t)\sigma_I+\pi(t)\sigma_{SI}-\sigma_I)dB(t), \end{aligned} \tag{30}$$

with initial condition $Y(0)=y>0$, where $k := r_R-\sigma_I\theta_I+\sigma_I^2$.

Stochastic control techniques & pension funds

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Model Uncertainty Aspects

- Uncertainty concerning the "true" statistical distribution of the state of the system.
- We assume that the controller is uncertain as to the true nature of the stochastic processes W and B in the sense that the exact laws of W and B are not known.
- There exists a "true" probability measure related to the true law of the process W and B, the controller is unaware of and a probability measure Q, which is his/her idea of what the exact laws of W and B look like.
- As the controller is uncertain about the validity of Q as a proper description of the futures states of the world, she seeks to make his/her decision robust.

Model Uncertainty Aspects

- He/She adopts a "cautionary" approach of seeking to maximize the worst possible scenario concerning the true description of the noise term. This is quantified as:

$$\inf_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[U_\varepsilon \left(Y(T) - \hat{A} \right) \middle| \mathcal{F}_t \right].$$

- As a result, the manager faces the robust control problem

$$\sup_{(\pi, b) \in \mathcal{A}^{\mathbb{F}}} \inf_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[U_\varepsilon \left(\widetilde{Y}(T) - \hat{A} \right) \middle| \mathcal{F}_t \right],$$

where $\widetilde{Y}(t)$ denotes the wealth process under the probability measure Q.

The Robust Control Problem

$$\sup_{(\pi,b)\in\mathcal{A}^{\mathbb{F}}} \inf_{Q\in\mathcal{Q}} J(t,y)$$

$$= \sup_{(\pi,b)\in\mathcal{A}^{\mathbb{F}}} \inf_{(\lambda_S,\lambda_I)\in\mathcal{Y}} \mathbb{E}_Q\left[U_\varepsilon\left(\widetilde{Y}(T)-\hat{A}\right)+\int_t^T \frac{\lambda_S^2(u)+\lambda_I^2(u)}{2\beta\Psi\left(u,\widetilde{Y}(u)\right)}du\right],$$

subject to the state dynamics

$$\begin{aligned}\frac{d\widetilde{Y}(t)}{\widetilde{Y}(t)} =&\Big[k+\pi(t)(\nu-r_N)+\pi(t)\sigma_S\lambda_S(t)+b(t)\sigma_I\theta_I-b(t)\sigma_I^2\\ &-\pi(t)\sigma_{SI}\sigma_I+(b(t)\sigma_I+\pi(t)\sigma_{SI}-\sigma_I)\lambda_I(t)-q\delta(t)\Big]dt\\ &+\pi(t)\sigma_S d\widetilde{W}(t)+(b(t)\sigma_I+\pi(t)\sigma_{SI}-\sigma_I)d\widetilde{B}(t),\end{aligned} \tag{31}$$

with initial condition $Y(s)=y>0$.

Theorem (GENERAL UTILITY)

Suppose that the fund manager has preference for robustness as described by the positive constant β. The optimal robust strategy is to invest in the stock, a proportion of the fund's wealth equal to

$$\pi^*(t,y) = -\left[\frac{\nu - r_N}{\sigma_S} - \theta_I \frac{\sigma_{SI}}{\sigma_S}\right] \frac{V_y}{y\sigma_S\left(V_{yy} - \beta\Psi V_y^2\right)}, \tag{32}$$

and in the inflation-indexed bond, proportion of the fund's wealth equal to

$$b^*(t,y) = -\frac{\theta_I - \sigma_I}{\sigma_I} \frac{V_y}{y\left(V_{yy} - \beta\Psi V_y^2\right)} - \pi^*(t,y)\frac{\sigma_{SI}}{\sigma_I} + 1. \tag{33}$$

On the other hand, Nature chooses the worst-case scenario defined by

$$\lambda_S^*(t,y) = \left[\frac{\nu - r_N}{\sigma_S} - \theta_I \frac{\sigma_{SI}}{\sigma_S}\right] \frac{\beta\Psi V_y^2}{V_{yy} - \beta\Psi V_y^2}, \tag{34}$$

and

$$\lambda_I^*(t,y) = (\theta_I - \sigma_I)\frac{\beta\Psi V_y^2}{V_{yy} - \beta\Psi V_y^2}. \tag{35}$$

In this case, the optimal robust value function is a smooth solution of the following nonlinear, second-order partial differential equation

$$V_t + (r_R - q\delta(t))yV_y - \frac{1}{2}\left[\left(\frac{\nu - r_N}{\sigma_S} - \theta_I \frac{\sigma_{SI}}{\sigma_I}\right)^2 + (\theta_I - \sigma_I)^2\right] \frac{V_y^2}{V_{yy} - \beta\Psi V_y^2} = 0 \tag{36}$$

with boundary condition $V(T,y) = U_\varepsilon\left(y - \hat{A}\right)$, assuming that such a solution exists.

A special solution

- We derive closed-form solutions for the special case of the exponential utility function, that is, the utility function of the form

$$U(y) = \frac{1}{\varepsilon}\left[1 - \exp\left(-\gamma\left(y - \hat{A}\right)\right)\right], \tag{37}$$

where $\gamma > 0$ stands for the risk-aversion parameter of the fund manager.

- We choose the following form for the state dependent scaling function

$$\Psi(t, y) = \frac{\varepsilon}{V_y(t, y)}, \tag{38}$$

- The parameter $\varepsilon > 0$ aims to penalize paths for which $\widetilde{Y}(T) < \hat{A}$.
- Other choices of Ψ are possible.

Theorem (Exponential Utility)

Assume Exponential preferences (Equation (37)) and the scaling function (Equation (38)). The value function for the robust control problem admits the form:

$$V(t,y) = \frac{1}{\varepsilon}\left[1 - \exp\left(-\gamma\left(y - \hat{A}\right) f(t) + g(t)\right)\right], \tag{39}$$

where

$$f(t) = e^{\alpha(t,T)}, \ \ \alpha(t,T) := r_R(T-t) - q\int_t^T \delta(u)du, \tag{40}$$

and

$$g(t) = \frac{1}{\varepsilon}\left(\gamma\hat{A}(1-f(t)) + K\int_t^T \frac{f(u)}{\gamma f(u) + \beta\varepsilon}du\right), \ \ K = \frac{1}{2}\left[\left(\frac{\nu - r_N - \theta_I\sigma_{SI}}{\sigma_S}\right)^2 + (\theta_I - \sigma_I)^2\right]. \tag{41}$$

Moreover, the optimal robust strategy for the fund manager is to invest in the stock market, a proportion of the fund's wealth equal to

$$\pi^*(t,y) = \left(\frac{\nu - r_N}{\sigma_S} - \theta_I\frac{\sigma_{SI}}{\sigma_S}\right)\frac{1}{y\sigma_S\left(\gamma f(t) + \beta\varepsilon\right)}, \tag{42}$$

and to invest in the inflation-indexed bond, proportion of the fund's wealth equal to

$$b^*(t,y) = \frac{\theta_I - \sigma_I}{\sigma_I}\frac{1}{y\left(\gamma f(t) + \beta\varepsilon\right)} - \pi^*(t,y)\frac{\sigma_{SI}}{\sigma_I} + 1. \tag{43}$$

On the other hand, Nature chooses the worst-case scenario $Q \in \mathcal{Q}$ that is defined by

$$\lambda_S^*(t) = -\left(\frac{\nu - r_N}{\sigma_S} - \theta_I\frac{\sigma_{SI}}{\sigma_S}\right)\frac{\beta\varepsilon}{\gamma f(t) + \beta\varepsilon}, \tag{44}$$

and

$$\lambda_I^*(t) = -\left(\theta_I - \sigma_I\right)\frac{\beta\varepsilon}{\gamma f(t) + \beta\varepsilon}. \tag{45}$$

Numerical study of the optimal investment strategy

Euler-Maruyama scheme → Monte-Carlo approach

E-M: For a time step of size $\Delta t = T/N$ with $N = 1000$ points, we define the step size in the Euler-Maruyama scheme as $\delta t = \Delta t$.

M-C: Simulate a large number M of paths of π^* and b^* in the time interval $[0, T]$ and at each time point we plot the average of $M = 10000$ different values.

μ, σ_S: SP500 stock index (previous year).
r_N : Secured Overnight Financing Rate
σ_I, σ_{SI} : USA inflation (2000-2020).
α, c : Actuarial table (Total population, USA 2017).

ε	r_R	r_N	σ_I	θ_I	σ_{SI}	σ_S	ν	β
1	0.115%	1.485%	0.02	0.238	0.052	0.0878	0.1298	0.2
q	γ	r_F	K	α	c	N	θ	$Y(0)$
0.0095	1.5	0.065	48	0.0195	0.0332	15	0.0524	8.87

Force of Mortality

Figure: Number of lives remaining in the fund portfolio at each instant of time according to Gompertz law of mortality $\mu(t) = \alpha e^{ct}$.

Effect of Robustness on Optimal Strategies

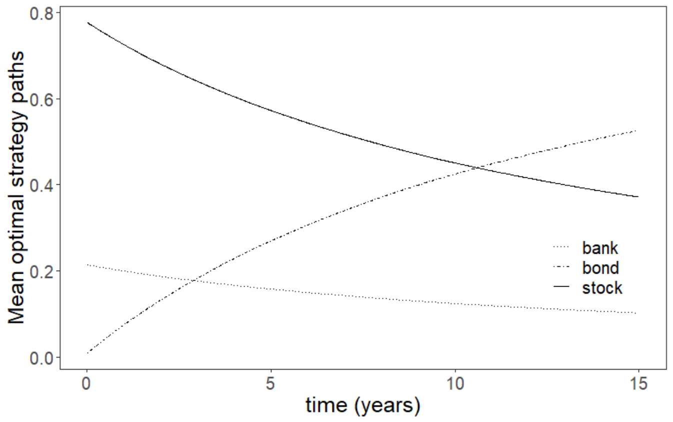

Figure: Average path of 10000 optimal investment strategy paths (bank, bond and stock) in the case of the exponential utility function. **Here we let** $\beta = 0.1$.

Effect of Robustness on Optimal Strategies

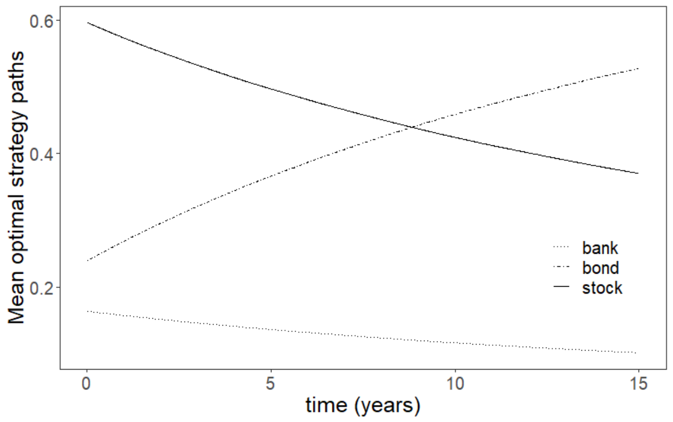

Figure: Average path of 10000 optimal investment strategy paths (bank, bond and stock) in the case of the exponential utility function. **Here we let** $\beta = 0.4$.

Effect of Robustness on Optimal Strategies

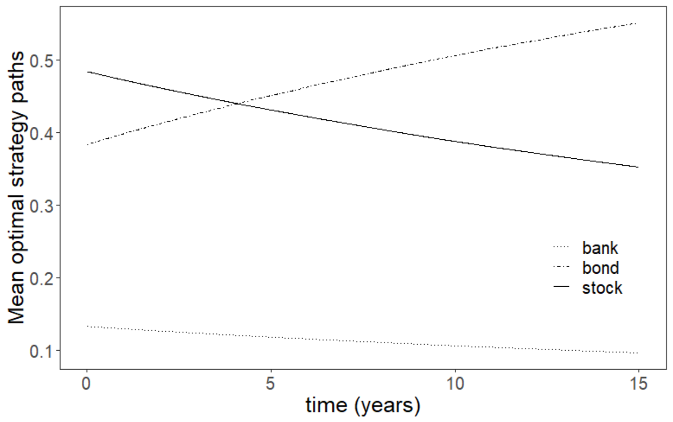

Figure: Average path of 10000 optimal investment strategy paths (bank, bond and stock) in the case of the exponential utility function. **Here we let** $\beta = 0.7$.

Effect of inflation

- Inflation plays a crucial role in our paper.
- In order to study its effect, we define the ratio

$$R(t) := \frac{V(t,y)}{\widetilde{V}(t,y)},$$

where:

- $V(t,y)$ stands for the value function of the robust control problem in the case where the inflation-adjusted bond is considered in the market;
- $\widetilde{V}(t,y)$ stands for the robust value function of the problem when the inflation-indexed bond is excluded from the market.

Effect of inflation

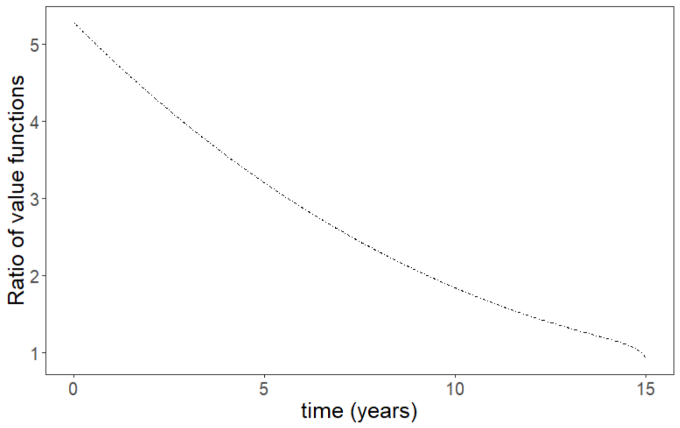

Figure: Ratio of value functions in the case of the exponential utility function.

Thank you for the attention!