# INFINITESIMAL PERTURBATION ANALYSIS ESTIMATES FOR MOMENTS OF THE SYSTEM TIME OF AN M/M/1 QUEUE

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(Received April 1986; revisions received November 1987, May 1988; accepted September 1988)

We consider infinitesimal perturbation analysis estimates for the derivative of moments of the system time in an M/M/1 queue in steady state. The unbiasedness of these estimates is established for derivatives with respect to the mean service time and the arrival rate. A similar result is obtained for the system time of the $i$th customer in a GI/G/1 queue with fixed initial conditions.

Perturbation Analysis (**PA**) is a method for sensitivity analysis of discrete event systems that provides gradient estimates of performance criteria with respect to parameters, from a *single* sample path. This is achieved by calculating sample path derivatives and using them to estimate the derivative of the performance criterion.

The origins of the method are in a paper by Ho, Eyler and Chien (1979) concerning the problem of buffer optimization in production lines. Research in this area has branched in two directions. The first is that of extending the applicability of the method to as complicated systems as possible. Since, in that case, the analytic investigation of the properties of **PA** estimates seemed hopeless, extensive simulation studies were carried out to validate the method (e.g., see Ho et al. 1984 and the references therein). The second direction is that of establishing analytic results about the statistical properties of **PA** estimates (e.g., unbiasedness and consistency). In order to do that, it was necessary to focus on simple and analytically tractable systems. In that spirit, Suri and Zazanis (1988) established the strong consistency of infinitesimal perturbation analysis (**IPA**) estimates for the derivative of the system time of an M/G/1 system under some conditions. This result was extended in Zazanis and Suri (1986) for a class of GI/G/1 queueing systems. In both papers, only the mean system time was considered.

The object of this paper is to clarify further some of the concepts involved in applying this method to single server queues, to extend the results to higher moments, and to demonstrate the unbiasedness of the estimates obtained, both from sample paths of *finite length* and in *steady state*. The distinction between unbiasedness in these two cases is stressed here (see also Zazanis 1987) because many results holding in the former case may not be valid in the latter (see, for example Stoyan 1983, p. 160–165). In this paper, unbiasedness for finite length estimates is established for GI/G/1 queues, while unbiasedness in steady state is established only for the M/M/1 queue by means of a direct computation. The restrictive Markovian hypothesis is the price paid for the directness and the simplicity of the arguments that establish the steady state result.

**IPA** estimates are known to be biased for a number of systems, performance measures and parameters of interest. For a discussion of this question the reader is referred to Heidelberger et al. (1988).

## 1. SYSTEM MODEL

Consider an M/M/1 queue with arrival rate $\lambda$ and mean service time $\theta$, and denote the utilization by $\rho = \lambda\theta$. We will consider the family of systems obtained when $\lambda$ is fixed and $\theta$ varies in $[a, b]$ where $0 < a < b < 1/\lambda$. Starting with an idle system, for any given value $\theta$, the sequence of system times converges weakly to the random variable $T_\theta$. Our performance measure will be $E[T_\theta^k]$, the $k$th moment of the system time of a customer entering a system in steady state. It is well known that the steady state system time of a customer is exponentially distributed

$$P(T_\theta \leq x) = 1 - e^{-x(1-\rho)/\theta}. \tag{1}$$

The $k$th moment of this exponential distribution is given by the expression

$$E[T_\theta^k] = k!\left(\frac{\theta}{1-\rho}\right)^k \tag{2}$$

*Subject classifications:* Queues, output process: perturbation analysis estimates. Queues, tandem: perturbation analysis for the M/M/1 queue. Queues, statistical analysis: unbiased perturbation analysis estimates.



0030-364X/90/3802-0364 $01.25
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and thus, by differentiating with respect to (w.r.t.) $\theta$

$$\frac{d}{d\theta} E[T_\theta^k] = k!\, k \frac{\theta^{k-1}}{(1-\rho)^{k+1}}. \tag{3}$$

In Section 4, we show that the **IPA** estimate in steady state is unbiased by computing its expected value and comparing it with (3).

Similarly, we will consider the family of systems obtained when $\theta$ is fixed and $\lambda$ varies in $[c, d]$ with $0 < c < d < 1/\theta$, and we will show that the **IPA** estimate of the derivative of $E[T_\lambda^k]$ w.r.t. $\lambda$ is also unbiased, by comparing its expected value in steady state with the true value

$$\frac{d}{d\lambda} E[T_\lambda^k] = k!\, k \left(\frac{\theta}{1-\rho}\right)^{k+1} \tag{4}$$

## 2. PERTURBATION ANALYSIS ESTIMATES FOR SYSTEM TIME MOMENTS

In this section, we obtain **IPA** estimates by considering a family of stochastic systems indexed by $\theta$, on the same probability space in an appropriate way. The reader is referred to Suri and Zazanis and (1988), Zazanis and Suri (1986), and Glynn (1987) for a further discussion of some of the ideas in this section.

Let $E_i$, $i = 1, 2, \ldots$, be an i.i.d. sequence of exponentially distributed random variables (r.v.) with unit mean. Let

$$X_{\theta,i} = \theta E_{2i-1}$$

$$A_i = \frac{1}{\lambda} E_{2i}, \quad i = 1, 2, \ldots \tag{5}$$

and let

$$T_{\theta,i} = X_{\theta,i} + \max[0, T_{\theta,i-1} - A_{i-1}], \quad i = 2, 3, \ldots \tag{6}$$

$$T_{\theta,1} = X_{\theta,1}.$$

Then $T_{\theta,i}$ represents the system time of the $i$th customer, $C_i$, in an M/M/1 queue in which the service time of $C_i$ is $X_{\theta,i}$, the interarrival time between $C_i$ and $C_{i+1}$ is $A_i$, and $C_1$ arrives to an empty system.

Also, let us designate by $L_{\theta,i}$ the index of the customer who initiates the busy period in which $C_i$ belongs. (A more explicit definition of $L_{\theta,i}$ in terms of the governing sequences $X_{\theta,i}$ and $A_i$, $i = 1, 2, \ldots$, is given in the Appendix). Evidently $1 \leq L_{\theta,i} \leq i$, the equality on the right holds in the case where $C_i$ finds the system empty, and the equality on the left holds in the case where the system has never emptied since the arrival of $C_1$. The subscript $\theta$ in $X_{\theta,i}$, $T_{\theta,i}$, and $L_{\theta,i}$ is used to indicate that we think of $\theta$ as varying in $[a, b]$, while $\lambda$ has a fixed value. When we want to think of $\theta$ as fixed and $\lambda$ varying in $[c, d]$, we will use the notation $A_{\lambda,i}$, $T_{\lambda,i}$, and $L_{\lambda,i}$.

An alternative expression for the system time of $C_i$ is given by

$$T_{\theta,i} = X_{\theta,i} + \sum_{j=L_{\theta,i}}^{i-1} (X_{\theta,j} - A_j) \tag{7}$$

(see Figure 1), with the sum in (7) taken to be equal to zero if ill-defined. From (6) one can see by induction that, for $h > 0$, $T_{\theta+h,i} > T_{\theta,i}$ for all $i$. Also, intuitively it is clear that $L_{\theta+h,i} \leq L_{\theta,i}$ for $i = 1, 2, \ldots$. (A rigorous argument for that is given in the

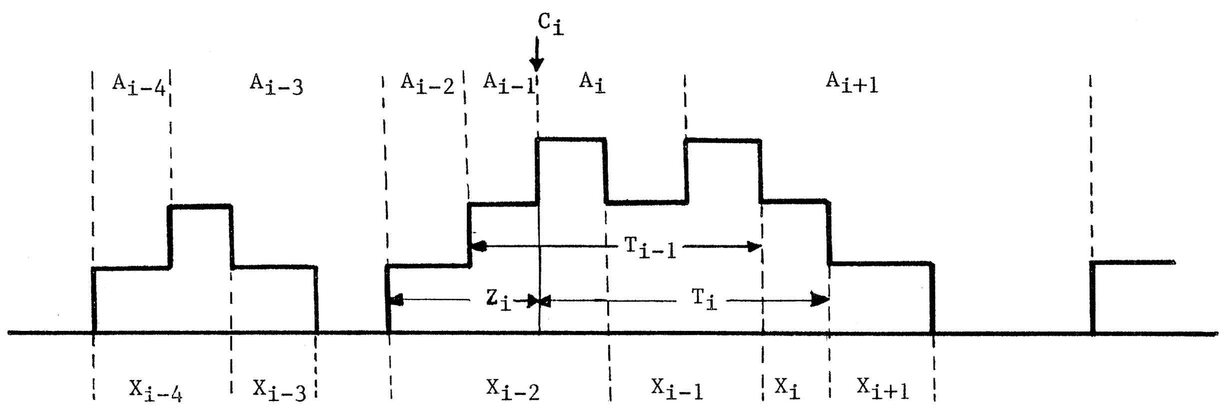


**Figure 1.** For the $i$th customer, here $L_i = i - 2$, $Z_i$ and $T_i$ are shown.

Appendix.) Hence, from (7) we get

$$0 \leqslant \frac{T_{\theta+h,i} - T_{\theta,i}}{h}$$

$$= \frac{1}{h}\left(X_{\theta+h,i} - X_{\theta,i} + \sum_{j=L_{\theta+j,i}}^{i-1} (X_{\theta+h,j} - A_j) - \sum_{j=L_{\theta,i}}^{i-1} (X_{\theta,j} - A_j)\right) \quad (8)$$

with ill-defined sums again taken to be equal to zero. Also, from (5) we have $X_{\theta+h,j} = X_{\theta,j}(1 + h/\theta)$, and therefore from (8) we get

$$\frac{T_{\theta+h,i} - T_{\theta,i}}{h} = \frac{1}{\theta} \sum_{j=L_{\theta+h,i}}^{i} X_{\theta,j} + \frac{1}{h} \sum_{j=L_{\theta+h,i}}^{L_{\theta,i}-1} (X_{\theta,j} - A_j). \quad (9)$$

When $h \to 0$, $L_{\theta+h,i} \to L_{\theta,i}$ w.p.1. Since $L_{\theta,i}$ takes on only integer values there exists, for almost all $\omega$, $\delta(\omega) > 0$ (depending on $i$ and $\theta$) such that $L_{\theta+h,i}(\omega) = L_{\theta,i}(\omega)$ for $0 \leqslant h < \delta(\omega)$ and thus

$$\lim_{h \to 0} \sum_{j=L_{\theta+h,i}}^{i} X_{\theta,j} = \sum_{j=L_{\theta,i}}^{i} X_{\theta,j} \quad \text{w.p.1}$$

and

$$\lim_{h \to 0} \frac{1}{h} \sum_{j=L_{\theta+h,i}}^{L_{\theta,i}-1} (X_{\theta,j} - A_j) = 0 \quad \text{w.p.1.}$$

Thus, from (9)

$$\frac{dT_{\theta,i}^k}{d\theta} = \lim_{h \to 0} \frac{T_{\theta+h,i} - T_{\theta,i}}{h}$$

$$= \frac{1}{\theta} \sum_{j=L_{\theta,i}}^{i} X_{\theta,j} \quad \text{w.p.1.} \quad (10)$$

(We omit the case $h < 0$ because it is entirely analogous.) Similarly, if we differentiate $T_{\theta,i}^k$ w.r.t. $\theta$ we get

$$\frac{dT_{\theta,i}^k}{d\theta} = k\, T_{\theta,i}^{k-1} \frac{dT_{\theta,i}}{d\theta}$$

$$= \frac{k}{\theta} T_{\theta,i}^{k-1} \sum_{j=L_{\theta,i}}^{i} X_{\theta,j} \quad \text{w.p.1.} \quad (11)$$

Define the random variable $Z_{\theta,i}$ to be the age of the busy period in which $C_i$ belongs (see Figure 1). More precisely, let

$$Z_{\theta,i} = \sum_{j=L_{\theta,i}}^{i-1} A_j. \quad (12)$$

Then, from (7)

$$\sum_{j=L_{\theta,i}}^{i} X_{\theta,j} = Z_{\theta,i} + T_{\theta,i} \quad (13)$$

and from (11) and (13) it follows that

$$\frac{dT_{\theta,i}^k}{d\theta} = \frac{k}{\theta} [T_{\theta,i}^k + T_{\theta,i}^{k-1} Z_{\theta,i}]. \quad (14)$$

The above equation gives the **IPA** estimate for $(d/d\theta)E[T_{\theta,i}^k]$.

It is not hard to show that this estimate is unbiased, that is, for all $i$ and $k$

$$E\left[\frac{dT_{\theta,i}^k}{d\theta}\right] = \frac{d}{d\theta} E[T_{\theta,i}^k]. \quad (15)$$

Indeed, we have, as is shown in the Appendix

$$0 \leqslant \frac{T_{\theta+h,i}^k - T_{\theta,i}^k}{h}$$

$$\leqslant \frac{k}{\theta}\left[\sum_{j=1}^{i} X_{b,j}\right]^k. \quad (16)$$

Since

$$E\left[\sum_{j=1}^{i} X_{b,j}\right]^k < \infty$$

for all $i$ and all $k$ and

$$\lim_{h \to 0} E\left[\frac{1}{h}(T_{\theta+h,i}^k - T_{\theta,i}^k)\right] = \frac{d}{d\theta} E[T_{\theta,i}^k]$$

an appeal to the dominated convergence theorem establishes (15). (We omit again the case $h < 0$, which is similar.)

The sequence $(T_{\theta,i}, Z_{\theta,i})$, $i = 1, 2, \ldots$ is, for each $\theta \in [a, b]$, a discrete time regenerative process (with regeneration points the indices of customers who initiate busy periods), and hence, it converges weakly to the two-dimensional random variable $(T_\theta, Z_\theta)$ (see Crane and Iglehart 1975). Consequently, the sequence of random variables $dT_{\theta,i}^k/d\theta$, $i = 1, 2, \ldots$ converges weakly to

$$\frac{dT_\theta^k}{d\theta} \stackrel{D}{=} \frac{k}{\theta} \{T_\theta^k + T_\theta^{k-1} Z_\theta] \quad (17)$$

where $\stackrel{D}{=}$ denotes equality in distribution. Equation 17 suggests **IPA** estimates using a fixed number of customers in steady state or a fixed number of regenerative cycles. The reader is referred to Suri and Zazanis for a detailed treatment of this aspect of **IPA**.

This is the **IPA** estimate for the derivative of the $k$th moment of the steady state system time. Of course, the fact that (15) holds for all $i$ does not guarantee that

$$E\left[\frac{dT_\theta^k}{d\theta}\right] = \frac{d}{d\theta} E[T_\theta^k]. \tag{18}$$

That this is the case for the M/M/1 queue is established in the next section by means of a direct computation. For further discussion of these issues, and for sufficient conditions that guarantee unbiasedness of steady state **IPA** estimates, assuming unbiasedness of finite length estimates, the reader is referred to Zazanis.

Derivatives w.r.t. the arrival rate $\lambda$ can be obtained in an entirely analogous fashion. The corresponding expression of the derivative of $T_{\lambda,i}^k$ w.r.t. $\lambda$ is

$$\frac{dT_{\lambda,i}^k}{d\lambda} = \frac{k}{\lambda} T_{\lambda,i}^{k-1}\left[\sum_{j=L_{\lambda,i}}^{i-1} A_{\lambda,j}\right] = \frac{k}{\lambda} T_{\lambda,i}^{k-1} Z_{\lambda,i} \tag{19}$$

where in the above equation we use (12), and the convention that a subscript $\lambda$ indicates that $\theta$ is thought to be fixed and $\lambda$ to vary. It is easy to establish in the same way that

$$E\left[\frac{dT_{\lambda,i}^k}{d\lambda}\right] = \frac{d}{d\lambda} E[T_{\lambda,i}^k], \quad i = 1, 2, \ldots \tag{20}$$

and to see that the sequence of random variables in (19) converges weakly to

$$\frac{dT_\lambda^k}{d\lambda} \overset{D}{=} \frac{k}{\lambda} T_\lambda^{k-1} Z_\lambda. \tag{21}$$

The above argument can be generalized immediately to the case where the first customer $C_1$ arrives to a system with an initial workload $W_0$, where $W_0$ is a r.v. independent of the sequence $E_i$, $i = 1, 2, \ldots$ and with distribution $F()$, which does not depend on $\theta$ or $\lambda$. Then (6) becomes

$$T_{\theta,1} = X_{\theta,1} + W_0 \tag{22}$$

and we distinguish three cases. In the first case, where $L_{\theta+h,i} > 1$, all other equations remain unchanged. In the second case, i.e., $L_{\theta,i} = 1$, where the system never empties because the arrival of the first customer (even with $h = 0$) requires the addition of $W_0$ to the left-hand side of (7), but no other changes, since we are only interested in the difference $T_{\theta+h,i} - T_{\theta,i}$. In the third case, $L_{\theta,i} > 1$, and $L_{\theta+h,i} = 1$ reduces to the first for sufficiently small $h$. Thus, (18) has been established for arbitrary initial conditions not depending on $\theta$ or $\lambda$.

Finally, we have essentially made no use of the fact that $X_{\theta,i}$ and $A_i$ are exponentially distributed. Thus, the results of this section can be generalized immediately to GI/G/1 systems satisfying the following conditions.

i. The service and interarrival time distributions depend on scale parameters $\theta$ and $\lambda$, respectively.

ii. The $k$th moment of the service time distribution $E[X_1^k]$ is finite. For derivatives w.r.t $\lambda$, $E[A_1^k]$ should also be finite. (We do not need the $(k + 1)$th moment since we do not deal with steady state results in this section.)

iii. Following Whitt (1974), let $U_n = U_{n-1} + A_{\lambda,n-1}$, $n = 2, 3, \ldots$, $U_1 = 0$, be the time of the $n$th arrival and $D_n$ be the time of the $n$th departure. Let $K = \bigcap_{j=1}^{\infty} \bigcap_{k=1}^{\infty} \{|U_j - D_k| > 0\}$. Then, the interarrival and service distributions should be such that $P(K) = 1$. This guarantees that $L_{\theta+h,i}(\omega) = L_{\theta,i}(\omega)$ for $0 \leq h < \delta(\omega)$, with $\delta(\omega) > 0$, for almost all $\omega$. In practice, for GI/G/1 systems, a sufficient condition is that either the interarrival or the service distribution be nonatomic.

## 3. UNBIASEDNESS OF PERTURBATION ANALYSIS ESTIMATES FOR THE M/M/1 QUEUE IN STEADY STATE

This section consists of two theorems where we show that, for the M/M/1 queue, Equation 18 and its counterpart for derivatives w.r.t the arrival rate $\lambda$ hold, thus establishing the unbiasedness of **IPA** estimates in steady state. Henceforth, we drop the subcript $\theta$ from $T_\theta$ and $Z_\theta$ so that no risk of confusion will arise.

**Theorem 1.** *For an M/M/1 queue in steady state, the infinitesimal perturbation analysis estimate of the derivative of the kth moment of the system time w.r.t. the mean service time,* $(d/d\theta)E[T^k]$, *is unbiased.*

**Proof.** The proof consists simply in evaluating the right-hand side of (17) and comparing it with the value of $(d/d\theta)E[T^k]$ given in (3). To this end, we need to compute $E[T^{k-1}Z]$. This computation is not straightforward because the r.v.'s $T$ and $Z$ are not independent and we proceed as follows.

$T$ is the system time of a customer $C$ who upon arrival finds the system in steady state, and $Z$ is the age of the busy period to which $C$ belongs. For an M/M/1 queue, when the number of customers in the system, $N$, is known, the future is conditionally independent of the past. To evaluate $E[T^{k-1}Z]$, let us condition the expectation upon the number of customers, say $N = n$, that $C$, upon arrival, already finds in the system. Because of the Markovian character of the system, as we argue above, $T$ and $Z$ are conditionally independent given $N$. Hence

$$E[T^{k-1}Z \mid N=n] = E[T^{k-1} \mid N=n]E[Z \mid N=n]. \tag{23}$$

Now $T$, given $N=n$, is the sum of $n+1$ independent exponential r.v.'s with mean $\theta$. Thus the distribution of $T$ is Erlang with $n+1$ stages. Hence

$$E[T^{k-1} \mid N=n] = \int_0^\infty x^{k-1} \frac{1}{\theta n!}\left(\frac{x}{\theta}\right)^n e^{-x/\theta}\,dx = \frac{(n+k-1)!}{n!}\theta^{k-1}. \tag{24}$$

Computing $E[Z \mid N=n]$ would be difficult if the system were not Markovian. Here, however, we can take advantage of the reversibility of the M/M/1 queue and note that the distribution of $Z$ given that $N=n$, is that of the length of a busy period starting with $n$ customers. Moreover, the length of a busy period starting with $n$ customers can be seen as the sum of $n$ i.i.d. random variables each being the length of an ordinary busy period (e.g., see Cox and Smith 1961). Since the expected length of a busy period is $\theta/(1-\rho)$ (e.g., Cox and Smith), we have

$$E[Z \mid N=n] = n\frac{\theta}{1-\rho}. \tag{25}$$

From (23), (24) and (25) we get

$$E[T^{k-1}Z \mid N=n] = \frac{(n+k-1)!}{(n-1)!}\frac{\theta^k}{1-\rho}. \tag{26}$$

Taking the expectation w.r.t. $N$, and using the fact that the steady state probability $N=n$ is $(1-\rho)\rho^n$, we have

$$E[T^{k-1}Z] = \sum_{n=0}^{\infty}(1-\rho)\rho^n E[T^{k-1}Z \mid N=n] \tag{27}$$

which, using (26), gives

$$E[T^{k-1}Z] = \theta^k \sum_{n=0}^{\infty} n(n+1)\ldots(n+k-1)\rho^n = \theta^k \rho \frac{d^k}{d\rho^k}\left(\frac{1}{1-\rho}\right) \tag{28}$$

or

$$E[T^{k-1}Z] = k!\,\theta^k \frac{\rho}{(1-\rho)^{k+1}}. \tag{29}$$

From (2), (17), and (29) we get

$$E\left[\frac{dT^k}{d\theta}\right] = \frac{k}{\theta}\left(k!\frac{\theta^k}{(1-\rho)^k} + k!\frac{\theta^k\rho}{(1-\rho)^{k+1}}\right) = k\,k!\frac{\theta^{k-1}}{(1-\rho)^{k+1}}. \tag{30}$$

Comparing (3) and (30) we establish that the **IPA** estimate is unbiased.

A similar analysis, given in the proof of the next theorem, shows that the **IPA** estimate for the derivative w.r.t the arrival parameter is unbiased as well.

**Theorem 2.** *For an M/M/1 queue in steady state, the **IPA** estimate of the derivative of the kth moment of the system time w.r.t. the arrival rate is unbiased.*

**Proof.** We must establish the equality of the expected value of (21) with (4). We have

$$E\left[\frac{dT^k}{d\lambda}\right] = \frac{k}{\lambda}E[T^{k-1}Z]. \tag{31}$$

But the right-hand side of (31) has been computed in (29). From (29) and (31) we get

$$E\left[\frac{dT^k}{d\lambda}\right] = k\,k!\frac{\theta^{k+1}}{(1-\rho)^{k+1}} \tag{32}$$

Comparing (32) with (4) concludes the proof of Theorem 2.

## APPENDIX

We start with a more precise definition for $L_{\theta,i}$. Let

$$L_{\theta,i} = \operatorname*{argmax}_{k\in\{1,2,\ldots,i\}}\left\{\sum_{j=k}^{i-1} X_{\theta,j} - A_j\right\} \tag{A1}$$

that is, let $L_{\theta,i}$ be the value of $k$ which maximizes the sum $\sum_{j=k}^{i-1}(X_{\theta,j} - A_j)$ with the convention that an ill-defined sum is equal to zero. (If the maximum is reached for more than one index, a null event for the M/M/1 system, then we will choose the smallest such $k$). From Figure 1, we can see that this definition is equivalent to the more intuitive one given in Section 2. From (A1) it follows that $L_{\theta+h,i} \leq L_{\theta,i}$ for all $\omega$ and $h > 0$, and hence, by definition of $L_{\theta,i}$ that

$$\sum_{j=L_{\theta+h,i}}^{i-1}(X_{\theta,j} - A_j) \leq \sum_{j=L_{\theta,i}}^{i-1}(X_{\theta,j} - A_j)$$

or

$$\sum_{j=L_{\theta+h,i}}^{L_{\theta,i}-1} (X_{\theta,j} - A_j) \leq 0 \tag{A2}$$

with equality holding only in the case where the above sum is ill-defined.

From (A2) and (9) it follows that

$$0 \leq \frac{1}{h}(T_{\theta+h,i} - T_{\theta,i})$$

$$\leq \frac{1}{\theta} \sum_{j=L_{\theta+h,i}}^{i} X_{\theta,j}.$$

We reinforce the above inequality as

$$0 \leq \frac{1}{h}(T_{\theta+h,i} - T_{\theta,i}) \leq \frac{1}{\theta} \sum_{j=1}^{i} X_{b,j}. \tag{A3}$$

We are ready to establish (16). We start with

$$0 \leq \frac{1}{h}[T^k_{\theta+h,i} - T^k_{\theta,i}]$$

$$\leq \frac{1}{h}\left[\sum_{m=0}^{k-1} T^m_{\theta+h,i} T^{k-m-1}_{\theta,i}\right][T_{\theta+h,i} - T_{\theta,i}]$$

$$\leq k T^{k-1}_{b,i} \frac{1}{h}[T_{\theta+h,i} - T_{\theta,i}] \tag{A4}$$

where, in the above inequalities, we have used the fact that $T_{\theta,i} < T_{\theta+h,i} \leq T_{b,i}$ for all $\omega$, when $\theta + h \leq b$. Also, from (7) we obviously have

$$T_{b,i} \leq \sum_{j=1}^{i} X_{b,j}. \tag{A5}$$

From the above and (9) then, (16) follows.

## ACKNOWLEDGMENT

This work was partly supported by the U.S. Office of Naval Research contracts N0001475-C-0648 and N0001479-C-0776, and by National Science Foundation grant ENG82-13680. The research for this work was conducted while the author was at the Division of Applied Sciences, Harvard University. The author thanks the associate editor and an anonymous referee for comments that led to improvements in the clarity of the paper.

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