stochastic processes and their applications

ELSEVIER Stochastic Processes and their Applications 74 (1998) 195-201

A Palm calculus approach to functional versions of Little's law

Michael A. Zazanis [1]

Department of Statistics, Athens University of Economics and Business, Athens 104 34, Greece

Received 2 August 1996; received in revised form 12 November 1997

Abstract

Using the generalized Campbell theorem, a functional (distributional) version of Little's law is obtained for general FIFO systems with non-anticipating arrivals. Our approach generalizes existing results and, in particular, points out the intimate relation between the Palm–Khintchine equations and the distributional law of Little. A multidimensional version of the Palm–Khintchine equations and the related version of Little's law for multiclass systems is also given, as well as an ordinal version for distributions.

Keywords: Stationary point processes; Palm distributions; Stationary queues; Little's law

1. Introduction

The distributional version of Little's law is a relation connecting the customer-stationary system time distribution with the stationary distribution of the number of customers in a queueing system. Such a relation will hold for a class of customers provided that (i) the customers of that class depart in the order they arrive (FIFO property) and (ii) they satisfy a certain "lack of anticipation" property (on which we will elaborate in Section 3). Early results on the distributional law of Little (for systems with Poisson arrivals) can be found in Haji and Newell (1971) and Brumelle (1972). For more recent work see Franken et al. (1982) (pp. 110–111), Miyazawa (1979), Keilson and Servi (1988) (for applications in the analysis of multiclass and vacation systems with Poisson arrivals), and Bertsimas and Mourtzinou (1993) where the distributional law is proposed as a tool for the analysis of a number of systems including multiclass queues.

In this paper we present a new, significantly shorter and more general proof of the functional (distributional) version of Little's law based on the generalized Campbell theorem which highlights the intimate connection between the Palm–Khintchine equations and the functional law which so far had been overlooked. Also, an "ordinal" (Halfin and Whitt, 1988) version of Little's law for distributions is given connecting

[1] Research supported in part by NSF grant EEC-9527519.

0304-4149/98/$19.00
PII S0304-4149(97)00116-6

the Palm distribution of the number of customers an arriving customer sees in the system to the Palm distribution of the number of arrivals during that customer's stay in the system.

2. The Palm–Khintchine equations

Suppose that on the probability space $(\Omega, \mathscr{F}, P)$, endowed with a measurable flow $\{\theta_t;\ t\in\mathbb{R}\}$, a real-valued simple point process A has been defined. We will assume that P is invariant under θ, i.e. $P\theta_t^{-1}=P$ and that A is compatible with the flow, i.e. $A(B,\theta_t\omega)=A(B+t,\omega)$ for all Borel sets B of the real line, and thus stationary. $\{T_n;\ n\in\mathbb{Z}\}$ denotes the points of A, P^0 the Palm transformation of P with respect to A, and E^0 the expectation with respect to P^0. The connection between the stationary and Palm distributions of the number of points in an interval $(0,t]$ is given by the following relations, known as the Palm–Khintchine equations, where λ denotes the rate of A:

$$P(A(0,t]>j)=\lambda\int_0^t P^0(A(0,u]=j)\,\mathrm{d}u, \quad j=0,1,2,\ldots . \tag{1}$$

For a proof we refer the reader to Khintchine's own account (Khintchine, 1955) and to Baccelli and Brémaud (1994). Define the probability generating functions $\boldsymbol{\Phi}(z,t)=Ez^{A(0,t]}$, $\boldsymbol{\Phi}^0(z,t)=E^0z^{A(0,t]}$. By virtue of Eq. (1) the above probability generating functions are related as follows:

$$\boldsymbol{\Phi}(z,t)=1-\lambda(1-z)\int_0^t \boldsymbol{\Phi}^0(z,u)\,\mathrm{d}u. \tag{2}$$

The above result can be extended easily to a "multidimensional" point process, i.e. a marked point process $\{T_n,\boldsymbol{c}_n\}$ with the mark sequence, $\{\boldsymbol{c}_n\}$, taking values in a finite set which without loss of generality, we will identify with the set of standard unit vectors in $\mathbb{R}^d$, $\{\boldsymbol{e}_1,\boldsymbol{e}_2,\ldots,\boldsymbol{e}_d\}$. Denote by $\{T_n^i\}$ the ith stream of points (with mark $\boldsymbol{e}_i$) and by $0<\lambda_i<\infty$, its rate. We assume that the marks have the "shadowing property", i.e. $\boldsymbol{c}_n=\boldsymbol{c}_0\circ\theta_{T_n}$ so that, under probability measure P, the marked point process is stationary and we will denote by P_i^0 the Palm transformation of P with respect to $\{T_n^i\}$. Also, $A_i(B)=\sum_{n\in\mathbb{Z}}\mathbf{1}(T_n\in B,\boldsymbol{c}_n=\boldsymbol{e}_i)$, and E_i^0 denotes expectation with respect to P_i^0. At the heart of our treatment of the multidimensional Palm–Khintchine equations lies the following elementary algebraic identity: Let $\boldsymbol{n}=(n_1,\ldots,n_d)\in\mathbb{Z}_+^d$, $\boldsymbol{z}=(z_1,\ldots,z_d)\in\mathbb{C}^d$, and define the monomial $\boldsymbol{z}^{\boldsymbol{n}}:=z_1^{n_1}\cdots z_d^{n_d}$. Let $N=n_1+\cdots+n_d$, $\mathscr{E}=\{\boldsymbol{e}_1,\boldsymbol{e}_2,\ldots,\boldsymbol{e}_d\}$ be the set of d standard unit vectors of $\mathbb{R}^d$, and $\{\boldsymbol{f}_j\}_{j=1}^N$ be a sequence of elements of $\mathscr{E}$ such that $\boldsymbol{f}_1+\cdots+\boldsymbol{f}_N=\boldsymbol{n}$. Set

$$\boldsymbol{n}(0)=0, \qquad \boldsymbol{n}(j)=\boldsymbol{f}_1+\cdots+\boldsymbol{f}_j, \qquad \boldsymbol{n}(N)=\boldsymbol{f}_1+\cdots+\boldsymbol{f}_N=\boldsymbol{n}.$$

With the above definitions one can easily check then the telescopic identity

$$\boldsymbol{z}^{\boldsymbol{n}}-1=\sum_{j=1}^N[\boldsymbol{z}^{\boldsymbol{n}(j)}-\boldsymbol{z}^{\boldsymbol{n}(j-1)}]=\sum_{j=1}^N[\boldsymbol{z}^{\boldsymbol{f}_j}-1]\boldsymbol{z}^{\boldsymbol{n}(j-1)}, \tag{3}$$

where, $\boldsymbol{z}^{\boldsymbol{f}_j}=z_k$ iff $\boldsymbol{f}_j$ is the unit vector in the kth direction.

Let $A=(A_1,\dots,A_d)$ and define the probability generating functions $\Phi(z,t):=Ez^{A(0,t]} = E[z_1^{A_1(0,t]}\cdots z_d^{A_d(0,t]}]$, $\Phi_i^0(z,t):=E_i^0 z^{A(0,t]}=E_i^0[z_1^{A_1(0,t]}\cdots z_d^{A_d(0,t]}]$, $i=1,2,\dots,d$. Then

Theorem 1. *The joint probability generating function of the number of points of the d substreams under the stationary probability measure P and under the Palm measures P_i^0, $i=1,\dots,d$, are connected by the relationship*

$$\Phi(z,t)=1-\sum_{i=1}^{d}(1-z_i)\lambda_i\int_0^t \Phi_i^0(z,u)\,\mathrm{d}u. \tag{4}$$

Proof. The following device will be useful in explaining the idea on which our proof is based. Suppose that the points T_n constitute arrival epochs of customers to an infinite capacity "delay box" in which each customer stays precisely t seconds and then leaves. The mark c_n associated with T_n designates the class of this customer. Let $X_i(s)$ designate the number of customers of class i in the system at time s and $X(s)$ the total number of customers of all classes. Consider the *right-continuous versions* of these processes given by $X_i(s)=A_i(s-t,s]$, $X(s)=\sum_{i=1}^d A_i(s-t,s]$, and define the process

$$\begin{aligned}Y(s)&=\int_{\mathbb{R}}\mathbf{1}(0\leqslant s-u<t)[(z^{c_0}-1)z^{A(0,s-u]}]\circ\theta_u A(\mathrm{d}u)\\ &=\sum_{n\in\mathbb{Z}}\mathbf{1}(T_n\leqslant s<T_n+t)(z^{c_n}-1)z^{A(T_n,s]},\end{aligned} \tag{5}$$

where by convention $(u,v]=\emptyset$ when $u\geqslant v$. When all the indicators in Eq. (6) vanish we have of course $Y(s)=X(s)=0$. Suppose now that $X(s)>0$ and let $M_1(s)=\inf\{n\colon T_n\leqslant s<T_n+t\}$, $M_2(s)=\sup\{n\colon T_n\leqslant s<T_n+t\}$, be respectively, the indices of the "oldest" and "youngest" customer in the system (we have of course $X(s)=M_2(s)-M_1(s)+1$). Now let $f_j=c_{M_2(s)+1-j}$, $j=1,2,\dots,M_1(s)$, be the class of the jth youngest customer present in the system and apply the algebraic identity (3) established above to conclude that

$$Y(s)=z^{A(s-t,s]}-1.$$

Taking expectations we obtain

$$EY(0)=Ez^{A(-t,0]}-1=Ez^{A(0,t]}-1, \tag{6}$$

the last equation following by stationarity. On the other hand, from Eq. (5) and the generalized Campbell Theorem (see Baccelli and Brémaud, 1994) we obtain

$$\begin{aligned}EY(0)&=\lambda E^0\int_{\mathbb{R}}\mathbf{1}(0\leqslant u<t)\left(\sum_{i=1}^{d}z_i\mathbf{1}(c_0=e_i)-1\right)z^{A(0,u]}\,\mathrm{d}u\\ &=\sum_{i=1}^{d}\lambda_i(z_i-1)E_i^0\int_0^t z^{A(0,u]}\,\mathrm{d}u=\sum_{i=1}^{d}\lambda_i(z_i-1)\int_0^t\Phi_i^0(z,u)\,\mathrm{d}u.\end{aligned} \tag{7}$$

Comparing Eqs. (7) and (8) completes the proof. □

As we will see presently, functional versions of Little's law can be obtained both for single and for multiclass systems by a straightforward extension of the above ideas. The reader will notice in particular that the above technique extends readily to countable mark spaces.

3. Functional versions of Little's law

Suppose that the point process considered in the previous section constitutes the arrival process of customers to a queueing system. Denote the system time of the nth customer by W_n and let $X(s)=\sum_{n\in\mathbb{Z}} \mathbf{1}(T_n \leqslant s < T_n + W_n)$ denote the number of customers in the system at time s. We will make the following two assumptions on the behavior of the queueing system in question without specifying in any other way its structure.

Assumption A.1 [*FIFO*]. The sequence of system times satisfies: $T_k+W_k \leqslant T_n+W_n P^0$-a.s. when $k\leqslant n$.

Assumption A.2 [*Lack of anticipation*]. Future arrivals neither affect the system time of customers already in the system nor provide any information about it, i.e., W_0 and $\{A(0,u]; u>0\}$ are P^0-independent.

Theorem 2. *Under the above assumptions the stationary number of customers in the system, $X(0)$, and the distribution of the system time of a typical customer, $F^0(x) := P^0(W_0 \leqslant x)$, are related via*

$$Ez^{X(0)} = 1 - \lambda(1-z)\int_0^\infty \overline{F^0}(u)\Phi^0(z,u)\,\mathrm{d}u, \tag{8}$$

where $\Phi^0(z,u) := E^0 z^{A(0,u]}$ and $\overline{F^0} = 1 - F^0$.

Proof. Consider the process $Y(s) := \sum_{n\in\mathbb{Z}} \mathbf{1}(T_n \leqslant s < T_n + W_n)(z-1)z^{A(T_n,s]}$. Because of the FIFO assumption this sum is equal to $\mathbf{1}(X(s)>0)\sum_{k=1}^{X(s)}(z-1)z^{k-1}$ which telescopes to give $Y(s)=z^{X(s)}-1$. Using the generalized Campbell theorem we have

$$Ez^{X(0)} - 1 = EY(0) = \lambda E^0 \int_0^{W_0} (z-1) z^{A(0,u]}\,\mathrm{d}u$$

$$= \lambda(z-1)\int_0^\infty E^0[\mathbf{1}(W_0>u) z^{A(0,u]}]\,\mathrm{d}u$$

$$= \lambda(z-1)\int_0^\infty \overline{F^0}(u)\Phi^0(z,u)\,\mathrm{d}u,$$

where, in the last equation we have used the lack of anticipation Assumption A.2. □

In particular, if $W_0 = t$ w.p.1 then $\overline{F^0}(u) = \mathbf{1}(u<t)$, $X(0) = A(-t,0]$, and Eq. (9) reduces to Eq. (2). We also point out that starting with Eq. (9) and using integration by parts in conjunction with Eq. (2) we obtain the following alternative statement of the distributional law: $Ez^{X(0)} = \int_0^\infty \Phi(z,u)F^0(\mathrm{d}u)$.

The distributional law can be readily extended to multiclass systems (assuming strict FIFO discipline across classes). Using the notation of Section 2, let in addition $X_i(s)$ be the number of customers of class i at time s, $F_i^0(x)$ the waiting time distribution for a typical customer of class $i = 1,2,\dots,d$, and $\mathbf{X}(s) = (X_i(s),\dots,X_d(s))$. Then an analysis paralleling that of Section 2 shows that

$$E\left[\prod_{k=1}^{d} z_k^{X_i(0)}\right] = 1 - \sum_{i=1}^{d} \lambda_i(1-z_i)\int_0^\infty E_i^0\left[\mathbf{1}(W_0>u)\prod_{k=1}^{d} z_k^{A_k(0,u]}\right]\mathrm{d}u.$$

From the above, together with the lack of anticipation assumption we obtain

$$E\mathbf{z}^{\mathbf{X}(0)} = 1 - \sum_{i=1}^{d}\lambda_i(1-z_i)\int_0^\infty \overline{F_i^0}(u)\Phi_i^0(\mathbf{z},u)\,\mathrm{d}u.$$

4. A ordinal version of Little's law for distributions

Halfin and Whitt (1989) established the following ordinal version of Little's law:

$$E^0X(0-) = E^0A(0,W_0] \tag{9}$$

which states that, in steady state, the average number of customers that are present in a queueing system just before the arrival of a typical customer is equal to the average number of arrivals during his system time. In this section we will show that, under additional assumptions, a distributional version of Eq. (10) holds as well, i.e. the (Palm) distribution of the number of customers in the system just before an arrival and the distribution of arrivals during a typical customer's stay in the system are the same.

Theorem 3. *Under the Assumption* A.1,

$$E^0z^{X(0-)} = E^0z^{A(0,W_0]}. \tag{10}$$

If in addition, Assumption A.2 *also holds, then* $E^0z^{X(0-)} = \int_0^\infty \Phi^0(z,u)F^0(\mathrm{d}u)$.

Proof. Define the process

$$\begin{aligned} Y(s) &:= (z-1)\int_{\mathbb{R}} \mathbf{1}(0<s-t\le W_0\circ\theta_t)z^{A(0,s-t)\circ\theta_t}A(\mathrm{d}t) \\ &= (z-1)\sum_{n\in\mathbb{Z}}\mathbf{1}(T_n<s\le T_n+W_n)z^{A(T_n,s)}. \end{aligned} \tag{11}$$

Note that this is the one-dimensional, left-continuous version of Eq. (6). As in Section 3, in view of the FIFO assumption, the sum in Eq. (12) telescopes to give

$$Y(s)=z^{X(s-)}-1. \tag{12}$$

From Mecke's definition of the Palm expectation (with respect to the arrival measure A)

$$\lambda E^0 Y(0)=E\int_{\mathbb{R}} \mathbf{1}(0<s\leqslant 1)Y(0)\circ\theta_s A(\mathrm{d}s)$$

$$=E\int_{\mathbb{R}^2} \mathbf{1}(0<s\leqslant 1, 0<s-t\leqslant W_0\circ\theta_t)(z-1)z^{A(t,s)}A(\mathrm{d}s)A(\mathrm{d}t), \tag{13}$$

where in the last expression we have taken into account that $Y(0)\circ\theta_s=Y(s)$. Write the right-hand side of the above equation as $E\int_{\mathbb{R}} f(t,\theta_t)A(\mathrm{d}t)$ where

$$f(t,\theta_t)=\int_{\mathbb{R}} \mathbf{1}(0<s\leqslant 1, 0<s-t\leqslant W_0\circ\theta_t)(z-1)z^{A(t,s)}A(\mathrm{d}s),$$

and hence,

$$f(t,\theta_0)=\int_{\mathbb{R}} \mathbf{1}(0<s\leqslant 1, 0<s-t\leqslant W_0)(z-1)z^{A\circ\theta_{-t}(t,s)}A\circ\theta_{-t}(\mathrm{d}s).$$

Following Schmidt and Serfozo (1995) apply the generalized Campbell theorem to obtain $\lambda E^0 Y_0=\lambda E^0\int_{\mathbb{R}} f(t,\theta_0)\,\mathrm{d}t$ or

$$E^0 Y(0)=E^0\int_{\mathbb{R}^2} \mathbf{1}(0<s\leqslant 1, 0<s-t\leqslant W_0)(z-1)z^{A\circ\theta_{-t}(t,s)}A\circ\theta_{-t}(\mathrm{d}s)\,\mathrm{d}t$$

$$=E^0\int_{\mathbb{R}^2} \mathbf{1}(0<s+t\leqslant 1, 0<s\leqslant W_0)(z-1)z^{A(0,s)}A(\mathrm{d}s)\,\mathrm{d}t$$

$$=E^0\int_{\mathbb{R}} \mathbf{1}(0<s\leqslant W_0)(z-1)z^{A(0,s)}A(\mathrm{d}s).$$

The left-hand side of the above string of equations is equal to $(E^0 z^{X(0-)}-1)$ (see Eq. (13)). On the other hand, the integral on the right-hand side above is

$$(z-1)\int_{(0,W_0]} z^{A(0,s)}A(\mathrm{d}s)=(z-1)(1+z+\cdots+z^{A(0,W_0]-1})$$

which in turn telescopes to $z^{A(0,W_0]}-1$. Formula (10) follows then directly from the above considerations. □

The extension to multiclass systems (always under the FIFO assumption) is straightforward. As before $E^0 z^{X(0-)}=E^0 z^{A(0,W_0]}$, or equivalently $E^0 z^{X(0-)}=\sum_i \frac{\lambda_i}{\lambda}\int_0^\infty \Phi_i^0(z,u)$ $F^0(\mathrm{d}u)$.

References

Baccelli, F., Brémaud, P., 1994. Elements of Queueing. Springer, New York.

Bertsimas, D., Mourtzinou, G., 1993. Multiclass queueing systems in heavy traffic: An asymptotic approach based on distributional and conservation laws. Working paper, Operations Research Center, MIT.

Brumelle, S.L., 1972. A generalization of $L = \lambda W$ to moments of queue length and waiting times. Oper. Res. 20, 1127–1136.

Franken, P., König, D., Arndt, U., Schmidt, V., 1982. Queues and Point Processes. Wiley, Chichester.

Haji, R., Newell, G.F., 1971. A relation between stationary queue and waiting-time distributions. J. Appl. Probab. 8, 617–620.

Halfin, S., Whitt, W., 1989. An extremal property of the FIFO discipline via an ordinal version of $L = \lambda W$. Commun. Statist. – Stochastic Models 5, 515–529.

Keilson, J., Servi, L.D., 1955. A distributional form of Little's law. Oper. Res. Lett. 7, 223–227.

Khintchine, Yu, A., 1955. Mathematical methods in the theory of queues. Charles Griffin & Co., London.

Miyazawa, M., 1979. A formal approach to queueing processes in the steady state and their applications. J. Appl. Probab. 16, 332–346.

Schmidt, V., Serfozo, R., 1995. Campbell's formula and applications to queueing. In: Dshalalow, J.H. (Ed.), Advances in Queueing: Theory, Methods, and Open Problems. CRC Press, Boca Raton, pp. 225–242.