Infinite Server Queues with Synchronized Departures Driven by a Single Point Process

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Abstract

We analyze an infinite-server queueing model with synchronized arrivals and departures driven by the point process $\{T_n\}$ according to the following rules. At time T_n, a single customer (or a batch of size β_n) arrives to the system. The service requirement of the ith customer in the nth batch is $\sigma_{i,n}$. All customers enter service immediately upon arrival but each customer leaves the system *at the first epoch of the point process $\{T_n\}$ which occurs after his service requirement has been satisfied.* For this system the queue length process and the statistics of the departing batches of customers are investigated under various assumptions for the statistics of the point process $\{T_n\}$, the incoming batch sequence $\{\beta_n\}$, and the service sequence $\{\sigma_{i,n}\}$. Results for the asymptotic distribution of the departing batches when the service times are long compared to the interarrival times are also derived.

KEYWORDS: STATIONARY AND ERGODIC POINT PROCESSES, GATED QUEUES, INFINITE SERVER QUEUES
SHORT TITLE: SYNCHRONIZED INFINITE SERVER QUEUES

1 Model Description: Dynamics and existence of a stationary version of the process

Consider a system where groups (or batches) of customers arrive at the epochs of a point process $\{T_n; n \in \mathbb{Z}\}$ defined on the whole real line. The nth group arrives at time T_n and consists of β_n customers. The ith customer of the nth group, which we will denote by $C_{i,n}$, $1 \leq i \leq \beta_n$, $n \in \mathbb{Z}$, remains in the system for $\sigma_{i,n}$ time units, and then departs *at the next arrival point after his service completion,* i.e. at time $T_{L(i,n)}$ where $L(i,n) := \inf\{k \in \mathbb{Z} : T_k > T_n + \sigma_{i,n}\}$.

In more descriptive terms we envision a shuttle bus which arrives at a certain facility at the epochs $\{T_n\}$ of a stationary and ergodic point process. Each time the shuttle bus arrives, it brings along a new group of passengers and delivers them to the facility. The ith passenger of the nth group will stay in this facility for $\sigma_{i,n}$ time units, and then he will move on to a waiting area from where he will be picked up by the *first shuttle that arrives.* We assume that the facility, the waiting area, and the shuttle, all have infinite capacity so that a new group of passengers can always be delivered to the facility and a departing shuttle will always be able to take along all passengers waiting to leave. We also assume that, when the shuttle arrives to the facility at time $\{T_n\}$, the new group of passengers is delivered to the facility instantaneously and the group of passengers waiting to leave also boards the shuttle instantaneously. We will denote the size of the departing group with the nth shuttle by χ_n.

In the sequel we will be referring to passengers in the facility as *customers in service*, and passengers in the waiting area, waiting for the shuttle, as *customers in the output buffer*. The facility together with the waiting area will be referred to as the system. Let $X(t)$ denote the number of customers in the system, $Y(t)$ the number of customers in service, and $Z(t)$ the number of customers in the output buffer at time t. If we denote by $\{R(t); t \in \mathbb{R}\}$ the *forward recurrence time process* associated with the point process $\{T_n\}$, i.e. $R(t) = \sum_{n\in\mathbb{Z}} \mathbf{1}(T_n \le t < T_{n+1})(T_{n+1} - t)$, then the number of customers in the system, the number of customers in service, and the number of customers in the output buffer can be expressed as follows

$$X(t) = \sum_{k=-\infty}^{\infty} \sum_{i=1}^{\beta_k} \mathbf{1}(T_k \le t < T_k + \sigma_{i,k} + R(T_k + \sigma_{i,k})) \,, \tag{1a}$$

$$Y(t) = \sum_{k=-\infty}^{\infty} \sum_{i=1}^{\beta_k} \mathbf{1}(T_k \le t < \sigma_{i,k} + T_k) \,, \tag{1b}$$

$$Z(t) = \sum_{k=-\infty}^{\infty} \sum_{i=1}^{\beta_k} \mathbf{1}(T_k + \sigma_{i,k} \le t < T_k + \sigma_{i,k} + R(T_k + \sigma_{i,k})) \,. \tag{1c}$$

Note that the above processes have been defined to have *right–continuous sample paths*. In particular we will denote by $X_n := X(T_n-)$, $Y_n := Y(T_n-)$, $Z_n := Z(T_n-)$, the corresponding values as seen by an arriving shuttle. It is easy to see that

$$X_n = \sum_{k=-\infty}^{n-1} \sum_{i=1}^{\beta_k} \mathbf{1}(T_k + \sigma_{i,k} > T_{n-1}) \,, \tag{2a}$$

$$Y_n = \sum_{k=-\infty}^{n-1} \sum_{i=1}^{\beta_k} \mathbf{1}(T_k + \sigma_{i,k} > T_n) \,, \tag{2b}$$

$$Z_n = \sum_{k=-\infty}^{n-1} \sum_{i=1}^{\beta_k} \mathbf{1}(T_{n-1} < T_k + \sigma_{i,k} \le T_n) \,. \tag{2c}$$

The above describes succinctly the dynamics of the process. It remains to be shown that, under natural stochastic assumptions, there exists a unique stationary version of this process. This is done in the next section, together with an analysis of the stationary number of customers in the system.

While this model has not been studied before in the literature, there is of course a related literature regarding infinite server queues. For general results on infinite server queues we refer the reader to [15], [3], and [2]. More specifically, infinite server queues with batch arrivals have been considered in [14], [7], [8], [9], [10]. We also mention the time–varying systems considered in [5] and [1] as well as the network of queues considered in [11]. Finally, in [13] and [12] the reader can also find results regarding matrix analytic techniques for the numerical computation of performance characteristics.

1.1 The stationary version of the process

For standard definitions regarding stationary point processes we refer the reader to [2]. We start with a probability space $(\Omega, \mathscr{F}, P)$ and a measurable flow $\{\theta_t\}$ on $(\Omega, \mathscr{F})$ such that P is invariant under θ, i.e. $P \circ \theta_t = P$ for all $t \in \mathbb{R}$. We also assume that a simple point process $\{T_n\}$, with corresponding counting measure N, has been defined on this space and that it is compatible with the flow $\{\theta_t\}$. Thus $N(B, \omega) = \sum_{n \in \mathbb{Z}} \mathbf{1}(T_n(\omega) \in B)$ for all $B \in \mathscr{B}(\mathbb{R})$ and $N(B, \theta_t(\omega)) = N(B + t, \omega)$ (see [2]).

Hence, under the probability measure P, the point process is stationary and we will assume it to have finite rate $\lambda > 0$. We use the standard numbering convention for the points of the process according to which T_0 is the first point to the left of, or precisely at, zero, i.e. $P(T_0 \leq 0 < T_1) = 1$. We denote by $\tau_n := T_{n+1} - T_n$ the time between arrivals. Also, P^0 denotes the Palm transformation of the measure P with respect to the point process. This can be done via Mecke's definition by letting

$$P^0(A) = \lambda E \sum_{\{n \in \mathbb{Z}: 0 < T_n \leq 1\}} \mathbf{1}(\theta_{T_n}(A))$$

for any $A \in \mathscr{F}$. Suppose that, in addition to the point process $\{T_n\}$, a stationary sequence of random elements $\{(\beta_n; \sigma_{1,n}, \sigma_{2,n}, \ldots, \sigma_{\beta_n,n}); n \in \mathbb{Z}\}$ has been defined on the probability space $(\Omega, \mathscr{F}, P)$. In fact, if we let $\mathbb{S}$ be the space of all sequences with non-negative elements, finitely many of which are non-zero, and $\mathscr{S}$ the collection of Borel sets of $\mathbb{S}$, consider a random element $S_0 : (\Omega, \mathscr{F}) \to (\mathbb{S}, \mathscr{S})$ and denote its components as $(\sigma_{1,0}, \sigma_{2,0}, \sigma_{3,0}, \ldots)$. The batch size is $\beta_0 := \inf\{i : \sigma_{j,0} = 0 \text{ for all } j > i\}$. If we set $S_n := S_0 \circ \theta_{T_n}$ then $\beta_n = \beta_0 \circ \theta_{T_n}$ and $\sigma_{i,n} = \sigma_{i,0} \circ \theta_{T_n}$. We thus have a stationary sequence of service times for the arriving batches compatible with the flow θ. We will also assume that $P^0(\beta_0 \geq 1) = 1$.

In order to show the existence of the stationary regime we consider the process $\{\widetilde{Y}(t); t \in \mathbb{R}\}$ defined on the same probability space via the expression

$$\widetilde{Y}(t) = \sum_{n \in \mathbb{R}} \mathbf{1}(T_n \leq t < T_n + \max_{i=1,2,\ldots,\beta_n} \{\sigma_{i,n}\}). \tag{3}$$

Note that the system defined by the above expression is an ordinary $G/G/\infty$ system. Also, since $P^0(\beta_0 < \infty) = 1$, it is easy to see that the difference between the sets $\{\omega : \widetilde{Y}(t) < \infty,\ t \in \mathbb{R}\}$ and $\{\omega : Y(t) < \infty,\ t \in \mathbb{R}\}$ is a set of probability zero. Furthermore, the process $\widetilde{Y}$ is finite with probability 1 provided that

$$E^0 \max_{i=1,2,\ldots,\beta_0} \{\sigma_{i,0}\} < \infty \tag{4}$$

(e.g. see [2]). Hence, provided that condition (4) holds, $Y_0 = X_0 < \infty$ P^0–a.s.

1.2 Notation

We will study this system under various assumptions for the distributional aspects of the input and service process. To this end, we will introduce a Kendall-type descriptor for these systems, namely $\langle A, S, B \rangle$, where A specifies the statistics of the arrival epochs, S the statistics of the service requirement for each customer, and B the statistics of the batch size. This descriptor will be used mostly in the case where the input process is a renewal process. In this case A will denote the inter-arrival distribution, S the distribution of the service requirements which will be assumed i.i.d., and B the distribution of the batch sizes (also assumed i.i.d.). Interarrival times, service times, and batch sizes are assumed to be independent of each other. Thus, for instance, $\langle M_\lambda, \delta_a, \text{Geo(p)}\rangle$ will denote a system where customers arrive at the epochs of a Poisson process with rate λ in batches, independent of the arrival process, geometrically distributed with probability of "success" p, and their service requirements are deterministic and equal to a. Similarly, $\langle GI, M_\mu, \delta_1, \rangle$ is a system where customers arrive according to a renewal process (with general interarrival time distribution) in batches of size 1 and whose service requirements are i.i.d. exponential random variables with rate μ. On the other hand, the notation $\langle G, G, G\rangle$ will refer to a system where the arrivals, batch sizes, and service requirements are jointly stationary with no independence assumptions made, whereas $\langle GI, GI, GI\rangle$ refers to the case where arrivals are renewal, batch sizes and batch requirements are i.i.d. and all these processes are mutually independent. $\langle G, GI, GI\rangle$ will refer to a system where arriving batch sizes and service requirements are both i.i.d., independent of each other and of the arrival process $\{T_n\}$, which however will be assumed to be an arbitrary stationary point process under P with rate $\lambda \in (0, \infty)$.

Also, since we will study in detail the departure process from this system, and in particular will pay attention to second-order characteristics of this process, we will use the symbols Var and Cov to denote the variance and covariance of various quantities *with respect to the stationary probability measure P* and the symbols Var^0, Cov^0 to denote the corresponding variances and covariances *with respect to the Palm probability measure P^0.*

1.3 The expected number in the system in the stationary framework

Let us now proceed to compute the expected number of customers in the system. We will establish the following

Proposition 1. *In the system $\langle G, G, G\rangle$ the expected number of customers in stationarity is given by*

$$EX(0) = \lambda E^0 \left[\sum_{i=1}^{\beta_0} \sigma_{i,0} + R(0) \circ \theta_{\sigma_{i,0}}\right]. \tag{5}$$

If we further assume that the service requirements are independent, identically distributed random variables with distribution $G(x) := P^0(\sigma_0 \le x)$, and the arriving batch sizes $\{\beta_n\}$ are i.i.d. random variables, and if we suppose that the arrival point process, the batch sizes, and the service requirements are independent of each other (the system $\langle G, GI, GI\rangle$ according to our notation) then the expected number of customers is given by

$$EX(0) = \lambda E^0 \beta_0 E^0[\sigma_{1,0} + R(0) \circ \theta_{\sigma_{1,0}}]. \tag{6}$$

In particular, for the system $\langle GI, GI, GI\rangle$, where the arrival process is renewal with interarrival time distribution F,

$$EX(0) = E^0 \beta_0 \int_0^\infty U(x) G(dx), \tag{7}$$

where $U := \sum_{k=0}^\infty F^{\star k}$ is the renewal function associated with the renewal arrival process.

Proof: Start with (1a) which we rewrite as

$$X(t) = \sum_{k=-\infty}^{\infty} \sum_{i=1}^{\infty} \mathbf{1}(\beta_0 \circ \theta_{T_k} \geq i) \mathbf{1}(T_k \leq t < T_k + \sigma_{i,0} \circ \theta_{T_k} + R(0) \circ \theta_{\sigma_{i,0}} \circ \theta_{T_k}) \; .$$

(Note that, by the composition rule for shifts [2, p.5], $R(T_k + \sigma_{i,k}) = R(0) \circ \theta_{\sigma_{i,0}} \circ \theta_{T_k} = R(0) \circ \theta_{T_k + \sigma_{i,0} \circ \theta_{T_k}}$.) Using Campbell's theorem (see [2, p.17]) we obtain (5).

The Palm expectation in (5) is finite provided that $E^0 \sum_{i=1}^{\beta_0} \sigma_{i,0} < \infty$ and $E^0 \sum_{i=1}^{\beta_0} R(0) \circ \sigma_{i,0} < \infty$. If we now assume that the service times are independent, identically distributed random variables, and also independent of the batch sizes and of the arrival process, then

$$EX(0) = \lambda \sum_{i=1}^{\infty} P^0(\beta_0 \geq i) E^0[\sigma_{i,0} + R(0) \circ \theta_{\sigma_{i,0}}]$$

from which (6) readily follows.

To establish the last part of the proposition we now assume in addition the arrival process to be renewal with interarrival distribution F, (independent of the service requirements which are i.i.d. with distribution G) and use Wald's lemma to obtain

$$\begin{aligned} E^0[\sigma_{1,0} + R(0) \circ \theta_{\sigma_{1,0}}] &= E^0 \left[\sum_{n=0}^{N[0,\sigma_{1,0})-1} \tau_n \right] = E^0 \tau_0 E^0 N[0, \sigma_{1,0}) \\ &= \lambda^{-1} \int_0^{\infty} U(x) G(dx). \end{aligned} \tag{8}$$

The first equation above is due to the fact that, under P^0, $\sigma_{1,0} + R(0) \circ \theta_{\sigma_{1,0}} = T_{L_{1,0}}$ with $L_{1,0} = \inf\{k : T_k > \sigma_{1,0}\} = N[0, \sigma_{0,1})$, whence $T_{L_{1,0}} = \tau_0 + \cdots + \tau_{N[0,\sigma_{0,1})-1}$. The last equality follows readily from the independence of the arrival process and service times since, conditioning on $\sigma_{1,0}$, we can see that $E^0 N[0, \sigma_{1,0}) = E^0 U(\sigma_{1,0})$. Equation (8), together with (6), yields (7). ∎

The expected number of customers in the system *as seen by an arriving shuttle* can also be obtained in terms of the statistics of the input process.

Proposition 2. *In the system $\langle G, G, G \rangle$ the expectation of the number of customers in the system under the Palm probability P^0 is given by*

$$E^0 X(0) \;=\; E^0 \sum_{i=1}^{\beta_0} N[0, \sigma_{i,0}). \tag{9}$$

Proof: The expectation $E^0 X(0)$ can be easily computed from (2a) using the invariance of the Palm probability measure P^0 under the shifts θ_{T_n} as follows

$$\begin{aligned} E^0 X(0) &= E^0 \sum_{n=1}^{\infty} \sum_{i=1}^{\beta_{-n}} \mathbf{1}(T_{-n} + \sigma_{i,-n} > T_{-1}) \;=\; \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} P^0(\beta_{-n} \geq i, T_{-n} + \sigma_{i,-n} > T_{-1}) \\ &= \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} P^0(\beta_{-n} \circ \theta_{T_n} \geq i, T_{-n} \circ \theta_{T_n} + \sigma_{i,-n} \circ \theta_{T_n} > T_{-1} \circ \theta_{T_n}) \end{aligned}$$

where, in the above equalities we have also used Fubini's theorem. However, since $\beta_{-n} \circ \theta_{T_n} = \beta_0$, $\sigma_{i,-n} \circ \theta_{T_n} = \sigma_{i,0}$, and $T_m \circ \theta_{T_n} = T_{m+n} - T_n$, P^0–a.s. for all $m \in \mathbb{Z}$. Thus

$$P^0(\beta_{-n} \circ \theta_{T_n} \geq i, T_{-n} \circ \theta_{T_n} + \sigma_{i,-n} \circ \theta_{T_n} > T_{-1} \circ \theta_{T_n}) = P^0(\beta_0 \geq i, -T_n + \sigma_{i,0} > T_{n-1} - T_n)$$

and

$$E^0 X(0) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} P^0(\beta_0 \geq i, \sigma_{i,0} > T_{n-1}) = E^0 \sum_{i=1}^{\infty} \mathbf{1}(\beta_0 \geq i) \sum_{n=1}^{\infty} \mathbf{1}(\sigma_{i,0} > T_{n-1}). \tag{10}$$

In the above string of equalities besides the invariance of P^0 under the aforementioned shift we have also used Fubini's theorem repeatedly together with the non-negativity of the random variables involved. Taking into account (10) and the fact that $\sum_{n=1}^{\infty} \mathbf{1}(\sigma_{i,0} > T_{n-1}) = N[0, \sigma_{i,0})$ we obtain (9).

∎

2 The number of customers in the system in the stationary framework for constant service times

2.1 Number of customers in the system for single Poisson arrivals and constant service times: The system $\langle M_\lambda, \delta_a, \delta_1 \rangle$

Of special interest is the case where customers arrive singly and their service time is constant and equal to a. Let $\{A_t; t \in \mathbb{R}\}$ denote the *backward recurrence time* or *age* process of the point process $\{T_n\}$ at time t, i.e. $A_t = \sum_{n \in \mathbb{Z}} \mathbf{1}(T_n \leq t < T_{n+1})(t - T_n)$. (For typographical convenience here we will use subscript notation for the process $\{A_t\}$.) Then the number of customers in the system at time t is given by the expression $X(t) = N(t - A_t - a, t - A_t]$. In particular, the number of customers at time 0 can be expressed as $X(0) = N(T_0 - a, T_0]$.

The joint probability generating function of the (stationary) number of customers in service and the number of customers in the waiting area, $f(w_1, w_2) := E w_1^{Y(0)} w_2^{Z(0)}$ can be computed easily if we distinguish the following two cases:

Case 1 $T_0 < -a$. Then $Y(0) = 0$, $Z(0) = N(T_0 - a, T_0]$.
Case 2 $T_0 \geq -a$. Then $Y(0) = N(-a, -T_0]$, $Z(0) = N(T_0 - a, -a]$.

Thus

$$f(w_1, w_2) \;:=\; E w_1^{Y(0)} w_2^{Z(0)} \;=\; \int_0^a y e^{-\lambda(a-t)(1-w_1)} e^{-\lambda t(1-w_2)} \lambda e^{-\lambda t} dt \;+\; e^{-\lambda a} w_2 e^{-\lambda a(1-w_2)}$$

and hence, we have the following

Proposition 3. *The joint probability generating function of the (stationary) number of customers in service and in the waiting area for the $\langle M_\lambda, \delta_a, \delta_1 \rangle$ system is given by*

$$f(w_1, w_2) \;=\; \frac{w_1}{1 + w_1 - w_2} e^{-\lambda a(1-w_1)} \;+\; e^{-\lambda a(2-w_2)} \left(w_1 - \frac{w_2}{1 + w_1 - w_2} \right). \tag{11}$$

In particular, if we set in turn $w_1 = 1$ and $w_2 = 1$ in the above expression we obtain the marginal distributions for the number of customers in service and the number of customers in the waiting area as follows

$$
\begin{aligned}
Ew_1^{Y(0)} &= e^{-\lambda a(1-w_1)}, &(12)\\
Ew_2^{Z(0)} &= \frac{1}{2-w_2} - \frac{(1-w_2)^2}{2-w_2} e^{-\lambda a(2-w_2)}. &(13)
\end{aligned}
$$

As expected, the number of customers in service is Poisson with mean λa. The number of customers in the waiting area has mean $EZ(0) = 1$, and variance $\mathsf{Var}(Z(0)) = 2(1 - e^{-\lambda a})$ as can be readily obtained from (13). The covariance between the two is $\mathsf{Cov}(Y(0), Z(0)) = e^{-\lambda a} - 1$ with corresponding correlation coefficient $\rho_{Y,Z} = -\sqrt{\frac{1-e^{-\lambda a}}{2\lambda a}}$.

We can also obtain the stationary number of customers in the waiting area in terms of the Erlang distribution functions defined by

$$
F_k(x) := 1 - \sum_{j=0}^{k-1} \frac{x^j}{j!} e^{-x}, \qquad k = 1, 2, \dots, \tag{14}
$$

as follows: Rewrite (13) as

$$
\begin{aligned}
Ew_2^{Z(0)} &= \frac{1}{2-w_2} - e^{-\lambda a} \frac{e^{-\lambda a(1-w_2)}}{2-w_2} + e^{-\lambda a} w_2 e^{-\lambda a(1-w_2)} \\
&= \sum_{k=0}^{\infty} \frac{w_2^k}{2^{k+1}} - e^{-\lambda a} \sum_{k=0}^{\infty} w_2^k \sum_{j=0}^{k} e^{-\lambda a} \frac{(\lambda a)^j}{j!} \frac{1}{2^{k-j+1}} + e^{-\lambda a} \sum_{k=1}^{\infty} w_2^k e^{-\lambda a} \frac{(\lambda a)^{k-1}}{(k-1)!}.
\end{aligned}
$$

Collecting terms in the above expression we establish the following

Proposition 4. *The stationary distribution of the number of customers in the waiting area for the $\langle M_\lambda, \delta_a, \delta_1 \rangle$ is given by*

$$
\begin{aligned}
P(Z(0) = 0) &= \frac{1}{2} F_1(2\lambda a), \\
P(Z(0) = k) &= \frac{1}{2^{k+1}} F_{k+1}(2\lambda a) + \frac{(\lambda a)^{k-1}}{(k-1)!} e^{-2\lambda a}, \qquad k \geq 1.
\end{aligned}
$$

2.2 The stationary number of customers in the system $\langle G, \delta_a, G \rangle$

Let Φ denote the *input measure* i.e. for any Borel set $B \in \mathscr{B}(\mathbb{R})$, $\Phi(B)$ denotes the number of customers whose arrival occurs in B, i.e. the measure defined by its values on intervals via the relationship

$$
\Phi(s, t] = \sum_{n \in \mathbb{Z}} \beta_n \mathbf{1}(s < T_n \leq t) .
$$

Suppose that $\sigma_{i,n} \equiv a$ for all n and all $i = 1, 2, \dots, \beta_n$. The number of customers present in the system at time 0, assuming that the system has been operating since the infinite past, is denoted by $X(0)$. Then

$$
X(0) = \Phi(T_0 - a, T_0] \quad P\text{–a.s.}.
$$

(As a consequence of the above, we also have that $X(0) = \Phi(T_0 - a, T_0]$, P^0–a.s. Thus, the Palm distribution of the number of customers in the system at a typical point of arrival is

$$P^0(X(0) = k) = P^0(\Phi(-a, 0] = k).$$

The stationary distribution of the number of customers in the system can be obtained by the Palm inversion formula (see [2]) as follows:

$$P(X(0) = k) = \lambda E^0 \int_{T_0}^{T_1} \mathbf{1}(X(s) = k) ds.$$

However, $X(s) = \Phi(-a, 0]$ P^0–a.s. and thus from the above we readily establish the following

Proposition 5. *The stationary number of customers in the system $\langle G, \delta_a, G\rangle$ is given by*

$$P(X(0) = k) = \lambda E^0[\tau_0 \mathbf{1}(\Phi(-a, 0] = k)].$$

3 The departure process when arrivals are (batch) Poisson and service times are constant: The system $\langle M_\lambda, \delta_a, GI\rangle$

Here we will focus our attention on the departure process and will derive results both for the statistics of departing batches and for the total number of departures in a time interval. In this section we will again assume, unless otherwise specified, that arrivals are Poisson with rate λ and service time are constant and equal to a. For the most part we will also assume that arriving batches $\{\beta_n\}$ are i.i.d. random variables, independent of the Poisson process and we will focus on the statistics of the departing batches, $\{\chi_n\}$. According to our convention we will denote this system by the descriptor $\langle M_\lambda, \delta_a, GI\rangle$. Occasionally, to underscore the essential aspects of the problem, we will restrict the analysis to the case where the batches are all of unit size i.e. to the case $\langle M_\lambda, \delta_a, \delta_1$. It will readily become clear that this involves no real loss of generality.

3.1 The distribution of departing batches

According to the dynamics of the process, at epoch T_n a batch of size β_n arrives and another batch, of size χ_n, leaves. Under the assumption of constant service times we have

$$\begin{aligned} \chi_n &= \Phi\left(T_{n-1} - a, T_n - a\right] \\ &= \Phi\left(T_{n-1} - a, T_{n-1} \wedge (T_n - a)\right] \ . \end{aligned} \tag{15}$$

We will denote by $\beta(z) = E^0[z^{\beta_0}]$, $\chi(z) = E^0[z^{\chi_0}]$, the Palm probability generating functions of the arriving and departing batches.

In order to simplify the discussion we will restrict ourselves for the moment to the case where customers arrive singly at the epochs of the Poisson process (i.e. the case where $\beta_n = 1$ w.p.1 and hence $\beta(z) = z$). Then the corresponding expression for the departing batches is

$$\chi_n = N(T_{n-1} - a, T_n - a]. \tag{16}$$

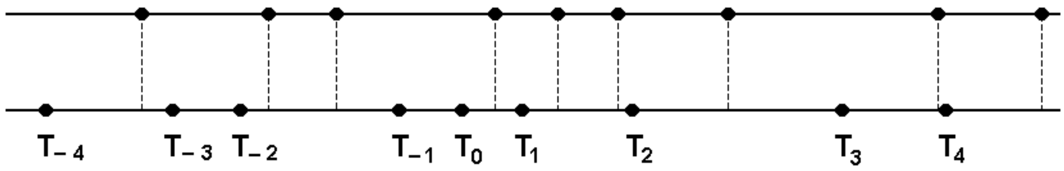

Figure 1: Departure batches when the arrival batches are of size one and service times are constant.

Despite the fact that, for $i \neq j$, χ_i and χ_j are obtained from the number of Poisson arrivals in the disjoint intervals $(T_i - a, T_{i+1} - a]$, $(T_j - a, T_{j+1} - a]$, as we will see in the sequel *these random variables are not independent.* This should come as no surprise since the disjoint intervals are not deterministic but functions of the points of the Poisson process itself.

Proposition 6. *For the $\langle M_\lambda, \delta_a, \delta_1 \rangle$ system, under P^0, the departing batches $\{\chi_n\}$ are identically distributed with probability generating function given by*

$$\chi(z) := E^0[z^{\chi_0}] = \frac{1}{2-z} - \frac{(1-z)^2}{2-z} e^{-\lambda a(2-z)} \ . \tag{17}$$

The corresponding distribution, with the same definitions as in (14), is given by

$$\begin{aligned} P^0(\chi_0 = 0) &= \frac{1}{2} F_1(2\lambda a) \ , \\ P^0(\chi_0 = k) &= \frac{1}{2^{k+1}} F_{k+1}(2\lambda a) \ + \ \frac{(\lambda a)^{k-1}}{(k-1)!} e^{-2\lambda a} \ , \quad k \geq 1 \ . \end{aligned} \tag{18}$$

Proof: Consider the departing batch $\chi_0 = N\,(T_{-1} - a, T_0 - a] = N\,(T_{-1} - a, -a]$ P^0–a.s.. Since $T_{-1} = -\tau_{-1}$ P^0–a.s. one can see that

$$E^0[z^{\chi_0} | \tau_{-1} = u] = \begin{cases} e^{-\lambda u(1-z)} & u < a \\ z e^{-\lambda a(1-z)} & u \geq a \ . \end{cases} \tag{19}$$

From this, taking into account that $P^0(\tau_{-1} \in du) = \lambda e^{-\lambda u} du$, we obtain (17). To establish (18) it suffices to note that (17) is the same expression as (13). ∎

If we let $a \to \infty$ in (17) we obtain $\chi(z) \to \frac{1}{2-z}$ and thus we have the following

Corollary 1. *In the $\langle M_\lambda, \delta_a, \delta_1 \rangle$ when $a \to \infty$, the departing batch size $\chi(z)$ converges in distribution to a geometric random variable with success probability $1/2$.*

An intuitive explanation for this result lies in the fact that, under P^0, $\chi_0 = N(T_{-1} - a, -a]$ according to (15). Thus, the limiting case of corollary 1 may be seen as a consequence of the fact that the number of Poisson (λ) points in an independent, exponentially (λ) distributed interval is geometric with success probability $1/2$. Nonetheless, the shortest complete proof of this is via proposition 6. Further such asymptotic results are given in section 3.4.

The more general situation, where customers arrive in batches, can be handled in precisely the same way. The final result for the p.g.f.'s of the departing batches is the same in all cases with z replaced by $\beta(z)$, the p.g.f. of the arriving batches. Thus we have the following

Corollary 2. *In the $\langle M_\lambda, \delta_a, GI \rangle$ if customers arrive in i.i.d. batches with p.g.f. given by $\beta(z)$, the p.g.f. of the departing batches under P^0 is given by*

$$\chi(z) := E^0[z^{\chi_0}] = \frac{1}{2-\beta(z)} - \frac{(1-\beta(z))^2}{2-\beta(z)} e^{-\lambda a(2-\beta(z))} \;. \tag{20}$$

From (20) we can easily see that the mean and the variance of the typical departing batch is

$$E^0\chi_0 = E^0\beta_0$$

and

$$\mathsf{Var}^0(\chi_0) = \mathsf{Var}^0(\beta_0) + 2(E^0\beta_0)^2 \left(1 - e^{-\lambda a}\right) \;,$$

with a corresponding (squared) coefficient of variation

$$C_\chi^2 = C_\beta^2 + 2\left(1 - e^{-\lambda a}\right) \;.$$

(In the above expression, $\mathsf{Var}^0(X)$ denotes the variance of a random variable X with respect to the Palm probability measure, i.e. $\mathsf{Var}^0(X) = E^0X^2 - (E^0X)^2$ and the squared coefficient of variation is defined as $C_X^2 := \frac{\mathsf{Var}^0(X)}{(E^0X)^2}$.) The expression for the coefficient of variation shows that the departing batches have greater variability than the arrival batches.

Corollary 3. *In the limit, as $a \to \infty$, the Palm distribution of the departing batches in the $\langle M_\lambda, \delta_a, GI \rangle$ system becomes*

$$\lim_{a\to\infty} E^0[z^{\chi_0}] = \frac{1}{2-\beta(z)}. \tag{21}$$

Note that the right hand side of (21) is the composition of the probability generating function (p.g.f.) of the arrival batches with the p.g.f. of a geometric distribution with probability of success $1/2$.

3.2 The covariance of the departing batches under P^0 in the system $\langle M_\lambda, \delta_a, \delta_1 \rangle$

As mentioned above, the sizes of departing batches are not independent. The joint statistics of the departing batches will be derived in section 3.4 in the more general case where service times are not deterministic. In this section we will examine the covariance of the sizes of departing batches. We start with the following

Lemma 1. *If $\{T_n\}$ is a Poisson process with rate λ and $a > 0$ then*

$$\mathsf{Cov}^0(N(-a,0], N(T_n - a, T_n]) = \lambda a P^0(T_n \le a) - nP^0(T_{n+1} \le a), \quad n = 0,1,2,\dots \tag{22}$$

where $\mathsf{Cov}^0(X,Y) := E^0[XY] - E^0XE^0Y$ denotes the covariance of any two random variables with respect to the Palm probability measure P^0.

Proof: The case $n = 0$ can be checked immediately since in that case the left hand side of (22) is $\mathsf{Var}^0(N(-a,0]) = \lambda a$ while the right hand side is also equal to λa since $P^0(T_0 \le a) = 1$. We can thus assume that $n \geq 1$. Conditioning on T_n we note that, if $T_n > a$ then $N(-a,0]$ and $N(T_n - a, T_n]$ are independent random variables and in this case it is easy to see that

$$E^0[N(-a,0]N(T_n-a,T_n]|T_n = v] = (1+\lambda a)\left(1+(n-1)\frac{a}{v}\right), \qquad \text{if } v > a.$$

On the other hand, when $T_n \leq a$, $N(T_n - a, T_n] = N(T_n - a, 0] + n$ and thus

$$\begin{aligned} E^0&[N(-a,0]N(T_n-a,T_n]|T_n = v] \\ &= E^0[(N(-a,T_n-a]+N(T_n-a,0])(N(T_n-a,0]+n)|T_n = v], \quad \text{if } v \leq a. \end{aligned}$$

Combining the above two cases and carrying out the computations gives

$$\begin{aligned} E^0&[N(-a,0]N(T_n-a,T_n] \mid T_n = v] \\ &= \begin{cases} (1+\lambda a)\left(1+(n-1)\dfrac{a}{v}\right) & \text{if} \quad v \leq a, \\ n+1+(n+3)\lambda a + \lambda^2 a^2 - 2\lambda v - \lambda^2 a v & \text{if} \quad v > a. \end{cases} \end{aligned} \tag{23}$$

Since $P^0(T_n \in dv) = \lambda \frac{(\lambda v)^{n-1}}{(n-1)!} e^{-\lambda v} dv$, from the above we obtain

$$\begin{aligned} E^0&[N(-a,0]N(T_n-a,T_n]] \\ &= 1+\lambda a+\lambda a(1+\lambda a)P^0(T_n > a) + (2\lambda a + (\lambda a)^2)P^0(T_n \leq a) - nP^0(T_{n+1} \leq a) \end{aligned}$$

or

$$E^0[N(-a,0]N(T_n-a,T_n]] = 1+3\lambda a+(\lambda a)^2 - \lambda a P^0(T_n > a) - nP^0(T_{n+1} \leq a).$$

Subtracting from the above expression $E^0 N(-a,0]\, E^0 N(T_n - a, T_n] = \left(E^0 N(-a,0]\right)^2 = (1+\lambda a)^2$ establishes the proof of the lemma. ∎

Using the above lemma we can readily obtain the covariance between two departing batches as follows

Proposition 7. *The covariance between two batches χ_0, χ_n, in the system $\langle M_\lambda, \delta_a, \delta_1 \rangle$ is given by*

$$\mathsf{Cov}^0(\chi_0, \chi_n) \;=\; \frac{(\lambda a)^n}{n!} e^{-\lambda a}, \qquad n = 1, 2, \ldots. \tag{24}$$

Proof: Write $\chi_1 = N(T_0 - a, T_1 - a] = N(T_0 - a, T_0] + N(T_0, T_1] - N(T_1 - a, T_1]$, or, since $N(T_0, T_1] = 1$

$$\chi_1 = N(T_0 - a, T_0] - N(T_1 - a, T_1] + 1, \tag{25}$$

$$\chi_n = N(T_{n-1} - a, T_{n-1}] - N(T_n - a, T_n] + 1, \tag{26}$$

the second equation above following by the same reasoning applied to $\chi_n = N(T_{n-1} - a, T_n - a]$.

Thus

$$\begin{aligned}
\mathsf{Cov}^0(\chi_1, \chi_n) &= \mathsf{Cov}^0(N(T_0 - a, T_0], N(T_{n-1} - a, T_{n-1}]) \\
&\quad + \mathsf{Cov}^0(N(T_1 - a, T_1], N(T_n - a, T_n]) \\
&\quad - \mathsf{Cov}^0(N(T_0 - a, T_0], N(T_n - a, T_n]) \\
&\quad - \mathsf{Cov}^0(N(T_1 - a, T_1], N(T_{n-1} - a, T_{n-1}]) \\
&= 2\mathsf{Cov}^0(N(-a, 0], N(T_{n-1} - a, T_{n-1}]) \\
&\quad - \mathsf{Cov}^0(N(-a, 0], N(T_n - a, T_n]) \\
&\quad - \mathsf{Cov}^0(N(-a, 0], N(T_{n-2} - a, T_{n-2}]),
\end{aligned}$$

the second equation above following by Palm stationarity. Hence, using the result of lemma 1, we obtain

$$\begin{aligned}
\mathsf{Cov}^0(\chi_0, \chi_n) &= 2\lambda a P^0(T_{n-1} \le a) - 2(n-1)P^0(T_n \le a) \\
&\quad - \lambda a P^0(T_n \le a) + nP^0(T_{n+1} \le a) \\
&\quad - \lambda a P^0(T_{n-2} \le a) + (n-2)P^0(T_{n-1} \le a).
\end{aligned}$$

Rearranging the above expression and simplifying gives $\mathsf{Cov}^0(\chi_1, \chi_n) = \frac{(\lambda a)^{n-1}}{(n-1)!} e^{-\lambda a}$. Equation (24) is a restatement of the above using stationarity. ∎

Proposition 8. *The sizes of the output batches, $\{\chi_n\}$, in the system $\langle M_\lambda, \delta_a, \delta_1 \rangle$ satisfy*

$$\mathsf{Var}^0\left(\sum_{k=0}^{n-1} \chi_k\right) = 2\lambda a \sum_{k=0}^{n-1} \frac{(\lambda a)^k}{k!} e^{-\lambda a} \;+\; 2n\left(1 - \sum_{k=0}^{n} \frac{(\lambda a)^k}{k!} e^{-\lambda a}\right). \tag{27}$$

Proof: We have

$$\begin{aligned}
\sum_{k=0}^{n-1} \chi_k &= N(-a, 0] + N(0, T_n] - N(T_n - a, T_n] \\
&= N(-a, 0] + n - N(T_n - a, T_n]
\end{aligned} \tag{28}$$

where, in the above equation it should be noted that the intervals $(-a, 0]$ and $(T_n - a, T_n]$ are not necessarily disjoint. Thus, taking into account that $E^0 N(-a, 0] = E^0 N(T_n - a, T_n] = 1 + \lambda a$ we have

$$\begin{aligned}
\mathsf{Var}^0\left(\sum_{k=0}^{n-1} \chi_k\right) &= E^0\left((N(-a, 0])^2 + (N(T_n - a, T_n])^2 - 2N(-a, 0]N(T_n - a, T_n]\right) \\
&= 2(1 + 3\lambda a + (\lambda a)^2) - 2E^0\left[N(-a, 0]N(T_n - a, T_n]\right].
\end{aligned} \tag{29}$$

From the above using lemma 1 we obtain

$$\mathsf{Var}^0\left(\sum_{k=0}^{n-1} \chi_k\right) = 2\lambda a P^0(T_n > a) + 2nP^0(T_{n+1} \le a).$$

This concludes the proof of the proposition. ∎

Note that, as $n \to \infty$, $P^0(T_n > a) \to 1$ while $nP^0(T_{n+1} \le a) \to 0$. Thus,

$$\lim_{n\to\infty} \mathsf{Var}^0 \left(\sum_{k=0}^{n-1} \chi_k \right) = 2\lambda a$$

which indicates the strong dependence that exists between the batches. In fact the following holds:

Corollary 4. *As $n \to \infty$ the sum of the sizes of the first n batches that depart at time 0 and after satisfy*

$$\sum_{k=0}^{n-1} \chi_k \; - \; n \xrightarrow{\mathrm{d}} \mathrm{Poi}(2\lambda a)$$

where $\mathrm{Poi}(c)$ denotes a Poisson random variable with mean $c > 0$ and $\xrightarrow{\mathrm{d}}$, as usual, convergence in distribution.

Proof: It is an immediate consequence of (28), together with the independent increments property of the Poisson process, and the fact that $\lim_{n\to\infty}(-a, 0] \cap (T_n - a, T_n] \downarrow \emptyset$. ∎

3.3 The number of departures in an interval for the stationary process and its index of dispersion for the departure process of the system $\langle M_\lambda, \delta_a, GI \rangle$

In this section we obtain the statistics of the stationary departure process within a time interval $(0, t]$.

Proposition 9. *Let $D(0, t]$ denote the number of departures in the interval $(0, t]$ for the system $\langle M_\lambda, \delta_a, GI \rangle$. Then*

$$\begin{aligned} \Psi(z) := E[z^{D(0,t]}] \;\; &= \;\; e^{-\lambda(t\vee a)} e^{-\lambda a(1-\beta(z))} \sinh(\lambda(t \wedge a)) \left(\beta(z) - \frac{1}{2 - \beta(z)} \right) \\ &\quad + e^{-\lambda t} \left(e^{\lambda t \beta(z)} - e^{\lambda(t\wedge a)\beta(z)} + \frac{e^{\lambda(t\wedge a)\beta(z)} - 1}{\beta(z)(2 - \beta(z))} + 1 \right) . \end{aligned} \tag{30}$$

Proof: In order to compute the probability generating function $Ez^{D(0,t]}$ we will examine the following two cases separately.

<u>Case 1:</u> <u>$t < a$.</u> Let A_s denote the age of the Poisson process at time s. Then

$$\begin{aligned} E[z^{D(0,t]} | A_0, A_t] \quad &= \;\; \mathbf{1}(A_t < t, A_0 > a - t + A_t)\beta(z)e^{-\lambda a(1-\beta(z))} \\ &\quad + \mathbf{1}(A_t < t, A_0 < a - t + A_t)e^{-\lambda(t+A_0-A_t)(1-\beta(z))} \\ &\quad + \mathbf{1}(A_t > t) \; . \end{aligned} \tag{31}$$

To see the above we start with the remark that $A_t > t$ implies that there are no Poisson points in the interval $[0, t]$, hence there can be no departures in that interval (since departures can occur only at the points of the process). This explains the last term on the right hand side of (31). Let us next examine

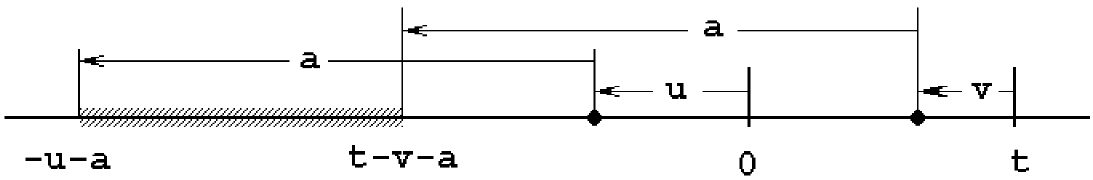

Figure 2: The first case.

the *first* term: $A_t < t$ means that there is at least one point of the process in $(0, t]$. However, since by assumption $t < a$, this means that only customers who arrived before 0 can leave in the interval $(0, t]$. Since there can be no arrivals in the interval $(-A_0, 0)$, we conclude that only customers who arrived before $-A_0$ are candidates for departure in $(0, t]$. Finally, the indicator of $A_0 + (t - A_t) > a$ guarantees that the customers who leave the system in $(0, t]$ are precisely those who arrived in the interval $(-A_0 - a, A_0]$. The number of batches who arrived during this interval is Poisson distributed with mean λa plus one batch that arrives at $-A_0$. This explains the first term on the right hand side of (31). Regarding the *second term* we have $A_0 + (t - A_t) < a$ which means that the customers who leave the system in $(0, t]$ are those who arrived in the system in the interval $(-A_0 - a, t - A_t - a)$. The number of such batches is Poisson distributed with mean $\lambda(t + A_0 - A_t)$. Note in particular that in this case the customers belonging to the batch that arrives at $-A_0$ leave after t.

Case 2: $t \geq a$. With the same notation as above

$$\begin{aligned} E[z^{D(0,t]}|A_0, A_t] \quad &= \mathbf{1}(A_t < t - a)\beta(z)e^{-\lambda(t-A_t)(1-\beta(z))} \\ &+ \mathbf{1}(t - a < A_t < t, A_0 > a - t + A_t)\beta(z)e^{-\lambda a(1-\beta(z))} \\ &+ \mathbf{1}(t - a < A_t < t, A_0 < a - t + A_t)e^{-\lambda(t+A_0-A_t)(1-\beta(z))} \\ &+ \mathbf{1}(A_t > t)\ . \end{aligned} \tag{32}$$

In order to determine the joint distribution of A_0 and A_t it suffices to keep in mind that, as long as the age of the arrival process at t, A_t, is less than t this means that there is at least one point of the Poisson process in the interval $(0, t]$ and hence to conclude that, on this event, A_0 and A_t are independent, exponential random variables with rate λ. If however $A_t > t$ this means that there is no Poisson point in $(0, t]$ and hence that $A_t = A_0 + t$ on that event. Combining the two cases above we can give the following representation for the random variables A_0, A_t: If η, ξ, are two independent exponential random variables with rate λ then $(A_0, A_t) \stackrel{\mathrm{d}}{=} (\xi, \min(\eta, t)) + \xi\mathbf{1}(\eta > t))$. The corresponding distribution function is

$$P(A_0 \leq u, A_t \leq v) = \begin{cases} (1 - e^{-\lambda u})(1 - e^{-\lambda v}) & v < t \\ 1 - e^{-\lambda u} - e^{-\lambda v} + e^{-\lambda(t+u)} & t \leq v < u + t \\ 1 - e^{-\lambda x} & u + t \leq v \end{cases}$$

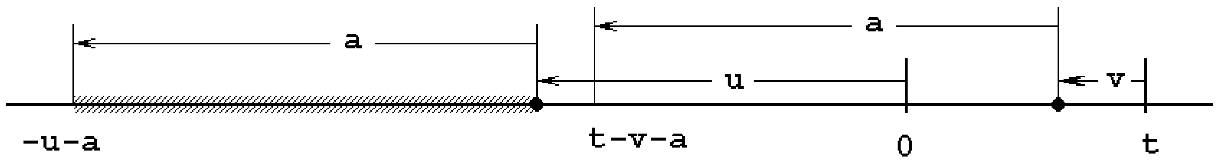

Figure 3: The second case.

Taking expectation with respect to A_0 and A_t in (31) we obtain

$$\begin{aligned}
E[z^{D(0,t]}] &= \int_{v=0}^{t}\int_{u=0}^{v+a-t} e^{-\lambda(t+u-v)(1-\beta(z))}\lambda^2 e^{-\lambda(v+u)}dudv \\
&\quad + \int_{v=0}^{t}\int_{u=v+a-t}^{\infty} \beta(z)e^{-\lambda a(1-\beta(z))}\lambda^2 e^{-\lambda(v+u)}dudv \;+\; e^{-\lambda t} \\
&= \frac{e^{-\lambda t(1-\beta(z))} - e^{-\lambda t}}{\beta(z)(2-\beta(z))} - \frac{e^{-\lambda a(2-\beta(z))}}{2-\beta(z)}\sinh(\lambda t) \\
&\quad + \beta(z)e^{-\lambda a(2-\beta(z)}\sinh(\lambda t) \;+\; e^{-\lambda t}
\end{aligned}$$

whence we obtain

$$\begin{aligned}
E[z^{D(0,t]}] &= e^{-\lambda a}\left(\beta(z) - \frac{1}{2-\beta(z)}\right)e^{-\lambda a(1-\beta(z))}\sinh(\lambda t) \\
&\quad + e^{-\lambda t}\left[1 + \frac{e^{\lambda t\beta(z)} - 1}{\beta(z)(2-\beta(z))}\right], \qquad \text{if } t < a. \qquad (33)
\end{aligned}$$

Similarly, when $t > a$, taking the corresponding expectations in (32) we obtain

$$\begin{aligned}
E[z^{D(0,t]}] &= \int_{v=0}^{t}\lambda e^{-\lambda v}\beta(z)e^{-\lambda(t-v)(1-\beta(z))}dv \\
&\quad + \int_{v=t-a}^{t}\int_{u=v+a-t}^{\infty}\beta(z)e^{-\lambda a(1-\beta(z))}\lambda^2 e^{-\lambda(u+v)}dudv \\
&\quad + \int_{v=t-a}^{t}\int_{u=0}^{v+a-t} e^{-\lambda(t+u-v)(1-\beta(z))}\lambda^2 e^{-\lambda(u+v)}dudv \;+\; e^{-\lambda t}
\end{aligned}$$

Thus in this case we have

$$\begin{aligned}
E[z^{D(0,t]}] &= e^{-\lambda t}\left[e^{\lambda t\beta(z)} - e^{\lambda t\beta(z)}\right] \;+\; \beta(z)e^{-\lambda a(1-\beta(z))}e^{-\lambda t}\sinh(\lambda a) \\
&= \frac{e^{-\lambda t}}{(2-\beta(z))\beta(z)}\left[e^{\lambda a\beta(z)} - 1\right] \;-\; \sinh(\lambda a)\frac{e^{-\lambda t}e^{-\lambda a(1-\beta(z))}}{2-\beta(z)} \;+\; e^{-\lambda t}
\end{aligned}$$

which we can rewrite as

$$\begin{aligned}
E[z^{D(0,t]}] &= e^{-\lambda t}\left(\beta(z) - \frac{1}{2-\beta(z)}\right)e^{-\lambda a(1-\beta(z))}\sinh(\lambda a) \\
&\quad + e^{-\lambda t}\left[1 + \frac{e^{\lambda a\beta(z)} - 1}{\beta(z)(2-\beta(z))}e^{\lambda t\beta(z)} - e^{\lambda a\beta(z)}\right], \qquad \text{if } t \geq a. \qquad (34)
\end{aligned}$$

Combining (33) and (34) into one equation, valid for all $t \geq 0$, we obtain (30). ∎

Evaluating the derivative of the probability generating function given in (30) at $z = 1$ we can verify that $\Psi'(1) = \lambda t E^0 \beta_0$ where $\beta'(1) = E^0\beta_0$ is the mean batch size. This is of course a direct consequence of stationarity. The variance of $D(0,t]$ can also be readily obtained:

$$\mathsf{Var}(D(0,t]) = \lambda t E^0[\beta_0^2] + 2(E^0[\beta_0])^2 \times \begin{cases} 1 - e^{-\lambda t} - e^{-\lambda a}\sinh(\lambda t), & 0 \leq t < a, \\ (\cosh(\lambda a) - 1), & t \geq a. \end{cases} \tag{35}$$

As a rough measure of comparison to the Poisson process we can also compute the *index of dispersion*, defined as the ratio $\frac{\mathsf{Var}(D(0,t])}{ED(0,t]}$ (see [4]).

$$I_a(t) = C_\beta^2 E^0[\beta_0] + 2E^0[\beta_0] \times \begin{cases} \dfrac{1 - e^{-\lambda t} - e^{-\lambda a}\sinh(\lambda t)}{\lambda t}, & t < a, \\[2ex] \dfrac{\cosh(\lambda a) - 1}{\lambda t}, & t \geq a. \end{cases}$$

where $C_\beta^2 := \frac{E^0[\beta_0^2]}{(E^0\beta_0)^2}$ is the squared coefficient of variation of the batch size distribution.

The next proposition characterizes the nature of the output process when the service time a is much larger that the mean interarrival time.

Proposition 10. *In the $\langle M_\lambda, \delta_a, GI\rangle$ system (Poisson-batch arrival model with deterministic service times equal to a) described in this section, when $a \to \infty$ the number of departures $D_a(0,t]$, in the time interval $(0,t]$, converges in distribution to a random variable that can be represented as follows: If Q_p is Poisson distributed with mean λt, Q_g is geometric with probability of success 1/2, and $\{\beta_n\}$, $\{\beta'_n\}$, are i.i.d. sequences distributed according to the distribution of the incoming batches, and if furthermore we assume that all the above random variables are independent, then*

$$D_a(0,t] \stackrel{\mathrm{d}}{\longrightarrow} \mathbf{1}(Q_p > 0)\left(\sum_{n=1}^{Q_p}\beta_n + \sum_{n=1}^{Q_g}\beta'_n\right). \tag{36}$$

Proof: If we let $a \to \infty$ in (33) we see that

$$\lim_{a\to\infty} E z^{D_a(0,t]} = e^{-\lambda t}\left(1 + \frac{e^{\lambda t\beta(z)} - 1}{\beta(z)(2 - \beta(z))}\right). \tag{37}$$

Furthermore the convergence is uniform in $z \in [0,1]$. Thus, by the continuity theorem [6], the right hand side of (37) is the p.g.f. of the number of departures in an interval of length t in the limit where the length of stay in the system for each individual customer goes to infinity. In fact the right hand side of (37) can be written as

$$K(z) := e^{-\lambda t} + (1 - e^{-\lambda t})\frac{e^{-\lambda t(1-\beta(z))} - e^{-\lambda t}}{(1 - e^{-\lambda t})\beta(z)}\frac{1}{2 - \beta(z)}. \tag{38}$$

Based on the above p.g.f. one can verify that the number of departures in an interval of length t has the stochastic representation claimed in (36). ∎

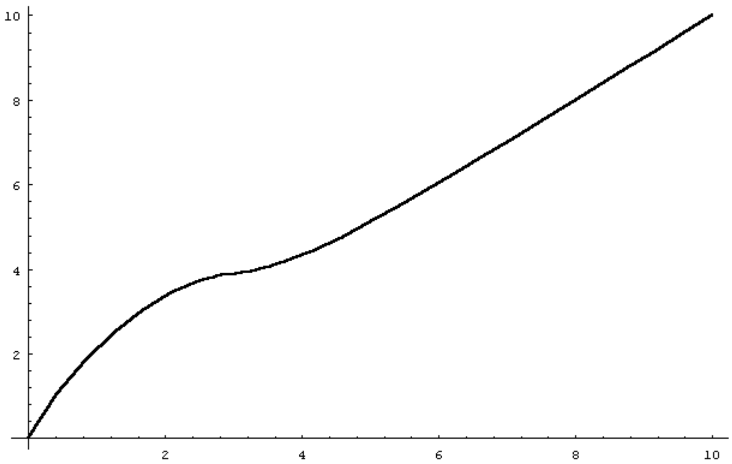

Figure 4: The variance function when $\lambda = E\beta = 1$, $a = 1$, $\mathsf{Var}(\beta) = 1$, and $t \in [0, 10]$

3.4 Joint statistics of departing batches in the system $\langle M_\lambda, GI, \delta_1 \rangle$

Proposition 11. *Let $\{\chi_i; i \in \mathbb{Z}\}$ denote the sequence of departing batches in a $\langle M_\lambda, GI, \delta_1 \rangle$ system. Assuming that customers arrive singly and according to a Poisson process with rate λ and that service times are i.i.d. with distribution G, the joint distribution of n consecutive batches is given by*

$$E^0 \left[z_1^{\chi_1} \cdots z_n^{\chi_n} \right] = \int_{0 \le T_1 \le \cdots \le T_n < \infty} \!\!\! \cdots \! \int dT_1 \cdots dT_n \lambda^n e^{-\lambda T_n} \prod_{i=1}^n \left[e^{-(1-z_i)\lambda \int_{T_{i-1}}^{T_i} \overline{G}(u) du} \right. \\ \left. \times \left(\overline{G}(T_n - T_{i-1}) \;+\; \textstyle\sum_{j=i}^n z_j G(T_{j-1}, T_j] \right) \right].$$

Proof: With each arriving customer we associate a point on the half plane $\mathbb{R} \times \mathbb{R}^+$ by means of the coordinates $\{(T_j, \sigma_j); j \in \mathbb{Z}\}$ where T_j is the arrival epoch of the jth customer and σ_j his service requirement. Thus if we consider the point process M on the half plane given by $M(A) = \sum_{j \in \mathbb{Z}} \delta_{(T_j, \sigma_j)}(A)$ where $\delta_{(t,x)}$ is the measure that assigns unit mass at the point (t, x) and A a Borel subset of $\mathbb{R} \times \mathbb{R}^+$, is Poisson with mean measure $\nu(dt \times dx) = \lambda dt G(dx)$. Consider the stripes $A_i := \{(t, x) : t < T_0, T_{i-1} \le x + t < T_i\}$, $i = 1, 2, \ldots, n$. Any customer whose arrival coordinates (T_j, σ_j) falls in stripe i will depart with the ith batch at time T_i. In fact the size of the ith batch, is given by the expression

$$\chi_i = M(A_i) + \sum_{k=0}^{i-1} \mathbf{1}(T_{i-1} \le T_k + \sigma_k < T_i). \tag{39}$$

From the above definition it is clear that A_i and A_j are disjoint when $i \ne j$ and hence that $M(A_i)$ is independent of $M(A_j)$. Also,

$$\begin{aligned} \int_{A_i} \nu(dt \times dx) &= \int_{t=-\infty}^0 \int_{T_{i-1}-t}^{T_i - t} \lambda dt G(dx) = \lambda \int_{s=0}^\infty [G(T_i + s) - G(T_{i-1} + s)] ds \\ &= \lambda \int_{T_{i-1}}^{T_i} \overline{G}(u) du. \end{aligned}$$

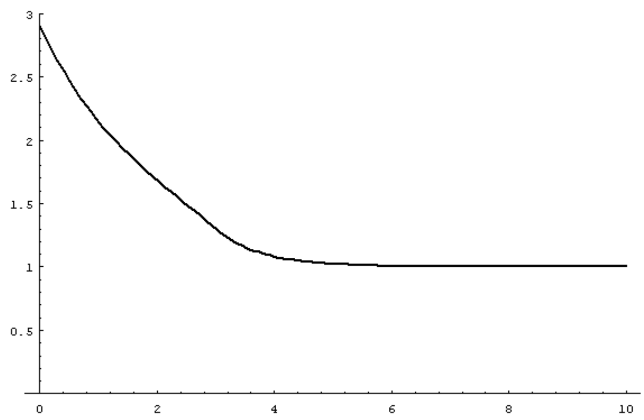

Figure 5: The corresponding index of dispersion with the same parameters as before. Note that when t is much smaller than $a = 3$ the point process is overdispersed while when t is much larger than a the dispersion approaches 1, which is the value for the Poisson process.

Then,

$$E^0\left[z_1^{\chi_1}\cdots z_n^{\chi_n} \mid T_1,\ldots,T_n\right] \;=\; \prod_{i=1}^{n}\left[e^{-(1-z_i)\lambda\int_{T_{i-1}}^{T_i}\overline{G}(u)du}\left(\overline{G}(T_n-T_{i-1})+\sum_{j=i}^{n} z_j G(T_{j-1},T_j]\right)\right]$$

From the above considerations together with the fact that the joint density of $(T_1,\ldots,T_n)$ is

$$\lambda^n e^{-\lambda T_n}\mathbf{1}(0\le T_1\le T_2\le\cdots\le T_n)$$

the proof of the proposition follows. ∎

Corollary 5. *Suppose that $\{G_a; a\in\mathbb{R}^+\}$ is a parametric family of distributions on $\mathbb{R}^+$ index by a parameter a and such that $\lim_{a\to\infty} G_a(x)=0$ for all $x\in\mathbb{R}^+$. Then we obtain*

$$\lim_{a\to\infty} E^0\left[z_1^{\chi_1}\cdots z_n^{\chi_n}\right]=\prod_{i=1}^{n}\frac{1}{2-z_i}\ , \qquad (40)$$

from which we conclude that, when a is very large (compared to $1/\lambda$) the output process consists of independent geometric batches with parameter $1/2$.

(We make no specific assumptions regarding the nature of the parameter though natural examples would be the case where a is a scale parameter i.e. $G_a(x)=G(x/a)$, or a location parameter i.e. $G_a(x)=G(x-a)$. In general, of course, a could be any type of parameter belonging to an open interval $I=(a_1,a_2)$ such that $\lim_{a\to a_2} G_a(x)=0$ for all $x\in\mathbb{R}^+$.)

Proof: The integrand in the expression for the joint batch distribution in proposition 11 is given by

$$\lambda^n e^{-\lambda T_n}\prod_{i=1}^{n}\left[e^{-(1-z_i)\lambda\int_{T_{i-1}}^{T_i}\overline{G}_a(u)du}\left(\overline{G}_a(T_n-T_{i-1})\;+\;\sum_{j=i}^{n} z_j G_a(T_{j-1},T_j]\right)\right]. \qquad (41)$$

As $a \to \infty$, $\overline{G}_a(u) \to 1$ for all real u and $\overline{G}_a(T_n - T_{i-1}) \to 1$, $G_a(T_{j-1}, T_j] \to 0$ w.p.1. Thus, letting $a \to \infty$ the integrand converges to

$$\lambda^n e^{-\lambda T_n} \prod_{i=1}^{n} e^{-(1-z_i)\lambda(T_i - T_{i-1})}.$$

Taking into account that $T_i - T_{i-1} = \tau_i$ and appealing to the Dominated Convergence Theorem we obtain

$$\lim_{a\to\infty} E^0 \left[z_1^{\chi_1} \cdots z_n^{\chi_n}\right] = \int_0^\infty \cdots \int_0^\infty d\tau_1 \cdots d\tau_n \lambda^n e^{-\lambda \sum_{i=1}^n \tau_i (2-z_i)}.$$

A straightforward computation completes the proof. ∎

To obtain a better idea of the correlation structure of the departing batches we compute the correlation between the sizes of two departing batches, one at time T_0 and the other at time T_n. As shown in figure 6, the size of the departing batch at time T_0, is the sum of the number of Poisson points in the shaded area A plus one if the customer who arrives at time T_{-1} finishes service before time T_0. Let us denote by ξ_A the number of points in the stripe A of figure 6, and by ξ_B the number of points in stripe B, i.e., to be more precise, $\xi_A = \sum_{k=2}^\infty \mathbf{1}(T_{-1} \le T_{-k} + \sigma_{-k} < T_0)$, $\xi_B = \sum_{k=2}^\infty \mathbf{1}(T_{n-1} \le T_{-k} + \sigma_{-k} < T_n)$. Let us also introduce the random variables $\tilde{\eta}_{-1} = \mathbf{1}(T_{-1} + \sigma_{-1} \le T_0)$, $\eta_{-1} = \mathbf{1}(T_{n-1} < T_{-1} + \sigma_{-1} \le T_n)$, and $\eta_i = \mathbf{1}(T_{n-1} < T_i + \sigma_i \le T_n)$, $i = 0, 1, 2, \ldots, n-1$. Then clearly

$$\begin{aligned} \chi_0 &= \xi_A + \tilde{\eta}_{-1} \\ \chi_n &= \xi_B + \eta_{-1} + \sum_{i=0}^{n-1} \eta_i. \end{aligned}$$

It is easy to see that the random variables $\{\eta_i; i = 0, 1, 2, \ldots, n-1\}$ are independent of $\xi_A, \xi_B, \eta_{-1}, \tilde{\eta}_{-1}$. Thus

$$\mathsf{Cov}^0(\chi_0, \chi_n) = \mathsf{Cov}^0(\xi_A, \xi_B) + \mathsf{Cov}^0(\xi_A, \eta_{-1}) + \mathsf{Cov}^0(\eta, \xi_B) + \mathsf{Cov}^0(\eta_{-1}, \tilde{\eta}_{-1}). \tag{42}$$

In order to compute the first term of the right hand side above we condition on $T_0 - T_{-1} = u$, $T_n - T_{n-1} = v$, $T_{n-1} - T_0 = w$. It is easy to see then that ξ_A, ξ_B are conditionally independent (and, given the above random variables, Poisson distributed) and thus

$$\begin{aligned} &E^0[\xi_A \xi_B | T_0 - T_{-1} = u, T_n - T_{n-1} = v, T_{n-1} - T_0 = w] \\ &\qquad = E^0[\xi_A | T_0 - T_{-1} = u] E^0[\xi_B | T_0 - T_{-1} = u, T_n - T_{n-1} = v, T_{n-1} - T_0 = w] \\ &\qquad = \left(\lambda \int_0^u \overline{G}(x) dx\right) \left(\lambda \int_{u+w}^{u+w+v} \overline{G}(x) dx\right). \end{aligned}$$

In view of the above we have

$$\mathsf{Cov}^0(\xi_A, \xi_B) = \lambda^2 E^0 \sigma_0 \mathsf{Cov}^0 \left(G_I(-T_{-1})\ ,\ G_I(T_n - T_{-1}) - G_I(T_{n-1} - T_{-1})\right) \tag{43}$$

where $G_I(x) := \frac{1}{E^0 \sigma_0} \int_0^x \overline{G}(y) dy$ is the integrated tail distribution that corresponds to the service distribution. Similarly,

$$\begin{aligned} \mathsf{Cov}^0(\xi_A, \tilde{\eta}_{-1}) &= \lambda E^0 \sigma_0 \mathsf{Cov}^0 \left(G_I(-T_{-1})\ ,\ G(T_n - T_{-1}) - G(T_{n-1} - T_{-1})\right), \\ \mathsf{Cov}^0(\xi_B, \eta_{-1}) &= \lambda E^0 \sigma_0 \mathsf{Cov}^0 \left(G(-T_{-1})\ ,\ G_I(T_n - T_{-1}) - G_I(T_{n-1} - T_{-1})\right), \\ \mathsf{Cov}^0(\eta_{-1}, \tilde{\eta}_{-1}) &= \mathsf{Cov}^0 \left(G(T_n - T_{-1}) - G(T_{n-1} - T_{-1})\ ,\ G(-T_{-1})\right). \end{aligned}$$

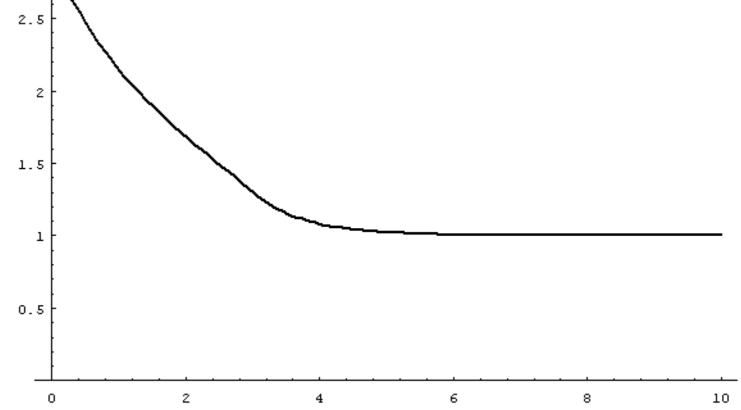

Figure 6: The computation of the covariance of the sizes of two batches

Using the above equations and (43) in (42) we obtain an expression for the covariance of the size of two batches. In particular, if the service time distribution is exponential, $G(x) = 1 - e^{-\mu x}$, then the above expressions simplify and the following proposition holds:

Proposition 12. *In the $\langle M_\lambda, M_\mu, \delta_1 \rangle$ model where customers arrive singly according to a Poisson process with rate λ and service times are independent, exponential, with rate μ then the stationary covariance of departing batches is given by*

$$\mathsf{Cov}(\chi_0, \chi_n) = pq^n, \qquad n = 1, 2, \ldots \tag{44}$$

where $p = \frac{\lambda}{\lambda+\mu}$ and $q = \frac{\mu}{\lambda+\mu}$.

4 Renewal arrivals and exponential service times: The system $\langle GI, M_\mu, GI \rangle$

In this section we assume that the arrival process is renewal whereas the service time distribution is exponential with rate μ. We will assume as usual that customers arrive in batches of size β_n where the sequence $\{\beta_n\}$ is i.i.d. with given distribution and corresponding probability generating function $\beta(z)$. Furthermore we will denote the number of customers in the system just prior to the nth arrival by $X_n := X(T_n-)$ and let $\phi(z) := E^0[z^{X_0}]$ denote the probability generating function of the number of customers in the system under the Palm measure P^0 at time $0-$ i.e. the p.g.f. of the event–stationary distribution just prior to a typical arrival. A first result which will play an important role in the sequel is the following

Proposition 13. *In the system $\langle GI, M_\mu, GI \rangle$, if the input batch size distribution is light-tailed i.e., for some $\epsilon > 0$, $\beta(1+\epsilon) < \infty$ then the departing batch size is also light-tailed and we have $\phi(1+\epsilon) < \infty$.*

Proof: Recalling (2a) we will show that, if $\epsilon > 0$, $\phi(1+\epsilon) = E^0(1+\epsilon)^{X_0} < \infty$. Indeed, $X_0 = \sum_{n=1}^{\infty} \sum_{i=1}^{\beta_{-n}} \mathbf{1}(T_{-n} + \sigma_{i,n} > 0)$ and thus

$$\begin{aligned} E^0[(1+\epsilon)^{X_0}] &= \prod_{n=1}^{\infty} \prod_{i=1}^{\beta_{-n}} \left((1+\epsilon)\mathbf{1}(T_{-n} + \sigma_{i,n} > 0) + \mathbf{1}(T_{-n} + \sigma_{i,n} \le 0)\right) \\ &= \prod_{n=1}^{\infty} \prod_{i=1}^{\beta_{-n}} \left(1 + \epsilon \mathbf{1}(T_{-n} + \sigma_{i,n} > 0)\right). \end{aligned}$$

In order to show that the above expectation is finite we consider first the conditional expectation given $\{(T_{-n}, \beta_{-n}); n = 1, 2, \ldots\}$. We have

$$E^0[(1+\epsilon)^{X_0}|(T_{-n}, \beta_{-n}); n = 1, 2, \ldots] = \prod_{n=1}^{\infty} \left(1 + \epsilon e^{\mu T_{-n}}\right)^{\beta_{-n}}.$$

(In the above equation note that T_{-n} is a negative). Taking expectations with respect to the batch sizes $\{\beta_{-n}; n \in \mathbb{N}\}$ we have

$$\begin{aligned} E^0[(1+\epsilon)^{X_0}|T_{-n}; n = 1, 2, \ldots] &= \prod_{n=1}^{\infty} \beta\left(1 + \epsilon e^{\mu T_{-n}}\right) = \prod_{n=1}^{\infty} \left(1 - \left(\beta(1) - \beta(1 + \epsilon e^{\mu T_{-n}})\right)\right) \\ &= \prod_{n=1}^{\infty} \left(1 - \beta'(1 + c_n)\epsilon e^{\mu T_{-n}})\right) \end{aligned}$$

where $c_n \in (0, \epsilon)$. Thus, since $\beta'(z)$ is an increasing function, we can write

$$E^0[(1+\epsilon)^{X_0}|T_{-n}; n = 1, 2, \ldots] \leq \prod_{n=1}^{\infty} \left(1 - \beta'(1)\epsilon e^{\mu T_{-n}})\right) \stackrel{\mathrm{d}}{=} \prod_{n=1}^{\infty} \left(1 - \eta e^{-\mu T_n}\right)$$

where we have used the mean value theorem for the differentiable function $\beta(z)$ and we have set $\eta := \beta'(1)\epsilon = \epsilon E^0\beta_0$. From the above we see that $E^0[(1+\epsilon)^{X_0}] < \infty$ provided that $E^0\left[\prod_{n=1}^{\infty}\left(1 - \eta e^{-\mu T_n}\right)\right] < \infty$. In view of the inequality $1 - x \leq e^{-x}$ which holds for all $x \in \mathbb{R}$, it is enough to show that

$$E^0\left[e^{-\eta \sum_{n=1}^{\infty} e^{-\mu T_n}}\right] < \infty. \tag{45}$$

Fix now $\delta > 0$ and consider a renewal process $\{\tilde{T}_n\}$ with interarrival times

$$\tilde{\tau}_n = \begin{cases} 0 & \text{if} \quad 0 \leq \tau_n < \delta, \\ \delta & \text{if} \quad \delta \leq \tau \end{cases}.$$

We thus have $\tilde{T}_n = \tilde{\tau}_1 + \cdots + \tilde{\tau}_n$ and hence $\tilde{T}_n \leq T_n$ a.s. for all n. In order to show (45) it then suffices to establish that

$$E^0\left[e^{-\eta \sum_{n=1}^{\infty} e^{-\mu \tilde{T}_n}}\right] < \infty.$$

Then, a moment's thought reveals that

$$\sum_{n=1}^{\infty} e^{-\mu \tilde{T}_n} = \sum_{k=0}^{\infty} \alpha^k V_k \tag{46}$$

where $\{V_k; k = 0, 1, 2, \ldots\}$ are independent, geometric random variables with common distribution $P(V = i) = P(\tau \geq \delta)\left(P(\tau < \delta)\right)^{i-1}$, $i = 1, 2, \ldots$ and $\alpha := e^{-\mu\delta} < 1$. We thus need to show that

$$E^0 e^{-\eta \sum_{k=0}^{\infty} \alpha^k V_k} = \prod_{k=0}^{\infty} \frac{1-q}{1 - qe^{-\eta\alpha^k}} < \infty.$$

Since none of the terms of the above infinite product is equal to zero, the above infinite product is finite if and only if the infinite product

$$\prod_{k=0}^{\infty} \frac{1 - qe^{-\eta\alpha^k}}{1-q} = \prod_{k=0}^{\infty} \left(1 - \frac{q}{p}(1 - e^{-\eta\alpha^k})\right)$$

converges properly, i.e. away from zero. A necessary and sufficient condition for this is the absolute convergence of the infinite series $\sum_{k=0}^{\infty} \frac{q}{p}(1-e^{-\eta\alpha^k})$. This series however converges absolutely as can be seen by comparing it with the geometric series $\sum_{k=0}^{\infty} \alpha^k$ (where $0 < \alpha < 1$), since

$$\lim_{k\to\infty} \frac{1-e^{\eta\alpha^k}}{\alpha^k} = \eta.$$

∎

A direct consequence of the above proposition is the following

Corollary 6. *If $\beta(z) < \infty$ for all $z \in \mathbb{R}$ then $\phi(z) < \infty$ for all $z \in \mathbb{R}$. In particular, if customers arrive singly then $\phi(z) < \infty$ for all $z \in \mathbb{R}$.*

4.1 Statistics of number of customers in the system and of departing batches when customers arrive individually: The system $\langle GI, M_\mu, \delta_1 \rangle$

From the above corollary it follows in particular that, if customers arrive singly, then $\phi(z)$ is analytic at $z = 1$ and hence that

$$\phi(z) = \sum_{k=0}^{\infty} \frac{(z-1)^k}{k!} \phi^{(k)}(1) \tag{47}$$

where the kth derivative of $\phi(z)$ at 1 is equal to the kth descending factorial moment of X_0:

$$\phi^{(k)}(1) = E^0[X_0(X_0-1)\cdots(X_0-k+1)]. \tag{48}$$

Let us denote by $\zeta(s) := E^0[e^{-s\tau_0}]$ the Laplace transform of the interarrival time distribution. Then, conditional on τ_0 and X_0, under P^0,

$$X_1 \overset{\mathrm{d}}{=} 1 + \sum_{i=1}^{X_0} \gamma_i \ ,$$

where the γ_i are independent Bernoulli random variables with probability of success $e^{-\mu\tau_0}$. Indeed, $X_n := X_{T_n-}$ and hence, at time T_1- the customer who arrives at $T_0 = 0$ is certainly present. Also, each one of the X_0 customers that were present at time T_0- will remain in the system with probability $e^{-\mu\tau_0}$, independently of each other. Thus

$$E^0[z^{X_1}|\tau_0, X_0] \ = \ z\left(e^{-\mu\tau_0}z + 1 - e^{-\mu\tau_0}\right)^{X_0} \ = \ z\sum_{k=0}^{X_0} e^{-k\mu\tau_0}(z-1)^k \binom{X_0}{k} \ .$$

Taking expectation with respect to τ_0 in the above gives

$$E^0[z^{X_1}|X_0] = zE^0 \sum_{k=0}^{\infty} \zeta(k\mu)\frac{(z-1)^k}{k!}\mathbf{1}(X_0 \geq k)X_0(X_0-1)\cdots(X_0-k+1) \ .$$

Note that, while the above sum is written as an infinite sum it has in fact a finite number of terms with probability 1. Observe however that $E^0[\mathbf{1}(X_0 \geq k)X_0(X_0-1)\cdots(X_0-k+1)] = \phi^{(k)}(1)$. This yields the following basic relationship:

$$\phi(z) \ = \ z\sum_{k=0}^{\infty} \zeta(\mu k)\frac{1}{k!}(z-1)^k\phi^{(k)}(1) \ . \tag{49}$$

If we differentiate n times term by term the power series on the right hand side of the above equation and we evaluate the result at $z = 1$ we obtain the recursive relation

$$\phi^{(n)}(1) = \phi^{(n)}(1)\zeta(\mu n) + n\phi^{(n-1)}(1)\zeta(\mu(n-1)) \tag{50}$$

whence we see that

$$\phi^{(n)}(1) = n! \prod_{i=0}^{n-1} \frac{\zeta(i\mu)}{1 - \zeta((i+1)\mu)} \qquad n = 0, 1, 2, \ldots.$$

which, with the usual assumption that an empty product is equal to 1 includes the case $\phi^{(0)}(1) = 1$. Thus we are ready to state the following

Proposition 14. *For the system with single arrivals, $\phi(z)$ is analytic for all $z \in \mathbb{R}$ with*

$$\phi(z) = \sum_{n=0}^{\infty} (z-1)^n \prod_{i=0}^{n-1} \frac{\zeta(i\mu)}{1 - \zeta((i+1)\mu)} \ . \tag{51}$$

Furthermore the p.g.f. of the size of departing batches is given by

$$\chi(z) = 1 + \sum_{n=1}^{\infty} (z-1)^n p^n \left(1 + \frac{1 - \zeta(n\mu)}{\zeta((n-1)\mu)}\right) \prod_{i=0}^{n-1} \frac{\zeta(i\mu)}{1 - \zeta((i+1)\mu)}. \tag{52}$$

Proof: Equation (51) is an immediate consequence of proposition 13, (47), and (50). To establish (52) it suffices to notice that

$$\chi(z) = (q + pz)\phi(q + pz) = (q + pz) \sum_{n=0}^{\infty} (z-1)^n p^n \prod_{i=0}^{n-1} \frac{\zeta(i\mu)}{1 - \zeta((i+1)\mu)}$$

and carry out the algebra. ∎

4.2 Statistics of number of customers in the system and departing batches when customers arrive in batches: The system $\langle GI, M_\mu, GI \rangle$

With the same notation as before we have that

Proposition 15. *If the p.g.f. of the batch size distribution is such that $\beta(1+\epsilon) < \infty$ where $\epsilon > 0$ then the corresponding probability generating function for the number of customers in the system exists for $z \in [0, 1+\epsilon]$ and is given by the power series*

$$\phi(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \phi^{(n)}(1)(z-1)^n \tag{53}$$

where the derivatives at 1 (which are equal to the corresponding descending factorial moments) are given by the recursive equations

$$\phi^{(n)}(1) = \frac{\zeta(n\mu)}{1 - \zeta(n\mu)} \sum_{k=0}^{n-1} \binom{n}{k} \phi^{(k)}(1)\beta^{(n-k)}(1), \quad n = 1, 2, \ldots, \qquad \phi^{(0)}(1) = 1. \tag{54}$$

The probability generating function for the departing batch size in steady state is given by

$$\chi(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \chi^{(n)}(1)(z-1)^n \tag{55}$$

where

$$\chi^{(n)}(1) = \frac{\phi^{(n)}(1)}{\zeta(n\mu)} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \zeta(k\mu). \tag{56}$$

Proof: The expansion around 1 in (53) is possible by virtue of proposition (13). From the relationship

$$E^0[z^{X_1}|X_0, \beta_0, \tau_0] = \left(1 - e^{-\mu\tau_0} + z e^{-\mu\tau_0}\right)^{X_0+\beta_0}$$

we obtain

$$\begin{aligned}
\phi(z) &= E^0\left[\phi(1 + (z-1)e^{-\mu\tau_0})\beta(1 + (z-1)e^{-\mu\tau_0})\right] \\
&= E^0\left[\sum_{n=0}^{\infty} \frac{1}{n!}(z-1)^n e^{-n\mu\tau_0} \sum_{k=0}^{n} \binom{n}{k} \phi^{(k)}(1)\beta^{(n-k)}(1)\right] \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}(z-1)^n \zeta(n\mu) \sum_{k=0}^{n} \binom{n}{k} \phi^{(k)}(1)\beta^{(n-k)}(1) \qquad (57)
\end{aligned}$$

and hence, taking into account (53), we see that

$$\phi^{(n)}(1) = \zeta(n\mu) \sum_{k=0}^{n} \binom{n}{k} \phi^{(k)}(1)\beta^{(n-k)}(1) \tag{58}$$

whence (55) follows readily.

Noting that

$$\chi(z) = E^0\left[(e^{-\mu\tau_0} + z(1 - e^{-\mu\tau_0}))^{X_0+\beta_0}\right] \tag{59}$$

we readily conclude that, if $\beta(1+\epsilon) < \infty$, then $\chi(1+\epsilon) < \infty$ hence the expansion in (56) is valid. From (59) we obtain

$$\begin{aligned}
\chi(z) &= \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} E^0[(1 - e^{-\mu\tau_0})^n] \sum_{k=0}^{n} \binom{n}{k} \phi^{(k)}(1)\beta^{(n-k)}(1) \\
&= \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \frac{\phi^{(n)}(1)}{\zeta(n\mu)} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \zeta(k\mu),
\end{aligned}$$

where in the last equation we have also taken into account (58). This last equation establishes (56). ∎

4.3 System time distribution for the system $\langle GI, M_\mu, GI \rangle$

The system time distribution for the typical customer in this system obviously does not depend on whether customers arrive individually or in batches. Thus, to keep our notation simple we will assume that customers arrive individually. As before, $\zeta(s) = E^0[e^{-s(T_1-T_0)}]$ denotes the Laplace transform of the interarrival time distribution and $R(t)$ the forward recurrence time of the arrival process at time t. Then the system time of a customer who arrives at $T_0 = 0$ and whose service requirement is σ_0 is given by $V_0 = \sigma_0 + R(\sigma_0)$.

Proposition 16. *If the service times $\{\sigma_n\}$ above are i.i.d. exponential with rate μ and independent of the arrival process, then the Laplace transform of the service time distribution, is given by*

$$\gamma(s) := E^0[e^{-sV_0}] = \frac{\zeta(s) - \zeta(s+\mu)}{1-\zeta(s+\mu)} \ . \tag{60}$$

Proof: Let $g(t,s) := E^0[e^{-sR(t)}]$, denote the Laplace transform of the forward recurrence time of the arrival process (which of course depends on t). A straightforward renewal theoretic argument gives

$$g(t,s) = E[e^{-s(T_1-t)}1(T_1 > t)] + \int_0^t g(t-u,s)dF(u) \ .$$

Taking further a Laplace transform with respect to t we have

$$\Gamma(\theta,s) \ := \ \int_0^\infty e^{-\theta t} g(t,s)dt \ = \ \frac{\zeta(s)-\zeta(\theta)}{\theta - s} - \Gamma(\theta,s)\zeta(\theta)$$

or

$$\Gamma(\theta,s) \ = \ \frac{\zeta(s)-\zeta(\theta)}{(\theta-s)(1-\zeta(\theta))}. \tag{61}$$

Since service times are exponential with rate μ then the Laplace transform of the system time of a typical customer is

$$\gamma(s) := E^0[e^{-sV_0}] = \int_0^\infty \mu e^{-\mu t} e^{-st} E^0[e^{-sR(t)}]dt = \mu\Gamma(\mu+s,s) \ . \tag{62}$$

From (61) and (62) equation (60) follows readily. ∎

5 Appendix – The distribution of departing batches under P for the system $\langle M_\lambda, \delta_a, \delta_1 \rangle$

Here we provide a sketch of the analysis for the distribution of departing batches under P when customers arrive singly according to a Poisson process with rate λ and service times are deterministic (equal to a).

The p.g.f. $E^0 z^{\chi_n}$ was obtained in proposition 6. Under P however, the random variables $\{\chi_n\}$ are no longer identically distributed. Using an analogous analysis we can establish the following

Proposition 17. *The* stationary *probability generating functions Ez^{χ_n} for the system $\langle M_\lambda, \delta_a, \delta_1 \rangle$ are* *given by the expressions:*

$$E[z^{\chi_n}] = \frac{1}{2-z} - \frac{(1-z)^2}{2-z} e^{-\lambda a(2-z)} \qquad n = 0, -1, -2, -3, \ldots,$$

$$E[z^{\chi_1}] = \frac{1}{(2-z)^2} - \frac{(1-z)^2}{2-z}(1+\lambda a) e^{-\lambda a(2-z)} \ ,$$

$$\begin{aligned} E[z^{\chi_n}] &= \left[\frac{1}{2-z} - \frac{(1-z)^2}{z(2-z)} - \frac{(-1)^{n-1}}{z(2-z)(1-z)^{n-1}}\right] e^{-\lambda a(2-z)} \\ &+ \sum_{k=0}^{n-2} \frac{(\lambda a)^k}{k!} \left(z^{k+1} - z^k\right) e^{-2\lambda a} \ + \ z^{-1} e^{-\lambda a} \sum_{k=0}^{n-2} \sum_{l=0}^{k} \frac{(-1)^l}{(1-z)^l} \frac{(\lambda a)^{k-l}}{(k-l)!} \\ &- \frac{(1-z)}{z} e^{-\lambda a} \sum_{k=0}^{n-2} \sum_{l=0}^{n-2-k} z^l \int_0^a \lambda e^{-\lambda u} \frac{(\lambda(a-u))^k (\lambda u)^l}{k! l!} du \qquad n = 2, 3, 4, \ldots \end{aligned}$$

It should be noted that batches departing before time 0 have the same distribution as under P^0. The distributions of the sizes of the batches that depart after time 0 are different, it can be shown however that, as $n \to \infty$,

$$E[z^{\chi_n}] \to \frac{1}{2-z} - \frac{(1-z)^2}{2-z} e^{-\lambda a(2-z)}.$$

(Compare this with (17).) The complicated expressions above which have been obtained by straightforward analysis are not particularly illuminating and the proof of this proposition will be omitted. More can be gleaned about the behavior of this process by examining the expected sizes of departing batches:

Proposition 18. *The stationary expectation of the departing batch size for the system $\langle M_\lambda, \delta_a, \delta_1 \rangle$ is given by the expression*

$$E\chi_n = \begin{cases} 1 & \textit{if} \quad n \le 0, \\ 2 - e^{-\lambda a} & \textit{if} \quad n = 1, \\ 1 - e^{-\lambda a} - \frac{(\lambda a)^{n-1}}{(n-1)!} e^{-\lambda a} + \sum_{j=0}^{n-1} \frac{(\lambda a)^j}{j!} e^{-2\lambda a} & \textit{if} \quad n \ge 2. \end{cases} \tag{63}$$

Proof. This can of course be obtained from the above expressions for the corresponding p.g.f.'s. It is however easier and more illuminating to obtain it by a direct argument as follows. We start with the fact that

$$E[\chi_n | \tau_{n-1} = u] = \begin{cases} \lambda u & \text{if} \quad u < a \\ 1 + \lambda a & \text{if} \quad u \ge a, \end{cases} \qquad n = 1, 0, -1, -2, -3, \ldots$$

and

$$E[\chi_n | \tau_{n-1} = u] = \begin{cases} \lambda u - P(T_{n-1} - a < 0 \le T_n - a | \tau_{n-1} = u) & \text{if} \quad u < a \\ 1 + \lambda a - P(T_{n-1} - a < 0 | \tau_{n-1} = u) & \text{if} \quad u \ge a, \end{cases} \qquad n = 2, 3, 4, \ldots$$

(This can be obtained by arguments similar to those in section 3.) Taking additionally into account that, due to the inspection paradox, while $P(\tau_n \in du) = \lambda e^{-\lambda u} du$ for all $n \in \mathbb{Z} \setminus \{0\}$, $P(\tau_0 \in du) = \lambda^2 u e^{-\lambda u} du$ and carrying out the elementary computations we establish the proposition. ∎

Finally we should point out that, as $\alpha \to \infty$, the corresponding stationary distribution for the size of the departing batches becomes

$$\lim_{a \to \infty} E[z^{X_n}] = \begin{cases} \dfrac{1}{2-z} & \text{if} \quad n \neq 1, \\ \\ \dfrac{1}{(2-z)^2} & \text{if} \quad n = 1. \end{cases} \tag{64}$$

Note that the first batch after the origin in the stationary case is an exception since the gap that precedes it is the sum of two independent exponentials with rate λ.

Acknowledgement. The author would like to thank an anonymous referee for many remarks that improved the exposition of the paper.

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