An $M^X/G/1$ Queueing System with Disasters and Repairs under a Multiple Adapted Vacation Policy

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Abstract

We consider a queueing system with batch Poisson arrivals subject to disasters which occur independently according to a Poisson process but affect the system only when the server is busy, in which case the system is cleared of all customers. Following a disaster that affects the system, the server initiates a repair period during which arriving customers accumulate without receiving service. The server operates under a Multiple Adapted Vacation policy. The stationary regime of this process is analyzed using the supplementary variables method. We obtain the probability generating function of the number of customers in the system, the fraction of customers who complete service, and the Laplace transform of the system time of a typical customer in stationarity. The stability condition for the system, and the Laplace transform of the time between two consecutive disasters affecting the system is obtained by analyzing an embedded Markov renewal process. The statistical characteristics of the batches that complete service without being affected by disasters and those of the partially served batches are also derived.

KEYWORDS: MULTIPLE ADAPTED VACATIONS, DISASTERS, SUPPLEMENTARY VARIABLES METHOD, BATCH ARRIVALS.

1 Introduction

Stochastic population models with *disasters*, or *catastrophes*, that model partial or nearly total extinctions have been studied for a long time (see for instance [5]). Among the first papers that considered the effect of disasters to queueing processes were Gelenbe, Glynn, and Sigman [11], which considers queues with negative customers (of which disasters are a special case), Chao [6] which considers a Jackson network in the presence of disasters and shows that its stationary distribution retains the product form, and Jain and Sigman [12] which considers the $M/G/1$ system under Poisson disasters and generalizes the Pollaczek–Khintchine formula for the steady state workload. The stationary distribution of the workload is also analyzed in Boxma et al. [4] for $M/G/1$ queues via a martingale argument for various clearing (i.e. disaster) rules.

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Since then queueing systems with disasters have been recognized as useful and natural models for communications systems and manufacturing operations subject to catastrophic failures. Dubin and Nishimura [8] examine a single server queue in the more general framework of customers arriving according to a Batch Markovian Arrival Process (BMAP) with disasters occurring according to a Markovian Arrival Process (MAP) and obtain the queue length and the sojourn time distribution. In [9] this model is extended to include repair times after the occurrence of disasters. The Laplace transform for the busy period and sojourn times of an $M/G/1$ queue system with disasters followed by repair times was studied in Yang et al. [19]. Yechiali [18] studies an $M/M/c$ queue subject to disasters and subsequent repair periods, during which arriving customers wait in line but abandon the system at an exponential rate. A variation of this model for the $M/M/1$ queue where customers perform synchronized abandonments was studied by Economou and Kapodistria [10]. Retrial queues with disasters were investigated by Artalejo and Corral [1] using the supplementary variables technique. Shin [14] considers multi-server retrial queues with an MMAP arrival process in the presence of disasters and negative customers and provides an algorithmic solution for the stationary queue length distribution using matrix geometric techniques. The transient behavior of the $M/M/1$ queue with disasters, followed by exponential repair periods is studied in Kumar et al. [13].

Despite the intense activity on queues with disasters during the last decade, there have been no studies of queues with vacations subject to disasters. In the present article we study such a system. Queues with vacations have been the subject of study for decades since they are particularly useful and flexible models for communications, computer, and manufacturing systems (see [16], [17]). We consider a single server queue with general service times, batch Poisson arrivals, vacations, and disasters which occur according to an independent Poisson process and which, if the server is busy serving customers, remove all customers from the system and cause a server breakdown which is followed by a repair period. Otherwise, if the server is on vacation, idle, or under repair, they have no effect. The vacation policy we consider is the so-called Multiple Adapted Vacation (MAV) policy first introduced by Takagi (see [16] and also [17] where the term MAV was coined) and, as we shall discuss in §2, is quite general. In §3 the system is analyzed using the supplementary variables method and the probability generating function (pgf) of the number of customers in the system in stationarity is given. Due to the effect of disasters (which do not affect the system during vacations) standard decomposition results do not hold in this case.

Batch arrivals are particularly important in modeling production process where units often arrive in lots. In §5 we analyze the effect of disasters on the size of completely served batches as well as that of partially served batches due to disasters and derive the corresponding pgf's. In §8 we obtain the Laplace transform of the delay of a typical customer who completes service as well as of a typical customer who is removed from the system. These results generalize existing results in the literature ([12], [4], [19]).

In [8] a more general model, in terms of the structure of the arrival, service time, and disaster process is presented. This is extended to include service breakdowns as a result of the occurrence of disasters in [9]. However neither paper studies vacations (whose effect in system performance is different from that of repair periods). Furthermore, their approach uses Matrix Geometric methods which are well suited for numerical implementation but give rise to expressions which are not as easy to interpret as the ones obtained under the more restrictive framework of Poisson arrivals.

The stability of the system is established in §7 via the analysis of a Markov-renewal process embedded at the busy period termination epochs. A closed form expression for the Laplace transform of the time between two consecutive disasters that affect the system is obtained.

2 The Model

The system consists of a single server to which customers arrive in batches according to a Poisson process with rate λ and are processed according to the FIFO discipline. Denote the size of nth batch by χ_n, $n = 1, 2, \ldots$. Batches are assumed to be i.i.d. with pgf (probability generating function) $\chi(z) := \sum_{n=1}^{\infty} \mathbb{P}(\chi_1 = n) z^n$. (Without loss of generality we assume that there are no empty batches.) The mean batch size is denoted by $m_\chi := \mathbb{E}\chi$. Each customer in the batch is served singly and the service requirements of the customers are assumed to be i.i.d. random variables with common distribution S which will be assumed absolutely continuous with corresponding density S' and hazard rate function $\mu(x) = \frac{S'(x)}{1-S(x)}$, $x \geq 0$. The Laplace transform of S will be denoted by $\hat{S}(s) := \int_0^\infty e^{-sx} dS(x)$. We do not assume that its mean is necessarily finite.

The server follows a MAV policy: At the end of the busy period the server either remains idle (with probability $1 - g_0$) or takes a vacation with probability g_0. If at the end of this vacation no customers have arrived then the server, independently of everything else, takes a new vacation with probability g_1, or remains idle and available to serve the first customer that arrives with probability $1 - g_1$. The process is repeated: Thus the policy is determined by the sequence of probabilities $\{g_k\}$, $k = 0, 1, 2, \ldots$. An alternative way to describe this policy is to assume that the number of vacations to be taken at the end of the ith busy period is a random variable ζ_i with distribution $\mathbb{P}(\zeta_i = k) = f_k$, $k = 0, 1, 2, \ldots$. If we set $F_k = \sum_{l=k}^{\infty} f_l$ then the probability that the server will take a vacation at the end of a busy period is $g_0 = F_1$ while the probability that he will take a vacation after k vacations have been completed with no arrivals is

$$g_k := \mathbb{P}(\zeta_i \geq k+1 | \zeta_i \geq k) = \frac{F_{k+1}}{F_k}, \quad k = 0, 1, 2, \ldots. \tag{2.1}$$

$F(z) := \sum_{k=0}^{\infty} f_k z^k$ denotes the pgf of the number of potential vacations. In some instances we will allow the cumulative distribution of the number of potential vacations in a string to be *defective*, i.e. $\lim_{k\to\infty} F_k < 1$ or equivalently $F(1) < 1$. This provides additional flexibility and allows us to include the case of repeated vacations (i.e. vacation strings that terminate only when a batch of customers arrives at the system) in the same framework. This corresponds to $F(z) \equiv 0$. On the other hand $F(z) \equiv 1$ corresponds to a system with no vacations.

Vacations have independent durations with common distribution function U, assumed again to be absolutely continuous with density U' and hazard rate $u(x) = \frac{U'(x)}{1-U(x)}$, $x \geq 0$. The corresponding Laplace transform will be denoted by $\hat{U}(s) = \int_0^\infty e^{-sx} dU(x)$ and the corresponding mean, *which is assumed finite*, by $m_U := \mathbb{E}U < \infty$.

Finally, the system is subject to catastrophic failures. These disasters occur according to a Poisson process with rate δ, independently of all other processes in the system, *provided that the server is busy, serving customers*. More precisely we assume that *potential disasters* occur according to a Poisson process, independent of all other processes in the system. If at the time a potential disaster occurs the server is busy serving customers then this becomes an *actual disaster* which removes all customers from the system and causes a server breakdown which is immediately followed by a repair period. If on the other hand a potential disaster occurs when the server is under repair, on vacation, or idle, then the disaster is not "realized" and has no effect on the system. Having clarified this point, we shall refer to actual disasters simply as disasters.

When a disaster occurs all customers present, including the one in service, are removed from the system and the server initiates a repair period. Repairs have i.i.d. durations with distribution function R, assumed absolutely continuous, with density function R', hazard rate $r(x) = \frac{R'(x)}{1-R(x)}$, $x \geq 0$, and Laplace transform $\hat{R}(s) = \int_0^\infty e^{-sx} dR(x)$. The mean repair time is also *assumed finite* and is denoted by m_R. During a repair period, any customers that may have arrived wait in line. As soon as the server is repaired, if there are customers waiting in line, a new busy period starts immediately, otherwise the server takes a string of vacations following the MAV policy described above.

For any probability distribution G on $[0, \infty)$ with finite mean m we will denote the corresponding equilibrium (or integrated tail) distribution by $G_e(x) := m^{-1} \int_0^x [1 - G(y)]dy$ and will use the fact that its Laplace transform is given by $\hat{G}_e(s) = \frac{1-\hat{G}(s)}{sm}$. For a discrete random variable, such as the batch size, χ, the equilibrium pgf is defined by $\chi_e(z) := \frac{1}{m_\chi} \frac{1-\chi(z)}{1-z}$.

We should point out that the assumption requiring the absolute continuity of the distribution functions of service, repair, and vacations times is necessary in order to provide a Markovian description of the system via the use of supplementary variables which leads to a system of ordinary differential equations for the occupation densities of states. However the final expressions obtained for the probability generating function of the stationary number of customers in the system and other quantities of interest are meaningful even when the aforementioned distribution functions are not absolutely continuous. Since the class of absolutely continuous distribution functions are dense within the space of probability distribution functions one could generalize standard results on the continuity of the dependence of the stationary distribution for the number of customers in the system, the workload etc. and establish rigorously that the results obtained under the absolute continuity assumption hold for arbitrary probability distribution functions.

3 Analysis via Supplementary Variables

Here we derive the steady-state differential-difference equations for the system by treating the elapsed service time, the elapsed repair time, and the elapsed vacation time as supplementary variables. From these we obtain the partial probability generating functions for the number of customers in the system and the state of the sever in stationarity. We will consider the processes

N_t : number of customers in the system at time t

$\mathcal{S}_t$: elapsed service time at time t (if server busy, otherwise 0)

$\mathcal{R}_t$: elapsed repair time at time t (if server on repair, otherwise 0)

$\mathcal{U}_t^j$: elapsed time of the jth vacation at time t (if server on jth vacation otherwise 0)

and the process $\{\xi_t\}$ taking values in the set $\{i, s, r\} \cup \{u_1, u_2, u_3, \ldots\}$. The elements of $\{i, s, r\}$ correspond to the server being *idle*, *serving* customers, or being under *repair* respectively, while the elements of $\{u_1, u_2, \ldots\}$ correspond to the server being on the first, second, etc. vacation within a string of vacations at the end of a busy cycle. Due to the presence of disasters the system has a regenerative structure with regeneration points the epochs when disasters occur and regenerative cycles of finite expected length. This is established in the detailed analysis given in §7.6 and in particular in Corollary 12. Therefore a stationary version of the process exists by virtue of standard results on regenerative processes (e.g. see [2]).

3.1 Balance equations

Suppose the process is stationary under the probability measure $\mathbb{P}$ and define the densities

$$
\begin{aligned}
P_0 &:= \mathbb{P}(N_0 = 0, \xi_0 = i),\\
P_n(x) &:= \lim_{h\downarrow 0} \frac{1}{h}\mathbb{P}(N_0 = n, \xi_0 = s;\ x < \mathcal{S}_0 \leq x + h),\\
W_n(x) &:= \lim_{h\downarrow 0} \frac{1}{h}\mathbb{P}(N_0 = n, \xi_0 = r; x < \mathcal{R}_0 \leq x + h),\\
V_{j,n}(x) &:= \lim_{h\downarrow 0} \frac{1}{h}\mathbb{P}(N_0 = n, \xi_t = u_j; x < \mathcal{U}_0^j \leq x + h), \quad j = 1, 2, \ldots.
\end{aligned}
$$

The balance equations satisfied in stationarity are

$$
\lambda P_0 = \sum_{j=1}^{\infty}(1-g_j)\int_0^\infty V_{j,0}(x)u(x)dx \ + \ (1-g_0)\left(\int_0^\infty P_1(x)\mu(x)dx + \int_0^\infty W_0(x)r(x)dx\right) \tag{3.1}
$$

$$
\frac{d}{dx}P_n(x) + (\lambda+\delta+\mu(x))\,P_n(x) = \mathbf{1}(n\geq 2)\lambda\sum_{k=1}^{n-1}\chi_k P_{n-k}(x), \quad x>0,\ n\geq 1 \tag{3.2}
$$

$$
\frac{d}{dx}W_0(x) + (\lambda + r(x))W_0(x) = 0 \tag{3.3}
$$

$$
\frac{d}{dx}W_n(x) + (\lambda + r(x))W_n(x) = \lambda\sum_{k=1}^{n}\chi_k W_{n-k}(x), \quad x>0, n\geq 1 \tag{3.4}
$$

$$
\frac{d}{dx}V_{j,0}(x) + (\lambda + u(x))V_{j,0}(x) = 0, \quad j = 1, 2, \ldots \tag{3.5}
$$

$$
\frac{d}{dx}V_{j,n}(x) + (\lambda + u(x))V_{j,n}(x) = \lambda\sum_{k=1}^{n}\chi_k V_{j,n-k}(x), \quad x>0,\ n\geq 1,\ j = 1, 2, \ldots \tag{3.6}
$$

The boundary conditions of the above system of differential equations are

$$
P_n(0) = \sum_{j=1}^{\infty}\int_0^\infty V_{j,n}(x)u(x)dx + \int_0^\infty P_{n+1}(x)\mu(x)dx + \int_0^\infty W_n(x)r(x)dx + \lambda\chi_n P_0, \ \ n\geq 1 \tag{3.7}
$$

$$
V_{1,0}(0) = g_0\int_0^\infty P_1(x)\mu(x)dx + g_0\int_0^\infty W_0(x)r(x)dx \tag{3.8}
$$

$$
V_{j,0}(0) = g_{j-1}\int_0^\infty V_{j-1,0}(x)u(x)dx, \quad j = 2, 3, \ldots \tag{3.9}
$$

$$
W_0(0) = \delta\sum_{n=1}^{\infty}\int_0^\infty P_n(x)dx \tag{3.10}
$$

with normalization condition

$$
P_0 + \sum_{n=1}^{\infty}\int_0^\infty P_n(x)dx + \sum_{n=0}^{\infty}\left(\int_0^\infty W_n(x)dx + \sum_{j=1}^{\infty}\int_0^\infty V_{j,n}(x)dx\right) = 1. \tag{3.11}
$$

3.2 Solution using partial probability generating functions

Define the partial probability generating functions

$$
\begin{aligned}
P(x;z) &:= \lim_{h\downarrow 0}\frac{1}{h}\mathbb{E}[z^{N_0};\xi_0=s,\, x<\mathcal{S}_0\le x+h] \;=\; \sum_{n=1}^{\infty} z^n P_n(x),\\
W(x;z) &:= \lim_{h\downarrow 0}\frac{1}{h}\mathbb{E}[z^{N_0};\xi_0=r,\, x<\mathcal{R}_0\le x+h] \;=\; \sum_{n=0}^{\infty} z^n W_n(x),\\
V_j(x;z) &:= \lim_{h\downarrow 0}\frac{1}{h}\mathbb{E}[z^{N_0};\xi_0=u_j,\, x<\mathcal{U}_0^j\le x+h] \;=\; \sum_{n=0}^{\infty} z^n V_{j,n}(x),\quad j=1,2,\ldots.
\end{aligned}
$$

The partial pgf's for the number of customers in the system in stationarity regardless of the value of the supplementary variables are then given by

$$
\begin{aligned}
P(z) &:= \mathbb{E}[z^{N_0};\xi_0=s]=\int_0^\infty P(x;z)dx, \qquad W(z):=\mathbb{E}[z^{N_0};\xi_0=r]=\int_0^\infty W(x;z)dx,\\
V_j(z) &:= \mathbb{E}[z^{N_0};\xi_0=u_j]=\int_0^\infty V_j(x;z)dx,\quad j=1,2,\ldots.
\end{aligned}\tag{3.12}
$$

Proposition 1. *Let* $\alpha(z):=\lambda(1-\chi(z))$, $\beta:=\hat{U}(\lambda)\in(0,1)$, $C:=\frac{1-F(\beta)}{1-\beta}\,\frac{1}{F(\beta)}$, *and*

$$
K(z):=C\left(1-\hat{U}(\alpha(z))\right)+1-\chi(z)=(1-\chi(z))\left(1+\lambda m_U C\hat{U}_e(\alpha(z))\right). \tag{3.13}
$$

The partial pgf for the number of customers in the system when the server is busy is given by

$$
P(z) \;=\; P(0;z)\,\frac{1-\hat{S}(\delta+\alpha(z))}{\delta+\alpha(z)} \tag{3.14}
$$

where

$$
P(0;z)=z\;\frac{\lambda P_0 K(z)-\delta P(1)\hat{R}(\alpha(z))}{\hat{S}(\delta+\alpha(z))-z}. \tag{3.15}
$$

The expression for $P(0;z)$ given in (3.15) depends on the unknown values of P_0 and $P(1)$. These can be determined using Rouché's theorem and the normalization condition (3.11) as we shall see in the sequel.

Proof. From (3.2), (3.4), (3.6), and (3.7) we obtain the linear first order PDE's

$$
\begin{aligned}
\frac{\partial}{\partial x}P(x;z)+(\alpha(z)+\delta+\mu(x))P(x;z) &= 0,\\
\frac{\partial}{\partial x}W(x;z)+(\alpha(z)+r(x))W(x;z) &= 0,\\
\frac{\partial}{\partial x}V_j(x;z)+(\alpha(z)+u(x))V_j(x;z) &= 0,
\end{aligned}\tag{3.16}
$$

and the equation

$$
\begin{aligned}
\sum_{n=1}^{\infty} z^n P_n(0) &= \sum_{j=1}^{\infty}\int_0^\infty \sum_{n=1}^{\infty} z^n V_{j,n}(x)u(x)dx+\int_0^\infty \sum_{n=1}^{\infty} z^n P_{n+1}(x)\mu(x)dx\\
&\quad +\int_0^\infty \sum_{n=1}^{\infty} z^n W_n(x)r(x)dx+\lambda\sum_{n=1}^{\infty}\chi_n z^n P_0
\end{aligned}
$$

which, taking into account that $\sum_{n=1}^{\infty} V_{j,n}(x)z^n = V_j(x;z) - V_{j,0}(x)$, $\sum_{n=1}^{\infty} W_n(x)z^n = W(x;z) - W_0(x)$, and $\sum_{n=1}^{\infty} P_{n+1}(x)z^n = z^{-1}P(x;z) - P_1(x)$, gives

$$\begin{aligned} P(0;z) &= \sum_{j=1}^{\infty}\int_0^{\infty} V_j(x;z)u(x)dx - \sum_{j=1}^{\infty}\int_0^{\infty} V_{j,0}(x)u(x)dx + z^{-1}\int_0^{\infty} P(x;z)\mu(x)dx \\ &\quad - \int_0^{\infty} P_1(x)\mu(x)dx + \int_0^{\infty} W(x;z)r(x)dx - \int_0^{\infty} W_0(x)r(x)dx + \lambda P_0\chi(z). \end{aligned} \tag{3.17}$$

Solving (3.16) we obtain

$$\begin{aligned} P(x;z) &= P(0;z)(1-S(x))e^{-(\delta+\alpha(z))x} \\ W(x;z) &= W(0;z)(1-R(x))e^{-\alpha(z)x} \\ V_j(x;z) &= V_j(0;z)(1-U(x))e^{-\alpha(z)x}, \quad j=1,2,\dots. \end{aligned} \tag{3.18}$$

The solution of (3.5) is

$$V_{j,0}(x) = V_{j,0}(0)(1-U(x))e^{-\lambda x}, \quad j=1,2,\dots \tag{3.19}$$

whence we obtain

$$\int_0^{\infty} V_{j,0}(x)u(x)dx = V_{j,0}(0)\hat{U}(\lambda), \quad j=1,2,\dots \tag{3.20}$$

since $\int_0^{\infty}(1-U(x))e^{-\lambda x}u(x)dx = \int_0^{\infty} e^{-\lambda x}dU(x)$. We thus get

$$V_{j,0}(0) = g_{j-1}\beta V_{j-1,0}(0), \quad j=2,3,\dots. \tag{3.21}$$

Note also that $V_j(0;z) = V_{j,0}(0)$ and $W(0;z) = W_0(0)$ because when the jth vacation begins, or when a repair begins, immediately after a disaster, there are necessarily no customers in the system.

Set $V_{j,0}(0) =: v_j$. Using (2.1), (3.21) gives

$$v_j = \frac{F_j}{F_1}\beta^{j-1}v_1, \quad j=1,2,\dots.$$

Thus

$$\sum_{j=1}^{\infty} v_j = \sum_{j=1}^{\infty}\frac{F_j}{F_1}\beta^{j-1}v_1 = \frac{v_1}{F_1}\,\frac{1-F(\beta)}{1-\beta}. \tag{3.22}$$

From (3.18) and (3.19) we obtain

$$\int_0^{\infty} V_j(x;z)u(x)dx = \int_0^{\infty} V_j(0;z)(1-U(x))e^{-\alpha(z)x}u(x)dx \;=\; v_j\hat{U}(\alpha(z)),\; j=1,2,\dots, \tag{3.23}$$

$$\int_0^{\infty} V_{j,0}(x)u(x)dx = \int_0^{\infty} V_{j,0}(0)(1-U(x))e^{-\lambda x}u(x)dx \;=\; v_j\beta,\; j=1,2,\dots, \tag{3.24}$$

$$\int_0^{\infty} P(x;z)\mu(x)dx = \int_0^{\infty} P(0;z)(1-S(x))e^{-(\delta+\alpha(z))x}\mu(x)dx \;=\; P(0;z)\hat{S}(\delta+\alpha(z)), \tag{3.25}$$

$$\int_0^{\infty} W(x;z)r(x)dx = \int_0^{\infty} W(0;z)(1-R(x))e^{-\alpha(z)x}r(x)dx \;=\; W(0;z)\hat{R}(\alpha(z)). \tag{3.26}$$

v_1 is obtained from (3.1), (3.8), and (3.9):

$$\lambda P_0 = \sum_{j=1}^{\infty} v_{j+1} \frac{1-g_j}{g_j} + \frac{1-g_0}{g_0} v_1.$$

Using (3.21) and the fact that $\frac{1-g_j}{g_j} = \frac{f_j}{F_{j+1}}$ we obtain

$$\lambda P_0 \;=\; \sum_{j=1}^{\infty} F_1^{-1} f_j \beta^j v_1 + v_1 \frac{f_0}{F_1} \;=\; \frac{v_1}{F_1} \left(F(\beta) - f_0\right) + v_1 \frac{f_0}{F_1} \;=\; \frac{v_1}{F_1} F(\beta). \tag{3.27}$$

The following argument clarifies the meaning of (3.27). When a busy period terminates by the departure of a customer leaving the system empty or when a repair period (following a disaster) terminates with no customers in the system, the server takes a string of vacations which may be *empty* (i.e. in fact the server does not take a vacation) with probability $f_0 = 1 - F_1$ or *non-empty* with probability F_1. The rate of non-empty vacation string initiations is v_1 and thus the rate of all vacation string initiations, (including empty strings) is v_1/F_1. Also, $F(\beta) := \sum_{j=0}^{\infty} f_j \hat{U}(\lambda)^j$ is the probability that no arrivals occur during a vacation string, *empty or otherwise*. Thus $v_1 F(\beta)/F_1$ is the rate of entering the idle state of the server whereas λP_0 is the exit rate for the same state. Equation (3.27) expresses the fact that these two should be equal.

Using (3.23)–(3.26) in (3.17) we obtain

$$\begin{aligned} P(0;z) \;=\;& \sum_{j=1}^{\infty} v_j \hat{U}(\alpha(z)) - \sum_{j=1}^{\infty} v_j \beta + z^{-1} P(0;z) \hat{S}(\delta + \alpha(z)) + W(0;z) \hat{R}(\alpha(z)) \\ & -v_1 + \sum_{j=1}^{\infty} (1-g_j) v_j \beta + \lambda P_0 \chi(z) - \lambda P_0. \end{aligned}$$

This, after some simplifications, using (3.22) and the fact that $\beta \sum_{j=1}^{\infty} g_j v_j = \sum_{j=2}^{\infty} v_j$, which is a consequence of (3.21), gives

$$P(0;z)\left(1 - z^{-1}\hat{S}(\delta + \alpha(z))\right) = \lambda P_0 \left(\hat{U}(\alpha(z)) - 1\right) C + W(0;z)\hat{R}(\alpha(z)) + \lambda P_0 \left(\chi(z) - 1\right). \tag{3.28}$$

Using (3.18) we write (3.10) as

$$W_0(0) \;=\; W(0;z) \;=\; \delta \sum_{n=1}^{\infty} \int_0^{\infty} P_n(x)dx \;=\; \delta \int_0^{\infty} P(x,1)dx \;=\; \delta P(1). \tag{3.29}$$

(3.28) and (3.29) establish (3.15). (3.14) follows from (3.12) and (3.18). □

Proposition 2. *The quantities P_0 and $P(1)$ in (3.15) satisfy the relationship*

$$\delta P(1)\hat{R}(\alpha(z_\delta)) = \lambda P_0 K(z_\delta) \tag{3.30}$$

where z_δ be the unique *solution of the equation*

$$z \;=\; \hat{S}(\delta + \lambda - \lambda\chi(z)) \tag{3.31}$$

in the open unit disk $|z| < 1$.

Proof. The existence of a unique solution z_δ of (3.31) in the open unit disk in the complex plane is established in Proposition 14 in the Appendix. Since $P(0; z)$ must be finite for all $|z| \le 1$, and since the denominator on the right hand side of (3.15) vanishes for $z = z_\delta$, the numerator must also vanish at this value, whence (3.30) follows. □

Remark 1. Consider a system with Poisson arrivals with rate λ in which customers arrive in batches with pgf $\chi(z)$. (This system operates with no vacations, no disasters, and no repairs.) Let Γ_0 denote the length the a busy period of this system starting with *a single customer* and denote by $\hat{\Gamma}_0(s) := \mathbb{E}[e^{-s\Gamma_0}]$ its Laplace transform. Note that, since we make no assumption regarding the stability of the system without vacations, Γ_0 may be a defective random variable with $\mathbb{P}(\Gamma_0 < \infty) < 1$ or, equivalently, $\hat{\Gamma}_0(0) < 1$. In any case $\hat{\Gamma}_0(s)$ is equal to the unique solution z_s of

$$z = \hat{S}(s + \lambda - \lambda\chi(z)) \tag{3.32}$$

inside the unit disk (see for instance [15] and [16, p.49]). Thus $z_s := \hat{\Gamma}_0(s)$ and $z_\delta = \hat{\Gamma}_0(\delta)$ which is equal to the probability that a busy period starting with a single customer does not suffer a disaster.

3.3 Partial Probability Generating Functions

We are now ready to derive explicit expressions for the partial pgf's for the number of customers in the system under stationarity according to the state of the server (busy, under repair, or on vacation). From these, the corresponding stationary probabilities can be obtained. To simplify the expressions we will set

$$\gamma := \frac{K(z_\delta)}{\hat{R}(\alpha(z_\delta))} = \frac{(1-\chi(z_\delta))\left(1 + \lambda m_U C\hat{U}_e(\alpha(z_\delta))\right)}{\hat{R}(\alpha(z_\delta))}. \tag{3.33}$$

When there are no repairs, $\gamma = K(z_\delta)$. If, in addition, there are no vacations then $K(z) = 1 - \chi(z)$ and $\gamma = 1 - \chi(z_\delta) = 1 - \chi(\hat{\Gamma}_0(\delta))$.

1. The partial pgf of the time stationary probabilities for the system size when the server is working.

$$P(z) = \lambda P_0 z \frac{\Theta(z)}{\hat{S}(\delta + \alpha(z)) - z} \frac{1 - \hat{S}(\delta + \alpha(z))}{\delta + \alpha(z)}. \tag{3.34}$$

where P_0 is given by (3.38). Note that $P(1) = \lambda P_0 \gamma / \delta$ is the *stationary probability that the server is busy serving customers.*

2. The partial pgf of the time stationary probabilities for the system size when the server is under repair. Taking into account (3.12), (3.18), and (3.30) we obtain

$$W(z) = \lambda P_0 \frac{K(z_\delta)}{\hat{R}(\alpha(z_\delta))} \frac{1 - \hat{R}(\alpha(z))}{\alpha(z)} = \lambda P_0 \gamma m_R \hat{R}_e(\alpha(z)). \tag{3.35}$$

Using de l' Hôpital's rule we obtain $W(1) = \lambda P_0 \gamma m_R$. This is the *steady state probability that the server is under repair.*

3. The partial pgf of the time stationary probabilities for the system size when the server is on the jth vacation.

$$\begin{aligned} V_j(z) &= \int_0^\infty V_j(x;z)dx \ = \ V_j(0;z)\int_0^\infty (1-U(x))e^{-\alpha(z)x}dx \\ &= V_j(0;z)\,\frac{1-\hat{U}(\alpha(z))}{\alpha(z)} \ = \ v_1\frac{F_j}{F_1}\beta^{j-1}\,m_U\hat{U}_e(\alpha(z)), \quad j=1,2,\ldots. \end{aligned}$$

Again, de l' Hôpital's rule gives $V_j(1) = v_j m_U$. Note that

$$V(1) = \sum_{j=1}^{\infty} V_j(1) = m_U \sum_{j=1}^{\infty} v_j = m_U\,\frac{v_1}{F_1}\,\frac{1-F(\beta)}{1-\beta} = \lambda P_0 m_U C\ . \tag{3.36}$$

This is the *probability that the server is on vacation.* Recall that the rate of non-empty vacation string initiations is given by v_1 (see the discussion following (3.27). If we denote by m_V the mean duration of a vacation string then (essentially by Little's law) $v_1 m_V = V(1)$, the stationary probability that the system is on vacation. This, taking into account (3.36), yields $m_V = \frac{m_U}{F_1}\frac{1-F(\beta)}{1-\beta}$.

Also $V(z) := \sum_{j=1}^{\infty} V_j(z)$ is given by

$$V(z) = \frac{v_1}{F_1}\frac{1-\hat{U}(\alpha(z))}{\alpha(z)}\sum_{j=1}^{\infty} F_j\beta^{j-1} = \lambda P_0\frac{1-F(\beta)}{(1-\beta)F(\beta)}\frac{1-\hat{U}(\alpha(z))}{\alpha(z)} = \lambda P_0 m_U C\hat{U}_e(\alpha(z)). \tag{3.37}$$

4. The stationary probability that the server is idle. P_0 can be determined by using the normalization condition $P_0 + P(1) + W(1) + \sum_{j=1}^{\infty} V_j(1) = 1$ which gives

$$P_0 = \left(1 \ + \ \frac{\lambda\gamma}{\delta} \ + \ \lambda\gamma m_R \ + \ \lambda m_U C\right)^{-1}. \tag{3.38}$$

5. The pgf of the number of customers in the system in stationarity. This is given by $\Phi(z) = P_0 + P(z) + W(z) + V(z)$. Thus

$$\Phi(z) \ = \ P_0\ \Big(1 + \lambda m_U C\hat{U}_e(\alpha(z)) + \gamma\lambda m_R\hat{R}_e(\alpha(z))\Big) + z\lambda P_0\,\frac{\Theta(z)}{\alpha(z)+\delta}\,\frac{1-\hat{S}(\delta+\alpha(z))}{\hat{S}(\delta+\alpha(z))-z} \tag{3.39}$$

with

$$\Theta(z) := K(z) - \gamma\hat{R}(\alpha(z)). \tag{3.40}$$

4 Performance Measures and Special Cases

4.1 Rate arguments

The rate at which disasters occur, d, is equal to the rate of repair initiations and is given by

$$d \ = \ \delta P(1) = W_0(0) = \lambda P_0\gamma. \tag{4.1}$$

The above is therefore the rate of busy period terminations due to disasters. We next determine the rate of *ordinary* busy period terminations, i.e. those due to service completions which leave the system empty. This rate is clearly $\int_0^\infty P_1(x)\mu(x)dx$. Equation (3.3) gives $W_0(x) = W_0(0)e^{-\lambda x}(1 - R(x))$ whence we obtain $\int_0^\infty W_0(x)r(x)dx = W_0(0)\hat{R}(\lambda)$ and thus, recalling that $g_0 = F_1$ and $V_{1,0}(0) = v_1$, (3.8) gives

$$\int_0^\infty P_1(x)\mu(x)dx = \frac{v_1}{F_1} - W_0(0)\hat{R}(\lambda) = \frac{v_1}{F_1} - \delta P(1)\hat{R}(\lambda) = \frac{v_1}{F_1} - \lambda P_0\gamma\hat{R}(\lambda) \tag{4.2}$$

where, in the above equalities we have also used (3.29), (3.30), and (3.33). Thus, the rate of ordinary busy period terminations (as opposed to terminations due to disasters) is given by

$$r_b := \int_0^\infty P_1(x)\mu(x)dx = \lambda P_0\left(\frac{1}{F(\beta)} - \gamma\hat{R}(\lambda)\right). \tag{4.3}$$

where, in the above equalities we have also used (4.1) and (3.27). Finally, the rate of initiations of busy periods, b, is given by

$$b = r_b + d = \lambda P_0\left(\frac{1}{F(\beta)} - \gamma(1 - \hat{R}(\lambda))\right). \tag{4.4}$$

(Indeed, each busy period always follows a previous one that was either terminated by the departure of a customer, or by the occurrence of a disaster.)

The proportion of customers that complete their service (as opposed to being eliminated from the system as a result of a disaster) is obtained from a ratio of rates argument as follows: The rate of customer departures due to service completion, r_c, is

$$r_c = \sum_{n=1}^\infty \int_0^\infty P_n(x)\mu(x)dx = \int_0^\infty P(x;1)\mu(x)dx = P(0;1)\hat{S}(\delta) = \lambda P_0\gamma\frac{\hat{S}(\delta)}{1 - \hat{S}(\delta)}. \tag{4.5}$$

Since the arrival rate of customers is λm_χ, the fraction of customers who complete their service, f_c is

$$f_c \;=\; \frac{r_c}{\lambda m_\chi} \;=\; P_0\frac{\gamma}{m_\chi}\frac{\hat{S}(\delta)}{1 - \hat{S}(\delta)}. \tag{4.6}$$

The rate of departures due to disasters is therefore $\lambda m_\chi - r_c = \lambda\left(m_\chi - P_0\gamma\frac{\hat{S}(\delta)}{1-\hat{S}(\delta)}\right)$. Hence, the average number of customers removed from the server by a disaster is

$$\frac{\lambda m_\chi - r_c}{d} \;=\; \frac{m_\chi}{\gamma P_0} - \frac{\hat{S}(\delta)}{1 - \hat{S}(\delta)}.$$

4.2 The pgf of the number of customers removed by a disaster

Proposition 3. *The number of customers removed from the system by a disaster has pgf*

$$\Phi_d(z) \;=\; \frac{P(z)}{P(1)} \;=\; z\frac{\delta}{\gamma}\frac{\Theta(z)}{\hat{S}(\delta + \alpha(z)) - z}\,\frac{1 - \hat{S}(\delta + \alpha(z))}{\delta + \alpha(z)}. \tag{4.7}$$

Proof. For a rigorous proof we will need Papangelou's theorem stated in the framework of Palm probabilities for stationary point processes. We refer the reader to [3]. (For a more elementary approach, one could use a conditional PASTA argument [20].) Denote by $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$, the filtration generated by the history of the process $\{(N_t,\xi_t)\}_{t\in\mathbb{R}}$. Recall that the sample paths of this process are assumed to be *right-continuous* $\mathbb{P}$–a.s. . We assume that under the probability measure $\mathbb{P}$ this process is stationary. Let $\{D_n\}_{n\in\mathbb{Z}}$, denote the disaster point process and let $\boldsymbol{\delta}_t$ denote its $\mathcal{F}-$stochastic intensity which can be explicitly expressed as $\boldsymbol{\delta}_t = \delta\,\mathbf{1}(\xi_t = s)$. Denote by $\mathbb{P}^0_D$ the Palm transformation of $\mathbb{P}$ with respect to the point process $\{D_n\}$. (Intuitively, $\mathbb{P}^0_D$ represents the "event-stationary" probability with events here being the disasters.)

By the definition of the Palm measure, $\mathbb{P}^0_D(D_0 = 0) = 1$. Thus, under $\mathbb{P}^0_D$, a typical disaster event happens at the time origin. The number of customers eliminated as a result of this disaster is N_{0-}, namely those that were present immediately prior to 0. Papangelou's theorem [3, p.213] connects event-stationary expectations via the stochastic intensity of the events $\{D_n\}$. Applying it to the $\mathcal{F}_t$–adapted process $\{z^{N_t}\}$ we obtain the following relationship

$$\mathbb{E}^0_D[z^{N_{0-}}] \;=\; \frac{\mathbb{E}[z^{N_0}\boldsymbol{\delta}_0]}{\mathbb{E}[\boldsymbol{\delta}_0]}\,. \tag{4.8}$$

The Palm expectation on the left hand side of the above is precisely the pgf of the number of customers eliminated by a disaster whereas, on the right, we have (time–) stationary expectations that can be evaluated in terms of the results of §3. Thus $\mathbb{E}[\boldsymbol{\delta}_0] = \mathbb{E}[\delta\mathbf{1}(\xi_0 = s)] = \delta\mathbb{P}(\xi_0 = s) = \delta P(1)$ and hence $\Phi_d(z) \;:=\; \mathbb{E}^0_D[z^{N_{0-}}] \;=\; \mathbb{E}[z^{N_{0-}}\mathbf{1}(\xi = s)\delta]/\mathbb{E}[\mathbf{1}(\xi_0 = s)\delta]$, which establishes the first equality in (4.7). The second equality follows readily from (3.14) and (3.15). □

4.3 The pgf of the system size at a departure epoch

A departing customer will leave behind l customers in the system at a departure epoch if and only if there are $l+1$ customers in the system just before the departure. Thus, if ϕ^+_l denotes the probability that a departing customer leaves behind l customers in the system from a stochastic intensity argument we obtain

$$\phi^+_l = D\int_0^\infty \mu(x)P_{l+1}(x)dx$$

where D is a normalizing constant. If $\Phi^+(z) := \sum_{l=0}^\infty \phi^+_l z^l$ denotes the corresponding pgf, taking into account that $\sum_{l=0}^\infty z^l P_{l+1}(x) = z^{-1}P(x;z)$, (3.25), and (3.15) we obtain

$$\Phi^+(z) = D\Theta(z)\frac{\hat{S}(\delta+\alpha(z))}{\hat{S}(\delta+\alpha(z)) - z} \tag{4.9}$$

with the normalizing constant D determined as $D = \frac{1-\hat{S}(\delta)}{\gamma\hat{S}(\delta)}$. Thus,

$$\Phi^+(z) = \frac{1}{\gamma}\Theta(z)\frac{\hat{S}(\delta+\alpha(z))}{\hat{S}(\delta)}\,\frac{1-\hat{S}(\delta)}{\hat{S}(\delta+\alpha(z)) - z}\;. \tag{4.10}$$

Let us now examine what happens when a) repair times are negligible and b) when disasters do not occur. In the first case $\hat{R}(s) \equiv 1$ and hence from (4.10) and (3.33) we obtain

$$\Phi^+(z) = \left(1 - \frac{K(z)}{K(z_\delta)}\right)\frac{\hat{S}(\delta+\alpha(z))}{\hat{S}(\delta)}\,\frac{1-\hat{S}(\delta)}{z - \hat{S}(\delta+\alpha(z))}\;.$$

In an ordinary $M/G/1$ system (with no batches or vacations) $K(z) = 1 - z$ and this gives

$$\Phi^+(z) = \frac{z - z_\delta}{1 - z_\delta}\,\frac{\hat{S}(\delta+\lambda-\lambda z)}{\hat{S}(\delta)}\,\frac{1-\hat{S}(\delta)}{z - \hat{S}(\delta+\lambda-\lambda z)}\;.$$

4.4 Special Cases

Different Vacation Policies. The MAV vacation policy encompasses most other vacation policies considered in practice. A few examples are provided below:

Single Vacations. The server takes a single vacation when it remains idle. This can be modeled by setting $g_0 = 0$, $g_1 = 1$ and $g_i = 0$ for $i = 2, 3, \ldots$. Then $F(z) = z$ and $C = 1/\beta$.

Bernoulli Vacations. Here $g_i = q$, $i = 0, 1, 2, \ldots$ where $q \in (0, 1)$. Set $p = 1 - q$. Then $F(z) = \frac{p}{1-qz}$ and $C = p/q$. In this case C is simply the mean number of vacations taken.

The system with non-terminating vacations. This case also falls under the framework we consider by assuming that the distribution $\{f_j\}$ is defective with $f_j = 0$, $F_j = \mathbb{P}(\zeta \geq j) = 1$, and hence $g_j = 1$, for $j = 1, 2, \ldots$ and $F(z) = 0$ for $z \neq 0$. Clearly, in this case the server is never idly waiting for a customer to serve. The rate equations (3.1), (3.8), (3.9) are still valid but with $P_0 = 0$. (3.15) can then be stated as

$$P(0; z) = z\, \frac{(1 - \hat{U}(\alpha(z)))v_1 \frac{1}{1-\beta} - \delta P(1)\hat{R}(\alpha(z))}{\hat{S}(\delta + \alpha(z)) - z}.$$

The argument based on Rouché's Theorem gives $\delta P(1) = \frac{1-\hat{U}(\alpha(z_\delta))}{\hat{R}(\alpha(z_\delta))} \frac{v_1}{1-\beta}$ and upon substituting in the equation above we obtain

$$P(0; z) = z\, \frac{v_1}{1-\beta} \frac{1 - \hat{U}(\alpha(z)) - \frac{\hat{R}(\alpha(z))}{\hat{R}(\alpha(z_\delta))}(1 - \hat{U}(\alpha(z_\delta)))}{\hat{S}(\delta + \alpha(z)) - z}.$$

Equations (3.18) are of course still valid. Also, $W(z) = \delta P(1) \frac{1-\hat{R}(\alpha(z))}{\alpha(z)} = \frac{1-\hat{U}(\alpha(z_\delta))}{\hat{R}(\alpha(z_\delta))} \frac{v_1}{1-\beta} m_R\, \hat{R}_e(\alpha(z))$ and $V(z) = v_1 m_U \frac{1}{1-\beta} \hat{U}_e(\alpha(z))$. Setting $\widetilde{\gamma} := \frac{1-\hat{U}(\alpha(z_\delta))}{\hat{R}(\alpha(z_\delta))}$, we obtain the following expression for the pgf of the number of customers in the system in stationarity:

$$\Phi(z) = \frac{v_1}{1-\beta} \left[z\, \frac{1 - \hat{U}(\alpha(z)) - \widetilde{\gamma}\hat{R}(\alpha(z))}{\delta + \alpha(z)} \frac{\hat{S}(\delta + \alpha(z))}{\hat{S}(\delta + \alpha(z)) - z} + \widetilde{\gamma} m_R \hat{R}_e(\alpha(z)) + m_U \hat{U}_e(\alpha(z)) \right]. \quad (4.11)$$

The value of v_1 is easily determined by the requirement that $\Phi(1) = 1$. Thus $\frac{v_1}{1-\beta} = \left(\widetilde{\gamma}\delta^{-1} + \gamma m_R + m_U\right)^{-1}$.

The system without repairs. If the repair period after the occurrence of a disaster has negligible duration then we may set $m_R = 0$, $\hat{R}(s) \equiv 1$, and $\gamma = K(z_\delta)$ in (3.38), (3.39), and (3.33) to obtain

$$\Phi(z) \;=\; P_0 + z\lambda P_0 \frac{K(z) - K(z_\delta)}{\delta + \alpha(z)} \frac{1 - \hat{S}(\delta + \alpha(z))}{\hat{S}(\delta + \alpha(z)) - z} + \lambda P_0 m_U\, C\, \hat{U}_e(\alpha(z))\,. \quad (4.12)$$

The system with no vacations and no repairs. Set $\hat{U}(s) \equiv 1$, $\hat{R}(s) \equiv 1$, $m_U = m_R = 0$, and $K(z) = 1 - \chi(z)$ in (3.38) and (3.39) to obtain

$$\Phi(z) \;=\; P_0 + z\lambda P_0 \frac{\chi(z_\delta) - \chi(z)}{\delta + \alpha(z)} \frac{1 - \hat{S}(\delta + \alpha(z))}{\hat{S}(\delta + \alpha(z)) - z}. \quad (4.13)$$

The $M/M/1$ queue with disasters. (No repairs or vacations.) Here we set $\hat{S}(s) = \mu/(\mu+s)$ and $\alpha(z) = \lambda(1-z)$. Also let $\rho = \lambda/\mu$. In this case $\gamma = 1 - z_\delta$ and $P_0 = 1 - \rho z_\delta$. (3.32) becomes $\lambda z^2 - (\lambda+\mu+\delta)z + \mu = 0$ and its root inside the unit disk is

$$z_\delta = \frac{\lambda+\mu+\delta - \sqrt{(\lambda+\mu+\delta)^2 - 4\lambda\mu}}{2\lambda}. \tag{4.14}$$

With the appropriate substitutions and simplifications (4.13) gives the stationary number of customers in the system as

$$\Phi(z) = \frac{1-\rho z_\delta}{1-\rho z_\delta z}, \tag{4.15}$$

a well known result. The rate of disasters is $\lambda P_0 \gamma = \lambda(1-z_\delta)(1-\rho z_\delta) = \delta\rho z_\delta$. The fraction of customers who complete their service is $(1-\rho z_\delta)(1-z_\delta)\frac{\frac{\mu}{\mu+\delta}}{1-\frac{\mu}{\mu+\delta}} = (1-\rho z_\delta)(1-z_\delta)\frac{\mu}{\delta} = z_\delta$. The average number of customers removed by each disaster is $\frac{\lambda(1-z_\delta)}{\delta\rho z_\delta} = \frac{1}{1-\rho z_\delta}$.

5 Batch Statistics

Here we examine the statistical processes of batches that complete service and of those that are partially served, as a result of a disaster happening while they are being served. Those batches that are flushed out of the system due to a disaster, without even starting service are clearly distributed according to the incoming batch distribution, $\mathbb{P}(\chi = n)$, $n = 1, 2, \ldots$.

Let us first consider those batches that complete service. We will not address this from scratch. Instead we will take advantage of the results already obtained.

Proposition 4. *The fraction of batches that are served completely is given by*

$$P_0\gamma\,\frac{\chi(\hat{S}(\delta))}{1-\chi(\hat{S}(\delta))}. \tag{5.1}$$

Proof. Recall that the stationary number of customers in the system is given by (3.39). Note that this expression depends on z_δ, given by (3.31), through γ, defined in (3.33). Now consider batches that remain bound together through the process *and served as a single customer*. The new service time has Laplace transform $\hat{S}_b(s) := \chi(\hat{S}(s))$ and the corresponding pgf for the number of customers in the system is

$$\begin{aligned}\Phi(z) \;=\; & P_{0,b}\left(1 + \lambda m_U C\hat{U}_e(\lambda-\lambda z) + \gamma\lambda m_R \hat{R}_e(\lambda-\lambda z)\right) \\ & + zP_{0,b}\left(K_b(z) - \gamma_b \hat{R}(\alpha(z))\right)\left(\frac{\lambda}{\lambda+\delta-\lambda z}\,\frac{1-\hat{S}_b(\delta+\lambda-\lambda z)}{\hat{S}_b(\delta+\lambda-\lambda z) - z}\right).\end{aligned}$$

In the above note that, while in (3.39) $K(z)$ was given by (3.13), here it is replaced by $K_b(z) := (1-z)(1 + \lambda m_U C\hat{U}_e(\lambda-\lambda z))$ since $\chi(z) = z$ (the batch size is 1). The corresponding fixed point equation in this case is

$$z_{\delta,b} \;=\; \hat{S}_b(\delta+\lambda-\lambda z_{\delta,b}) \tag{5.2}$$

and $\gamma_b = \frac{K_b(z_{\delta,b})}{\hat{R}(\alpha(z_{\delta,b}))}$. Finally, referring to (3.38), we see that $P_{0,b} = (1 + \delta^{-1}\lambda\gamma_b + \lambda\gamma_b m_R + \lambda m_U C)^{-1}$ and hence that $P_{0,b}$ depends on the batch size distribution only through γ_b. We claim that

$$z_{\delta,b} = \chi(z_\delta). \tag{5.3}$$

Indeed, using (3.31), $\chi(z_\delta) = \chi(\hat{S}(\delta+\lambda-\lambda\chi(z_\delta))) = \hat{S}_b((\delta+\lambda-\lambda\chi(z_\delta))$. Then

$$\gamma_b = \frac{(1-z_{0,b})(1+\lambda m_U C\hat{U}_e(\lambda-\lambda z_{0,b}))}{\hat{R}(\lambda-\lambda z_{0,b})} = \frac{(1-\chi(z_\delta))(1+\lambda m_U C\hat{U}_e(\alpha(z_\delta)))}{\hat{R}(\alpha(z_\delta))} = \gamma.$$

Hence we also have $P_{0,b} = P_0$. From equation (4.6) we infer that the rate of customer departures in a system with batch size equal to 1 is

$$\lambda P_0 \gamma \frac{\hat{S}(\delta)}{1-\hat{S}(\delta)}. \tag{5.4}$$

Consider now the original system, with batch arrivals, but with all the customers in the batch "glued together" into a single customer with service time $\chi(\hat{S}(s))$. We can then use (5.4) but with service times with Laplace transform $\chi(\hat{S}(\cdot))$ to see that the departure rate of *fully served* batches from the server is

$$\lambda P_0 \gamma \frac{\chi(\hat{S}(\delta))}{1-\chi(\hat{S}(\delta))}. \tag{5.5}$$

Dividing this with λ, the arrival rate of batches gives (5.1). □

5.1 The batch size distribution for completely served batches

It is easy to see via an elementary conditioning argument that the Laplace transform of a typical customer who completes service is given by $\frac{\hat{S}(s+\delta)}{\hat{S}(\delta)}$. More generally, we will determine the joint statistics for batches that complete service in terms of service duration of the customers they contain and batch size distribution.

Proposition 5. *Let Δ, χ, and σ_i, $i = 1, 2, \ldots$ be independent random variables. Δ is exponentially distributed with rate δ, χ is distributed according to the incoming batch size distribution, and the σ_i are distributed according to the service time distribution. Then the joint distribution of batch size and service times for fully served batches is given by*

$$\mathbb{E}\left[z^\chi e^{-\sum_{i=1}^n s_i\sigma_i} \mid \Delta > \sum_{i=1}^{\chi}\sigma_i\right] = \frac{1}{\chi(\hat{S}(\delta))}\sum_{n=1}^{\infty}\mathbb{P}(\chi=n)z^n\prod_{i=1}^{n}\hat{S}(s_i+\delta). \tag{5.6}$$

If $\tilde{\chi}$ denotes a random variable with the distribution of the typical fully served *batch then*

$$\mathbb{P}(\tilde{\chi}=n) = \mathbb{P}(\chi=n)\frac{\hat{S}(\delta)^n}{\chi(\hat{S}(\delta))} \quad n=1,2,\ldots, \quad \textit{with corresponding pgf} \quad \tilde{\chi}(z) = \frac{\chi(z\hat{S}(\delta))}{\chi(\hat{S}(\delta))}. \tag{5.7}$$

Finally, the joint transform of the number of customers in a typical fully served batch and the time required to serve them is

$$\frac{\chi(z\hat{S}(s+\delta))}{\chi(\hat{S}(\delta))}. \tag{5.8}$$

Proof. Condition on χ and σ_i, $i = 1, \ldots, \chi$ to obtain

$$\mathbb{E}\left[z^\chi e^{-\sum_{i=1}^{\chi} s_i\sigma_i}\,\mathbf{1}(\Delta > \textstyle\sum_{i=1}^{\chi}\sigma_i) \mid \chi; \sigma_1,\ldots,\sigma_\chi\right] = z^\chi e^{-\sum_{i=1}^{\chi}(s_i+\delta)\sigma_i}.$$

The above remark may be used to obtain the joint distribution of the batch size and the service times of the customers included in it. Indeed,

$$\mathbb{E}\left[z^{\chi}\, e^{-\sum_{i=1}^{n} s_i\sigma_i} \mid \Delta > \sum_{i=1}^{\chi}\sigma_i\right] \;=\; \frac{\mathbb{E}\left[z^{\chi}e^{-\sum_{i=1}^{\chi} s_i\sigma_i}\, \mathbf{1}(\Delta > \sum_{i=1}^{\chi}\sigma_i)\right]}{\mathbb{P}(\Delta > \sum_{i=1}^{\chi}\sigma_i)} = \frac{\mathbb{E}\left[z^{\chi}\prod_{i=1}^{\chi}\hat{S}(s_i+\delta)\right]}{\chi(\hat{S}(\delta))}$$

whence (5.6) follows. Comparing powers of z on the right and left hand side of (5.6) we have

$$\mathbb{E}\left[\mathbf{1}(\chi = n)\, e^{-\sum_{i=1}^{n} s_i\sigma_i} \mid \Delta > \sum_{i=1}^{\chi}\sigma_i\right] \;=\; \mathbb{P}(\chi = n)\,\frac{\prod_{i=1}^{n}\hat{S}(s_i+\delta)}{\chi(\hat{S}(\delta))}.$$

The distribution of the number of customers in a fully served batch is then obtained from the above by setting $s_i = 0$ for all i. (5.8) is also obtained from (5.6) by setting $s_i = s$ for all i. □

5.2 The number of customers served in a partial batch

We consider batches during the service times of which a disaster happens. Let χ be a random variable with the typical batch size distribution and σ_i, $i = 1, 2, \ldots$, a sequence of i.i.d. service times with distribution $S(x)$. Δ, as before, is an independent exponential random variable with rate δ.

Proposition 6. *Let $\bar{\chi}$ denote the batch size of a typical batch during the service time of which a disaster occurs and denote by φ the number of customers of this batch who have already completed their service times and departed when the disaster occurs. (Clearly, $0 \le \varphi \le \bar{\chi} - 1$.) Then*

$$\begin{aligned}\mathbb{P}(\bar{\chi} = m) &= \mathbb{P}(\chi = m)\,\frac{1-\hat{S}(\delta)^n}{1-\chi(\hat{S}(\delta))}, \quad m = 1, 2, \ldots \quad \text{with corresponding pgf}\\ \bar{\chi}(z) &= \frac{\chi(z) - \chi(z\hat{S}(\delta))}{1-\chi(\hat{S}(\delta))}.\end{aligned} \tag{5.9}$$

Also, the distribution of the number of customers already served in the partially served batch is

$$\begin{aligned}\mathbb{P}(\varphi = n) &= \mathbb{P}(\chi \ge n+1)\frac{\hat{S}(\delta)^n(1-\hat{S}(\delta))}{1-\chi(\hat{S}(\delta))}, \quad n = 0, 1, 2, \ldots \quad \text{with corresponding pgf}\\ \bar{\varphi}(z) &= \frac{\chi_e(z\hat{S}(\delta))}{\chi_e(\hat{S}(\delta))}.\end{aligned} \tag{5.10}$$

Proof. Define $\tau_0 = 0$, $\tau_n = \tau_{n-1} + \sigma_n$, $n = 1, 2, \cdots, \chi$. The event that a batch affected by a disaster consists of m parts and that n of them have completed service (with $n < m$) is $\{\varphi = n, \bar{\chi} = n\} = \{\tau_n \le \Delta < \tau_{n+1}\,, \bar{\chi} = m\}$. Thus

$$\begin{aligned}\mathbb{P}(\varphi = n, \bar{\chi} = m) &= \mathbb{P}(\tau_n \le \Delta < \tau_{n+1}\,, \chi = m \mid \Delta < \tau_{\chi}) \;=\; \frac{\mathbb{P}(\tau_n \le \Delta < \tau_{n+1}\,, \chi = m)}{\mathbb{P}(\Delta < \tau_{\chi})}\\ &= \frac{\mathbb{E}[\mathbf{1}(\tau_n \le \Delta < \tau_{n+1})\mathbf{1}(\chi = m)]}{1-\mathbb{E}[e^{-\delta\tau_{\chi}}]}\end{aligned}$$

and hence, the joint distribution of φ and $\bar{\chi}$ is

$$\mathbb{P}(\varphi = n, \bar{\chi} = m) \;=\; \frac{\hat{S}(\delta)^n\left(1-\hat{S}(\delta)\right)}{1-\chi(\hat{S}(\delta))}\,\mathbb{P}(\chi = m)\mathbf{1}(m \ge n+1). \tag{5.11}$$

Hence, adding the above over all n we obtain the marginal distribution of $\bar{\chi}$ as given in (5.9), while adding over all m we obtain the marginal distribution of φ as given in (5.10). The corresponding pgf for $\bar{\chi}$ is straightforward. The pgf for φ is $\sum_{n=0}^{\infty} z^n \mathbb{P}(\varphi = n) = \frac{1-\chi(z\hat{S}(\delta))}{1-\chi(\hat{S}(\delta))} \frac{1-\hat{S}(\delta)}{1-z\hat{S}(\delta)}$ which, taking into account the definition of $\chi_e(z)$, gives the second part of (5.10). □

6 The Busy Period

6.1 The pgf of the Number in the System at a Busy Period Initiation Epoch

Denote by ψ_n the probability that the typical busy period in stationarity starts with n customers present and let $\{t_l\}$, $l = 1, 2, \dots$ be the initiation epochs of the busy periods. The system size process, $\{N_t; t \geq 0\}$, (number of customers in the system) is defined as a process with right–continuous paths a.s.. Then N_{t_l} is the number of customers in the system at the initiation epoch of the lth busy period and

$$\psi_n := \lim_{m\to\infty} \frac{1}{m} \sum_{l=1}^{m} \mathbf{1}(N_{t_l} = n), \quad n = 1, 2, \dots.$$

Denote by $\Psi(z) := \sum_{n=1}^{\infty} \psi_n z^n$ the corresponding pgf.

Proposition 7. *The number of customers present at the initiation of the typical busy period has pgf given by*

$$\Psi(z) = \frac{\Theta(z) - \Theta(0)}{\Theta(1) - \Theta(0)}. \tag{6.1}$$

Proof. Denote by $b_n := \lim_{t\to\infty} \frac{1}{t} \sum_{l=1}^{\infty} \mathbf{1}(t_l \leq t; N_{t_l} = n)$ the rate of busy period initiations that start with n customers present. The rate of all busy period initiations, $b := \lim_{t\to\infty} \frac{1}{t} \sum_{l=1}^{\infty} \mathbf{1}(t_l \leq t) = \sum_{n=1}^{\infty} b_n$ has already been obtained in (4.4). A ratio of rates argument gives

$$\psi_n = \frac{b_n}{b}, \quad n = 1, 2, \dots,$$

and thus

$$\Psi(z) = \frac{1}{b} \sum_{n=1}^{\infty} b_n z^n \tag{6.2}$$

In view of the analysis of §3

$$b_n = \lambda P_0 \chi_n + \int_0^\infty W_n(x) r(x) dx + \sum_{j=1}^{\infty} \int_0^\infty V_{j,n}(x) u(x) dx, \quad n = 1, 2, \dots \tag{6.3}$$

Taking into account (6.3) and (3.7) the infinite sum in (6.2) is equal to

$$\begin{aligned}
\sum_{n=1}^{\infty} b_n z^n &= \sum_{n=1}^{\infty} z^n \left(\lambda P_0 \chi_n + \int_0^\infty W_n(x) r(x) dx + \sum_{j=1}^{\infty} \int_0^\infty V_{j,n}(x) u(x) dx \right) \\
&= \sum_{n=1}^{\infty} z^n P_n(0) - \sum_{n=1}^{\infty} z^n \int_0^\infty P_{n+1}(x) \mu(x) dx \\
&= P(0; z) \left(1 - z^{-1} \hat{S}(\delta + \alpha(z))\right) + \int_0^\infty P_1(x) \mu(x) dx
\end{aligned} \tag{6.4}$$

whereas b can be obtained by setting $z = 1$ in (6.4). Hence

$$\Psi(z) = \frac{P(0;z)\left(1 - z^{-1}\hat{S}(\delta + \alpha(z))\right) + \int_0^\infty P_1(x)\mu(x)dx}{P(0;1)\left(1 - \hat{S}(\delta)\right) + \int_0^\infty P_1(x)\mu(x)dx}\ . \tag{6.5}$$

Also, using (3.15), (3.33), and (4.3), the numerator of (6.5) is written as

$$\begin{aligned}\sum_{n=1}^{\infty} b_n z^n &= \delta P(1)\hat{R}(\alpha(z)) - \lambda P_0 K(z) + \lambda P_0\left(F(\beta)^{-1} - \hat{R}(\lambda)\right)\\ &= \lambda P_0 \gamma\left(\hat{R}(\alpha(z)) - \hat{R}(\lambda)\right) + \lambda P_0\left(F(\beta)^{-1} - K(z)\right) \end{aligned} \tag{6.6}$$

whereas the denominator of (6.5) can be obtained from (6.6) evaluated at $z = 1$ as

$$b = \lambda P_0 \gamma\left(1 - \hat{R}(\lambda)\right) + \lambda P_0 F(\beta)^{-1}.$$

Thus, substituting into (6.5), we obtain the pgf of the number present at a busy period initiation epoch as

$$\Psi(z) = \frac{\gamma\left(\hat{R}(\alpha(z)) - \hat{R}(\lambda)\right) - K(z) + F(\beta)^{-1}}{\gamma\left(1 - \hat{R}(\lambda)\right) + F(\beta)^{-1}} = \frac{\Theta(z) - \Theta(0)}{\Theta(1) - \Theta(0)}. \tag{6.7}$$

In the above $\Theta(z)$ is given by (3.40) and hence $\Theta(1) = -\gamma$ and $\Theta(0) = F(\beta)^{-1} - \gamma\hat{R}(\lambda)$. □

6.2 The Laplace Transform of the Length of the Typical Busy Period

The expression obtained in the previous subsection for the pgf of the number of customers in the system at the initiation epoch of a busy period allows us to determine easily the Laplace transform of the length of the typical busy period. This will be obtained in terms of the solution of equation (3.32) which gives the Laplace transform of the length of a busy period starting with single customer, in an $M^X/G/1$ queue without disasters. We begin with the following

Remark 2. *Let X, Δ, be independent random variables. If $F(s) := \mathbb{E}[e^{-sX}]$ and Δ is exponentially distributed with rate δ then* $\mathbb{E}[e^{-s(X\wedge\Delta)}|X < \Delta] = \frac{F(s+\delta)}{F(\delta)}$, $\mathbb{E}[e^{-s(X\wedge\Delta)}|\Delta < X] = \frac{\delta}{\delta+s}\frac{1-F(s+\delta)}{1-F(\delta)}$, $\mathbb{E}[e^{-s(X\wedge\Delta)}] = \frac{\delta}{\delta+s} + \frac{s}{\delta+s}F(s+\delta)$.

Proposition 8. *The Laplace transform of the length of a typical busy period, $\hat{B}(s)$, in the system we examine is given by*

$$\hat{B}(s) = \frac{\delta}{\delta + s} + \frac{s}{s + \delta}\hat{\Gamma}(s + \delta) \tag{6.8}$$

where $\hat{\Gamma}(s) = \Psi\left(\hat{\Gamma}_0(s)\right)$, $\hat{\Gamma}_0(s)$ is given by (3.32), and $\Psi(z)$ by (6.7).

Note that the distribution of B is always proper ($\hat{B}(0) = 1$), even when Γ_0 is defective.

Proof. At an initiation epoch of a typical busy period the distribution of the number of customers present has pgf given by (6.7). Suppose that, at this point the disaster mechanism is "shut off". Let Γ denote the resulting busy period length with corresponding Laplace transform given by

$$\hat{\Gamma}(s) = \sum_{k=1}^{\infty} \hat{\Gamma}_0(s)^k \, \psi_k = \Psi(\hat{\Gamma}_0(s)).$$

(Γ is a defective random variable if the system without disasters is not stable.) If Δ is an independent exponential random variable with rate δ, the length of the busy period of the actual system, B, has Laplace transform given by $\hat{B}(s) = \mathbb{E}[e^{-s(\Gamma \wedge \Delta)}]$. An appeal to Remark 2 completes the proof. □

7 System Stability and Cycle Analysis

Intuitively it is clear that the presence of disasters guarantees the stability of the system regardless of the value of the arrival rate. Here we establish this fact by a rigorous argument which is interesting in its own right since it provides insight into the structure of the sample paths of this system. A byproduct of this argument will be the derivation of the Laplace transform of the time between two consecutive disasters.

We can envision the sample paths of the system as consisting of cycles. Suppose for the sake of concreteness that at time 0 a disaster has just occurred. Then there follows an inactive time consisting of a repair period and, possibly, of a string of vacations, if there where no arrivals during the repair period. If no arrivals occur during the vacation string then there follows an idle period which is terminated with the arrival of the first batch. This will be called a *repair inactive period* and is followed by a busy period which either terminates normally, with the departure of a customer leaving the system empty, or as a result of a disaster. If terminated by a disaster the inactive period that follows will be again a repair inactive period. Otherwise it will be an *ordinary inactive period* consisting only of a string of vacations and possibly of an idle period. An ordinary inactive period is followed by a busy period whose duration and characteristics, depending as they do on the number of customers initially present will have a different distribution from that following a repair inactive period.

7.1 Markov Renewal Setting

Let $\{H_n; n = 0, 1, 2, \ldots\}$ denote the point process of busy period terminations and $\{\xi_n\}$ be a sequence of random variables with values in $\{0, 1\}$ such that $\xi_n = 0$ if the nth busy period termination is due to a disaster and $\xi_n = 1$ if due to a customer departure. Clearly $H_0 = 0$ and $\xi_0 = 0$ since at time 0 we assume that a disaster has occurred. $\{(H_n, \xi_n); n \in \mathbb{N}_0\}$ is a Markov Renewal process since $\mathbb{P}(H_{n+1} - H_n \leq x, \xi_{n+1} = i \mid H_k, \xi_k; k = 0, 1, \ldots, n) = \mathbb{P}(H_{n+1} - H_n \leq x, \xi_{n+1} = i \mid \xi_n)$ for $x \geq 0$ and $i \in \{0, 1\}$. This is because at the epoch of a busy period termination the only relevant bit of information regarding the future evolution of the system is whether the busy period terminated normally or as a result of a disaster. Define the Markov renewal transition kernel $[Q_{ij}(x)]$, $i, j \in \{0, 1\}$, $x \geq 0$ and consider its Laplace transform, defined as $\hat{Q}_{ij}(s) := \int_0^\infty e^{-sx} dQ_{ij}(x)$, which we next proceed to determine.

7.2 Ordinary inactive periods

Assume that a busy period just ended normally as a result of a customer departure and set the time origin at this point. Let $\{U_i\}$ be an i.i.d. sequence of random variables distributed according to the vacation distribution function, $U(\cdot)$ and set $\mathcal{V}_0 = 0$, $\mathcal{V}_i := \mathcal{V}_{i-1} + U_i$, $i = 1, 2, \ldots$. The process $\{\mathcal{V}_i\}$ is an ordinary renewal process. Also let $\{E_i\}$, $i = 1, 2, \ldots$, denote the successive arrival times of batches, which constitute a Poisson process with rate λ and counting measure N (i.e. $N(B) = \sum_{i=1}^{\infty} \mathbf{1}(E_i \in B)$ for any Borel subset B of $\mathbb{R}$). Let ζ denote the maximum number of vacations, a (possibly defective) random variable. Then the length of the inactive period, L_o, is

$$L_o \;=\; \sum_{n=1}^{\zeta} \mathbf{1}(\mathcal{V}_{n-1} \le E_1 < \mathcal{V}_n)\mathcal{V}_n \;+\; \mathbf{1}(E_1 > \mathcal{V}_\zeta)E_1. \tag{7.1}$$

In the above expression we do not wish to exclude the case where ζ, the number of potential vacations, is a defective random variable. Set $\mathcal{Z} := \{\zeta = \infty\}$. Then on $\mathcal{Z}$ we set $\mathcal{V}_\zeta = +\infty$ and clearly, on $\mathcal{Z}$, $\mathbf{1}(E_1 > \mathcal{V}_\zeta) = 0$. In particular in the case of repeated vacations $\mathcal{Z} = \Omega$. The number of the customers present at the end of the inactive period, C_o, is

$$C_o \;=\; \sum_{n=1}^{\zeta} \mathbf{1}(\mathcal{V}_{n-1} \le E_1 < \mathcal{V}_n) \left(\chi_0 + \sum_{i=1}^{N(E_1, \mathcal{V}_n]} \chi_i \right) \;+\; \mathbf{1}(E_1 > \mathcal{V}_\zeta)\chi_0. \tag{7.2}$$

7.3 The joint distribution of the length of an inactive period and of the number of customers present when it ends.

The joint distribution of the length of an inactive period and the number of customers present at its end is obtained. This is used in order to determine the elements of the Markov renewal kernel, $\hat{Q}_{11}(s)$ and $\hat{Q}_{10}(s)$ defined above.

Proposition 9. *Setting $\phi_o(s, z) := \mathbb{E}\left[z^{C_o} e^{-sL_o}\right]$, we have*

$$\phi_o(s, z) \;=\; \frac{1 - F(\hat{U}(s + \lambda))}{1 - \hat{U}(s + \lambda)} \left(\hat{U}(s + \alpha(z)) - \hat{U}(s + \lambda) \right) \;+\; \chi(z)\frac{\lambda}{\lambda + s} F(\hat{U}(\lambda + s)). \tag{7.3}$$

Also the corresponding elements of the Laplace transform of the Markov renewal kernel are

$$\hat{Q}_{11}(s) \;=\; \tfrac{1 - F(\hat{U}(s+\lambda))}{1 - \hat{U}(s+\lambda)} \left(\hat{U}(s + \alpha(z_{s+\delta})) - \hat{U}(s + \lambda) \right) \;+\; \chi(z_{s+\delta}) \tfrac{\lambda}{\lambda+s} F(\hat{U}(\lambda + s)) \tag{7.4}$$

and

$$\hat{Q}_{10}(s) \;=\; \tfrac{\delta}{\delta+s} \left[\tfrac{1 - F(\hat{U}(s+\lambda))}{1 - \hat{U}(s+\lambda)} \left(\hat{U}(s) - \hat{U}(s + \alpha(z_{s+\delta})) \right) \;+\; (1 - \chi(z_{s+\delta}))\, \tfrac{\lambda}{\lambda+s} F(\hat{U}(\lambda + s)) \right]. \tag{7.5}$$

Proof. From (7.1) and (7.2) we have

$$z^{C_o} e^{-sL_o} = \sum_{n=1}^{\zeta} \mathbf{1}(\mathcal{V}_{n-1} \le E_1 < \mathcal{V}_n) e^{-s\mathcal{V}_n} z^{\chi_0 + \sum_{i=1}^{N(E_1, \mathcal{V}_n]} \chi_i} \;+\; e^{-sE_1} z^{\chi_0} \mathbf{1}(E_1 > \mathcal{V}_\zeta).$$

To obtain the joint transform of L_o and C_o, the length of the ordinary inactive period and the number of customers present when the subsequent busy period begins, we need to evaluate the expectation of the right hand side of the above equation. To do this we take advantage of the independence of various quantities involved and obtain the following conditional expectations.

$$\begin{aligned}&\mathbb{E}[z^{C_o}e^{-sL_o} \mid \zeta;\, \mathcal{V}_n,\, n=1,2,\ldots;\, E_i,\, i=1,2,\ldots] \\ &\quad = \chi(z)\left(\sum_{n=1}^{\zeta}\mathbf{1}(\mathcal{V}_{n-1}\le E_1<\mathcal{V}_n)e^{-s\mathcal{V}_n}\chi(z)^{N(E_1,\mathcal{V}_n]} + e^{-sE_1}\mathbf{1}(E_1>\mathcal{V}_\zeta)\right).\end{aligned}$$

Next, take expectation with respect to the post–E_1 Poisson arrival process to obtain

$$\begin{aligned}&\mathbb{E}[z^{C_o}e^{-sL_o} \mid \zeta;\, \mathcal{V}_n,\, n=1,2,\ldots;\, E_1] \qquad (7.6)\\ &\quad = \chi(z)\left(\sum_{n=1}^{\zeta}\mathbf{1}(\mathcal{V}_{n-1}\le E_1<\mathcal{V}_n)e^{-s\mathcal{V}_n}e^{-\alpha(z)(\mathcal{V}_n-E_1)} + e^{-sE_1}\mathbf{1}(E_1>\mathcal{V}_\zeta)\right).\end{aligned}$$

Now take expectation with respect to the first Poisson arrival epoch, E_1.

$$\begin{aligned}&\mathbb{E}\left[\mathbf{1}(\mathcal{V}_{n-1}\le E_1<\mathcal{V}_n)e^{-s\mathcal{V}_n}e^{-\alpha(z)(\mathcal{V}_n-E_1)} \mid \zeta;\, \mathcal{V}_n,\, n=1,2,\ldots\right] \qquad (7.7)\\ &\quad = e^{-(s+\alpha(z))\mathcal{V}_n}\int_{\mathcal{V}_{n-1}}^{\mathcal{V}_n}\lambda e^{-\lambda t}e^{\lambda(1-\chi(z))t}dt = \frac{e^{-(s+\alpha(z))\mathcal{V}_n}}{\chi(z)}\left(e^{-\lambda\chi(z)\mathcal{V}_{n-1}}-e^{-\lambda\chi(z)\mathcal{V}_n}\right)\end{aligned}$$

Also $\mathbb{E}\left[e^{-sE_1}\mathbf{1}(E_1>\mathcal{V}_\zeta) \mid \zeta;\, \mathcal{V}_n,\, n=1,2,\ldots\right] = \frac{\lambda}{\lambda+s}e^{-s\mathcal{V}_\zeta}$. Next, taking expectation with respect to the vacation sequence,

$$\mathbb{E}\left[e^{-(s+\alpha(z))\mathcal{V}_n}\left(e^{-\lambda\chi(z)\mathcal{V}_{n-1}}-e^{-\lambda\chi(z)\mathcal{V}_n}\right) \mid \zeta\right] = \hat{U}(s+\lambda)^{n-1}\left(\hat{U}(s+\alpha(z))-\hat{U}(s+\lambda)\right) \qquad (7.8)$$

and also $\mathbb{E}\left[\frac{\lambda}{\lambda+s}e^{-s\mathcal{V}_\zeta} \mid \zeta\right] = \frac{\lambda}{\lambda+s}\hat{U}(s+\lambda)$. Putting together (7.6), (7.7), and (7.8) we obtain

$$\mathbb{E}\left[z^{C_o}e^{-sL_o}\right] = \mathbb{E}\left[\sum_{n=1}^{\zeta-1}\hat{U}(s+\lambda)^{n-1}\left(\hat{U}(s+\lambda-\lambda\chi(z))-\hat{U}(s+\lambda)\right) + \chi(z)\frac{\lambda}{\lambda+s}\hat{U}(\lambda+s)^{\zeta}\right].$$

Using the geometric sum $\sum_{n=1}^{\zeta-1}\hat{U}(s+\lambda)^{n-1} = \frac{1-\hat{U}(s+\lambda)^{\zeta}}{1-\hat{U}(s+\lambda)}$, and evaluating the above expectation we obtain (7.3).

Let K_o denote the length of the busy period that follows the ordinary idle period, assuming that the disaster mechanism has been deactivated. Then the joint transform of (L_o, K_o) is given by $\mathbb{E}[e^{-s_1L_o-s_2K_o}] = \phi_o(s_1, \hat{\Gamma}_0(s_2))$. Therefore,

$$\mathbb{E}[e^{-s_1L_o-s_2K_o}\mathbf{1}(\Delta>K_o)] = \mathbb{E}[e^{-s_1L_o-s_2K_o}e^{-\delta K_o}] = \phi_o(s_1, \hat{\Gamma}_0(s_2+\delta)).$$

Setting $\hat{\Gamma}_0(s+\delta) =: z_{s+\delta}$ (cf. Remark 1) we have $\hat{Q}_{11}(s) = \phi_o(s, z_{s+\delta})$ whence (7.4) follows.

Similarly, the joint transform of (L_o, K_o) *on the event* $\Delta < K_o$*, i.e. on the event that a disaster happens during the busy period,* is

$$\begin{aligned}\mathbb{E}[e^{-s_1L_o-s_2\Delta}\mathbf{1}(\Delta<K_o)] &= \mathbb{E}\left[e^{-s_1L_o}\int_0^{K_o}e^{-s_2x}\delta e^{-\delta x}dx\right] = \frac{\delta}{\delta+s_2}\mathbb{E}\left[e^{-s_1L_o}\left(1-e^{-(s_2+\delta)K_o}\right)\right]\\ &= \frac{\delta}{\delta+s_2}\left(\phi_o(s_1,1)-\phi_o(s_1,\hat{\Gamma}_0(s_2+\delta))\right)\end{aligned}$$

Hence $\hat{Q}_{10}(s) = \frac{\delta}{\delta+s}\left(\phi_o(s,1)-\phi_o(s,z_{s+\delta})\right)$ from which we obtain (7.5). □

7.4 Repair inactive periods

In a repair inactive period, we define the *modified* renewal process $\{\mathcal{V}_i^r\}$ as follows: $\mathcal{V}_0^r = 0$, $\mathcal{V}_1^r := R$, $\mathcal{V}_i^r := \mathcal{V}_{i-1}^r + U_i$, $i = 2, 3, \ldots$. Here $R, U_1, U_2, U_3, \ldots$, are all independent random variables, the first distributed according to the repair time distribution, and the subsequent distributed according to the vacation distribution. Then, denoting by L_r and C_r the length of the repair inactive period and the number of customers in the system when it ends, we have

$$z^{C_r} e^{-sL_r} = \sum_{n=1}^{\zeta+1} \mathbf{1}(\mathcal{V}_{n-1}^r \leq E_1 < \mathcal{V}_n^r) e^{-s\mathcal{V}_n^r} z^{\chi_0 + \sum_{i=1}^{N(E_1, \mathcal{V}_n^r]} \chi_i} \;+\; e^{-sE_1} z^{\chi_0} \mathbf{1}(E_1 > \mathcal{V}_{\zeta+1}^r).$$

Hence, setting $\phi_r(s,z) := \mathbb{E}\left[z^{C_r} e^{-sL_r}\right]$, we have the following proposition whose proof is very similar to that of Proposition 9 and we will omit.

Proposition 10. *The joint transform of the length of a repair inactive period and the number of customers in the system when it ends is*

$$\begin{aligned} \phi_r(s,z) \;&=\; \hat{R}(s+\alpha(z)) - \hat{R}(s+\lambda) \\ &\quad + \hat{R}(s+\lambda)\left[\tfrac{1-F(\hat{U}(s+\lambda))}{1-\hat{U}(s+\lambda)}\left(\hat{U}(s+\alpha(z)) - \hat{U}(s+\lambda)\right) \;+\; \chi(z)\tfrac{\lambda}{\lambda+s}F(\hat{U}(\lambda+s))\right]. \end{aligned} \tag{7.9}$$

The corresponding elements of the Laplace transform of the Markov renewal kernel are given by

$$\begin{aligned} \hat{Q}_{01}(s) \;&=\; \hat{R}(s+\alpha(z_{s+\delta})) - \hat{R}(s+\lambda) \\ &\quad + \hat{R}(s+\lambda)\left[\tfrac{1-F(\hat{U}(s+\lambda))}{1-\hat{U}(s+\lambda)}\left(\hat{U}(s+\alpha(z_{s+\delta})) - \hat{U}(s+\lambda)\right) \;+\; \chi(z_{s+\delta})\tfrac{\lambda}{\lambda+s}F(\hat{U}(\lambda+s))\right] \end{aligned} \tag{7.10}$$

and

$$\begin{aligned} \hat{Q}_{00}(s) \;&=\; \tfrac{\delta}{\delta+s}\Bigg[\hat{R}(s) - \hat{R}(s+\alpha(z_{s+\delta})) \;+\; \hat{R}(s+\lambda)\Bigg[\tfrac{1-F(\hat{U}(s+\lambda))}{1-\hat{U}(s+\lambda)}\left(\hat{U}(s) - \hat{U}(s+\alpha(z_{s+\delta}))\right) \\ &\quad + \;(1-\chi(z_{s+\delta}))\tfrac{\lambda}{\lambda+s}F(\hat{U}(\lambda+s))\Bigg]\Bigg]. \end{aligned} \tag{7.11}$$

7.5 Embedded Markov chain

Consider now the two-state Markov chain $\{\xi_n\}$, $n = 0, 1, 2, \ldots$, defined above. Its transition probability matrix can be determined from the Laplace transform of the Markov-Renewal kernel as follows. If we set $p_{11} := \mathbb{P}(\xi_{n+1} = 1 \mid \xi_n = 1)$ then, using (7.4), we have

$$p_{11} \;=\; \hat{Q}_{11}(0) \;=\; \tfrac{1-F(\beta)}{1-\beta}\left(\hat{U}(\alpha(z_\delta)) - \beta\right) + F(\beta)\chi(z_\delta) = F(\beta)K(z_\delta). \tag{7.12}$$

Similarly, with $p_{00} := \mathbb{P}(\xi_{n+1} = 0 \mid \xi_n = 0)$, using (7.11),

$$\begin{aligned} p_{00} := Q_{00}(s) \;&=\; 1 - \hat{R}(\alpha(z_\delta)) + \hat{R}(\lambda)\left[\tfrac{1-F(\beta)}{1-\beta}\left(1 - \hat{U}(\alpha(z_\delta))\right) + (1-\chi(z_\delta))\,F(\beta)\right] \\ &=\; 1 - \hat{R}(\alpha(z_\delta)) + \hat{R}(\lambda)F(\beta)K(z_\delta). \end{aligned} \tag{7.13}$$

Hence the transition probability matrix of $\{\xi_n\}$ is given by

$$P \;=\; \begin{bmatrix} p_{00} & 1-p_{00} \\ 1-p_{11} & p_{11} \end{bmatrix} \;=\; \begin{bmatrix} \hat{R}(\alpha(z_\delta))(1-\gamma\hat{R}(\lambda)F(\beta)) & 1-\hat{R}(\alpha(z_\delta))(1-\gamma\hat{R}(\lambda)F(\beta)) \\ 1-F(\beta)K(z_\delta) & F(\beta)K(z_\delta) \end{bmatrix}.$$

The stationary distribution of the above chain is

$$\pi_0 = \frac{\gamma}{\frac{1}{F(\beta)} + \gamma(1-\hat{R}(\lambda))} = \frac{d}{b}, \qquad \pi_1 = \frac{\frac{1}{F(\beta)} - \gamma\hat{R}(\lambda)}{\frac{1}{F(\beta)} + \gamma(1-\hat{R}(\lambda))} = \frac{r_b}{b}. \tag{7.14}$$

In the above π_0 gives the fraction of busy periods that terminate as a result of a disaster whereas π_1 the fraction that terminate normally, as a result of a customer departure that leaves the system empty. The number of customers present in the system at the initiation a busy period has pgf given by

$$\begin{aligned} \phi_1(z) &:= \phi_o(0,z) = 1 - F(\beta)K(z) \quad \text{if the preceding busy period ends normally,} \\ \phi_0(z) &:= \phi_r(0,z) = \hat{R}(\alpha(z)) - \hat{R}(\lambda)F(\beta)K(z) \quad \text{if the preceding busy period ends due to a disaster.} \end{aligned}$$

Comparing with (6.7), we see that $\Psi(z) = \pi_0\phi_0(z) + \pi_1\phi_1(0)$.

7.6 The Laplace transform of the time between two consecutive disasters

Proposition 11. *The Laplace transform of the time between two consecutive disasters is given by*

$$D(s) := \frac{\delta}{\delta+s}\left[\hat{R}(s) - \hat{R}(s+\alpha(z_{s+\delta}))\frac{s}{s+\alpha(z_{s+\delta})}\,\frac{1+(\lambda+s)m_U C(s)\hat{U}_e(s)}{1+(\lambda+s)m_U C(s)\hat{U}_e(s+\alpha(z_{s+\delta}))}\right] \tag{7.15}$$

where

$$C(s) = \frac{1}{F(\hat{U}(\lambda+s))}\,\frac{1-F(\hat{U}(\lambda+s))}{1-\hat{U}(\lambda+s)}. \tag{7.16}$$

The corresponding mean time between disasters is finite and equal to $-D'(0) = \frac{1}{\delta} + m_R + \frac{1}{\lambda\gamma} + \frac{1}{\gamma}m_U C$.

Corollary 12. *The system is stable provided that the mean vacation time m_U and the mean repair time m_R are finite. The finiteness of the mean service time is not a necessary condition for the stability of the system.*

Proof. Assuming that a disaster occurs at time 0, let D_1 denote the first time a disaster occurs again. This can be represented in terms of the Markov-renewal process defined in the beginning of this section as $D_1 = \sum_{n=1}^{\infty} H_n \mathbf{1}(\xi_1 = 1, \xi_2 = 1, \ldots, \xi_{n-1} = 1, \xi_n = 0)$. Hence

$$\mathbb{E}[e^{-sD_1} \mid \xi_0 = 0] = \sum_{n=1}^{\infty} \mathbb{E}[e^{-sH_n}\mathbf{1}(\xi_1 = 1, \xi_2 = 1, \ldots, \xi_{n-1} = 1, \xi_n = 0) \mid \xi_0 = 0].$$

Denote the increments of the Markov-renewal process $(H_n, \xi_n)_{n\in\mathbb{N}}$ by $\{h_n\}$: $h_n = H_n - H_{n-1}$, $n = 1, 2, \ldots$. (We assume of course that $H_0 = 0$ and $\xi_0 = 0$.) By the defining property of the Markov-renewal process, the random variables $h_1, h_2, \ldots, h_n$ are *conditionally independent* given $\xi_0, \xi_1, \ldots, \xi_{n-1}, \xi_n$. More specifically,

$$\mathbb{E}[e^{-s(h_1+h_2+\cdots+h_n)} \mid \xi_0, \xi_1, \ldots, \xi_{n-1}, \xi_n] = \prod_{k=1}^{n} \mathbb{E}[e^{-sh_k} \mid \xi_{k-1}, \xi_k] = \prod_{k=1}^{n} \frac{\hat{Q}_{\xi_{k-1},\xi_k}(s)}{p_{\xi_{k-1},\xi_k}}.$$

and therefore, for $n \geq 2$,

$$\begin{aligned}
&\mathbb{E}[e^{-sH_n}\mathbf{1}(\xi_1 = 1, \xi_2 = 1, \ldots, \xi_{n-1} = 1, \xi_n = 0) \mid \xi_0 = 0] \\
&= \mathbb{E}[e^{-s(h_1+\cdots+h_n)} \mid \xi_1 = 1, \ldots, \xi_{n-1} = 1, \xi_n = 0 \mid \xi_0 = 0]\, \mathbb{P}(\xi_1 = 1, \ldots, \xi_{n-1} = 1, \xi_n = 0 \mid \xi_0 = 0) \\
&= \frac{\hat{Q}_{01}(s)}{p_{01}} \left(\prod_{k=1}^{n-1} \frac{\hat{Q}_{11}(s)}{p_{11}} \right) \frac{\hat{Q}_{10}(s)}{p_{10}} \cdot p_{01} p_{11}^{n-1} p_{10} \;=\; \hat{Q}_{01}(s)\hat{Q}_{11}(s)^k \hat{Q}_{10}(s), \qquad n = 2, 3, \ldots.
\end{aligned}$$

When $n = 1$,

$$\mathbb{E}[e^{-sH_1}\mathbf{1}(\xi_1 = 0) \mid \xi_0 = 0] = \hat{Q}_{00}(s).$$

Therefore, summing for all n,

$$D(s) \;:=\; \hat{Q}_{00}(s) + \sum_{k=0}^{\infty} \hat{Q}_{01}(s)\hat{Q}_{11}(s)^k \hat{Q}_{10}(s) \;=\; \hat{Q}_{00}(s) + \hat{Q}_{01}(s)(1 - \hat{Q}_{11}(s))^{-1}\hat{Q}_{10}(s).$$

Using (7.4), (7.5), (7.10), (7.11), in the above we obtain (7.15). □

8 Stationary Workload and System Times

Let Ω denote the stationary workload in the system and T the time a typical customer arriving in stationarity spends in the system regardless of whether he eventually departs as a result of a disaster or upon completing his service. Similarly let T_d denote the system time of a typical customer arriving in stationarity given that his departure is caused by a disaster and T_s that of a typical customer, given that he completes his service. $\hat{T}(s)$, $\hat{T}_d(s)$, and $\hat{T}_s(s)$ denote the corresponding Laplace transforms.

Theorem 13. *The Laplace transform of a typical customer's system time arriving in stationarity and who does not suffer a disaster is given by*

$$\hat{T}_s(s) = \frac{\lambda P_0}{f_c} \left(\hat{\kappa}(s) - \gamma \hat{R}(s) \right) \frac{\hat{S}(s+\delta)}{s - \alpha(\hat{S}(\delta + s))} \, \chi_e(\hat{S}(s+\delta)) \tag{8.1}$$

where f_c is given in (4.6). Similarly, in stationarity, the Laplace transform of a the system time of a customer who is removed by a disaster is

$$\hat{T}_d(s) \;=\; \frac{\delta}{\delta + s} \frac{1}{1 - f_c} \left(P_0 \left(1 + \frac{\lambda\gamma}{\delta} + \lambda m_U C \hat{U}_e(s) + \gamma \lambda m_R \hat{R}_e(s) \right) - f_c \hat{T}_s(s) \right). \tag{8.2}$$

and finally the Laplace transform of the system time of a customer in stationarity, regardless of whether he completes service or not is

$$\begin{aligned}
\hat{T}(s) \;&=\; \frac{\delta}{\delta + s} P_0 \left(1 + \frac{\lambda\gamma}{\delta} + \lambda m_U C \hat{U}_e(s) + \gamma \lambda m_R \hat{R}_e(s) \right) \\
&\quad + \frac{s}{\delta + s} \lambda P_0 \left[\hat{\kappa}(s) - \gamma \hat{R}(s) \right] \frac{\hat{S}(s+\delta)}{s - \alpha(\hat{S}(s+\delta))} \chi_e(\hat{S}(s+\delta)).
\end{aligned} \tag{8.3}$$

In the above P_0 is given by (3.38) and $\hat{\kappa}(s) := \frac{s}{\lambda}\left(1 + \lambda m_U C \hat{U}_e(s)\right)$. The Laplace transform of the stationary workload is given by

$$\begin{aligned}
\hat{\Omega}(s) \;&=\; P_0 \frac{\lambda\gamma + \delta - s}{\alpha(\hat{S}(s)) + \delta - s} \\
&\quad + P_0 \frac{\delta - s}{\alpha(\hat{S}(s)) + \delta - s} \left(\lambda\gamma m_R \hat{R}_e(\alpha(\hat{S}(s))) + \lambda m_U C \hat{U}_e(\alpha(\hat{S}(s))) \right).
\end{aligned} \tag{8.4}$$

Proof. Let $\bar{\mathcal{S}}_t$ denote the *residual* service time of the customer in service at time t, when $\xi_t = s$ (i.e. when the server is busy) otherwise set $\bar{\mathcal{S}}_t = 0$. Similarly let $\bar{\mathcal{R}}_t$, and $\bar{\mathcal{U}}_t^j$, $j = 1, 2, \ldots$, denote the *residual life* of the repair process, or vacation time at time t (provided again the server is under repair or on vacation). Also, denote by Q_t the number of customers in the system at time t, *excluding any that may be receiving service at that time*.

Suppose that at time 0 the system is in the stationary regime and a batch containing a tagged customer arrives. We proceed in two steps. *In the first step*, in order to avoid the complications that arise from the possibility of disasters, we imagine that at the arrival instant the disaster mechanism shuts off and disasters no longer occur in the system. We first determine the Laplace transform of the sojourn time of the tagged customer under these circumstances. The system time of the tagged customer consists of four parts, namely:

1) A delay due to the residual service time of the customer that may be in service at time 0, which is equal to $\mathcal{T}_1 := \mathbf{1}(\xi_0 = s)\bar{\mathcal{S}}_0$.

2) A delay that ensues if the server happens to be on vacation or under repair a time 0. This delay is equal to $\mathcal{T}_2 := \mathbf{1}(\xi_0 = r)\bar{\mathcal{R}}_0 + \sum_{j=1}^{\infty} \mathbf{1}(\xi_0 = u_j)\bar{\mathcal{U}}_0^j$.

3) The sum of the service requirements of all customers in the system at time 0 *not receiving service*, Q_0. This is the total number of customers if the server is on vacation or under repair, or the number of customers in queue, excluding the one in the server, if the server is busy at time 0.

4) Finally, the sum of the service requirements of all customers preceding the tagged customer in the batch that arrived at time 0, together with the tagged customer's own service requirement. The Laplace transform of this last delay is $\chi_e(\hat{S}(s))\hat{S}(s)$ from a discrete renewal theoretic argument (i.e. a size biasing argument), since we consider not a "typical" batch but a typical customer.

From (3.18) the joint transform of Q_0, $\mathcal{T}_1$ and $\mathcal{T}_2$ is

$$
\begin{aligned}
\mathbb{E}[z^{Q_0} e^{-s_1 \mathcal{T}_1 - s_2 \mathcal{T}_2}] \quad = \quad & P_0 + z^{-1} P(0; z) \int_0^\infty (1 - S(x)) e^{-x(\delta + \alpha(z))} \int_0^\infty e^{-s_1 y} \frac{S'(x+y)}{1 - S(x)} dy dx \\
& + W_0(0) \int_0^\infty (1 - R(x)) e^{-x\alpha(z)} \int_0^\infty e^{-s_2 y} \frac{R'(x+y)}{1 - R(x)} dy dx \\
& + \sum_{j=1}^{\infty} v_j \int_0^\infty (1 - U(x)) e^{-x\alpha(z)} \int_0^\infty e^{-s_2 y} \frac{U'(x+y)}{1 - R(x)} dy dx.
\end{aligned}
$$

Taking into account (3.15), (3.22), (3.27), and (3.29),

$$
\begin{aligned}
& \mathbb{E}[z^{Q_0} e^{-s_1 \mathcal{T}_1 - s_2 \mathcal{T}_2}] \\
& \quad = P_0 + z^{-1} P(0; z) \tfrac{\hat{S}(s_1) - \hat{S}(\delta + \alpha(z))}{\alpha(z) + \delta - s_1} + W(0; z) \tfrac{\hat{R}(s_2) - \hat{R}(\alpha(z))}{\alpha(z) - s_2} + \textstyle\sum_{j=1}^{\infty} V_j(0) \tfrac{\hat{U}(s_2) - \hat{U}(\alpha(z))}{\alpha(z) - s_2} \\
& \quad = P_0 + \lambda P_0 \tfrac{K(z) - \gamma \hat{R}(\alpha(z))}{\hat{S}(\delta + \alpha(z)) - z} \tfrac{\hat{S}(s_1) - \hat{S}(\delta + \alpha(z))}{\alpha(z) + \delta - s_1} + \lambda P_0 \gamma \tfrac{\hat{R}(s_2) - \hat{R}(\alpha(z))}{\alpha(z) - s_2} + \lambda P_0 C \tfrac{\hat{U}(s_2) - \hat{U}(\alpha(z))}{\alpha(z) - s_2}. \qquad (8.5)
\end{aligned}
$$

Let us then denote by T_1 the part of the system time of the tagged customer that would have been subject to disasters had the disaster mechanism not been switched off (namely delays described in 1, 3, and 4 above) and by T_2 the part that would not have been subject to disasters. We can obtain the joint Laplace transform of T_1 and T_2 from (8.5) by replacing z with $\hat{S}(s_1)$ and then, to account for the system time of the customers preceding the tagged customer in the batch, including the tagged customer, by multiplying the result by $\chi_e(\hat{S}(s_1)\hat{S}(s_1)$. We thus obtain

$$\begin{aligned}\mathbb{E}[e^{-s_1T_1-s_2T_2}] &= \mathbb{E}[\hat{S}(s_1)^{Q_0}e^{-s_1\mathfrak{T}_1-s_2\mathfrak{T}_2}]\,\chi_e(\hat{S}(s_1)\hat{S}(s_1)\\ &= \Bigg[P_0-\lambda P_0\frac{K(\hat{S}(s_1))-\gamma\hat{R}(\alpha(\hat{S}(s_1)))}{\alpha(\hat{S}(s_1))+\delta-s_1}\\ &\quad+\lambda P_0\gamma\frac{\hat{R}(s_2)-\hat{R}(\alpha(\hat{S}(s_1)))}{\alpha(\hat{S}(s_1))-s_2}+\lambda P_0 C\frac{\hat{U}(s_2)-\hat{U}(\alpha(\hat{S}(s_1)))}{\alpha(\hat{S}(s_1))-s_2}\Bigg]\,\chi_e(\hat{S}(s_1)\hat{S}(s_1)\end{aligned}$$

or, after simplifying,

$$\begin{aligned}\mathbb{E}[e^{-s_1T_1-s_2T_2}] &= \frac{\lambda P_0}{\alpha(\hat{S}(s_1))-s_2}\Bigg[\Big(\hat{\kappa}(\alpha(\hat{S}(s_1)))-\gamma\hat{R}(\alpha(\hat{S}(s_1)))\Big)\frac{s_2-s_1+\delta}{\alpha(\hat{S}(s_1))-s_1+\delta}\\ &\quad-\hat{\kappa}(s_2)+\gamma\hat{R}(s_2)\Bigg]\,\chi_e(\hat{S}(s_1)\hat{S}(s_1). \end{aligned}\tag{8.6}$$

Now comes the *second step*, namely restoring the disaster mechanism after time 0. The tagged customer will manage to obtain service and leave the system provided that $T_1<\Delta$ where Δ is again an independent exponential random variable with rate δ. Trivially, $\mathbb{E}[e^{-s_1T_1+s_2T_2}|T_1<\Delta]=\frac{\mathbb{E}[e^{-(s_1+\delta)T_1+s_2T_2}]}{\mathbb{E}[e^{-\delta T_1}]}$. From (8.6) we have

$$\begin{aligned}\mathbb{E}[e^{-(s_1+\delta)T_1-s_2T_2}] &= \frac{\lambda P_0}{\alpha(\hat{S}(s_1+\delta))-s_2}\Bigg[\Big(\hat{\kappa}(\alpha(\hat{S}(s_1+\delta)))-\gamma\hat{R}(\alpha(\hat{S}(s_1+\delta)))\Big)\frac{s_2-s_1}{\alpha(\hat{S}(s_1+\delta))-s_1}\\ &\quad-\hat{\kappa}(s_2)+\gamma\hat{R}(s_2)\Bigg]\,\chi_e(\hat{S}(s_1+\delta)\hat{S}(s_1+\delta).\end{aligned}\tag{8.7}$$

Setting $s=s_1=s_2$ above we obtain

$$\mathbb{E}[e^{-(s+\delta)T_1-sT_2}]=\lambda P_0\frac{\hat{\kappa}(s)-\gamma\hat{R}(s)}{s-\alpha(\hat{S}(s+\delta))}\chi_e(\hat{S}(s+\delta))\hat{S}(s+\delta)\tag{8.8}$$

Setting $s=0$ in (8.8) we obtain, after some simplifications,

$$\mathbb{E}[e^{-\delta T_1}]=P_0\gamma\frac{\hat{S}(\delta)}{m_\chi(1-\hat{S}(\delta))}.\tag{8.9}$$

(Notice that this is the fraction of customers that complete service, f_c, as given in (4.6.) Thus, the Laplace transform of a customer who does not suffer a disaster is given by the ratio of the right hand sides of (8.8) and (8.9) thus obtaining (8.1).

In similar fashion, starting with the relation

$$\mathbb{E}[e^{-s_1\Delta-s_2T_2}|T_1>\Delta]=\frac{\mathbb{E}[e^{-s_2T_2}]-\mathbb{E}[e^{-(s_1+\delta)T_1-s_2T_2}]}{1-\mathbb{E}[e^{-\delta T_1}]}\,\frac{\delta}{\delta+s_1}$$

(where Δ is again an independent exponential random variable with rate δ) and using the fact that $\mathbb{E}[e^{-s_2T_2}] = \frac{\lambda P_0}{s_2}\left(\gamma\frac{s_2+\delta}{\delta} + \hat{\kappa}(s_2) - \gamma\hat{R}(s_2)\right)$ which follows by setting $s_1 = 0$ in (8.6) we see that the Laplace transform of the system time of a customer removed by a disaster is

$$\hat{T}_d(s) = \lambda P_0 \frac{\frac{1}{s}\left(\gamma + \frac{\gamma s}{\delta} + \hat{\kappa}(s) - \gamma\hat{R}(s)\right) - \frac{\hat{\kappa}(s)-\gamma\hat{R}(s)}{s-\alpha(\hat{S}(s+\delta))}\chi_e(\hat{S}(s+\delta))\hat{S}(s+\delta)}{1 - P_0\gamma\frac{\hat{S}(\delta)}{m_\chi(1-\hat{S}(\delta))}} \frac{\delta}{\delta+s}$$

which, after some rearrangement, gives (8.2).

Finally, the Laplace transform of the typical customer's system time, regardless of whether the customer suffers a disaster or not can be obtained by the fact that $\hat{T}(s) = f_c\hat{T}_s(s) + (1-f_c)\hat{T}_d(s)$ which yields (8.3).

The argument for the stationary workload is similar. Begin with the joint distribution of Q_0 and $\bar{\mathcal{R}}_0$:

$$\begin{aligned}\mathbb{E}[z^{Q_0}e^{-s\bar{\mathcal{R}}_0}] &= P_0 + P_0\lambda m_U C\hat{U}_e(\alpha(z)) + P_0\gamma\lambda m_R\hat{R}_e(\alpha(z)) \\ &\quad + z^{-1}P(0;z)\int_0^\infty e^{-x(\alpha(z)+\delta)}(1-S(x))\int_0^\infty e^{-ys}\frac{S'(x+y)}{1-S(x)}dydx.\end{aligned}$$

Substituting z with $\hat{S}(s)$ in the above we obtain (8.4). □

Remark 3. *When the batches are of size 1, i.e. $\chi(z) \equiv 1$ and $\alpha(z) \equiv \lambda - \lambda z$ then (8.3) can be obtained from (3.39) by setting $z = 1 - \frac{s}{\lambda}$, in agreement with the distributional version of Little's law which holds in this case.*

8.1 Special Cases

No repairs and no vacations: When there are no repairs and no vacations (an $M^X/G/1$ queue with disasters) (8.4) reduces to

$$\hat{\Omega}(s) = \frac{\delta}{\delta + \alpha(z_\delta)} \frac{\alpha(z_\delta) + \delta - s}{\alpha(\hat{S}(s)) + \delta - s}.$$

When in addition the batch size is always 1, i.e. $\chi(z) = z$, then $\hat{\Omega}(s) = \frac{\delta}{\delta+\lambda-\lambda z_\delta}\frac{\delta+\lambda-\lambda z_\delta - s}{\delta+\lambda-s-\lambda\hat{S}(s)}$, which is the expression obtained in Jain and Sigman ([12]) and in [4]. The system time of a customer who completes service in this case is

$$\hat{T}_s(s) = \frac{s - \lambda(1-z_\delta)}{s - \lambda + \lambda\hat{S}(s+\delta)} \frac{\hat{S}(s+\delta)}{\hat{S}(\delta)} \frac{1-\hat{S}(\delta)}{1-z_\delta}.$$

No vacations, batch size 1: This is the model considered in [19]. The Laplace transform of the system time for customers that complete service is given by

$$\hat{T}_s(s) = \left(s\hat{R}(\lambda - \lambda z_\delta) - (\lambda - \lambda z_\delta)\hat{R}(s)\right) \frac{1}{s - \lambda + \lambda\hat{S}(s+\delta)} \frac{\hat{S}(s+\delta)}{\hat{S}(\delta)} \frac{1-\hat{S}(\delta)}{1-z_\delta}. \tag{8.10}$$

In particular, when the service time is exponential with rate μ (so that $\hat{S}(s) = \frac{\mu}{s+\mu}$ and z_δ is given by (4.14)) equation (8.10) reduces to

$$\hat{T}_s(s) = \frac{s_+\hat{R}(s) - s\hat{R}(s_+)}{s - s_+} \frac{|s_-|}{s + |s_-|} \tag{8.11}$$

where $s_\pm = \frac{1}{2}\left(\lambda - \mu - \delta \pm \sqrt{(\lambda+\mu+\delta)^2 - 4\lambda\mu}\right)$. Note that for all positive values of λ, μ, and δ, it holds that $s_- < 0 < s_+$. We also have in this case the following interesting representation: the term $\frac{s_+\hat{R}(s) - s\hat{R}(s_+)}{s - s_+}$ is (8.11) is the Laplace transform of the random variable $(R-E)^+$ where R is a random variable with the repair duration distribution while E is an exponential random variable with rate s_+, independent of R. To this we add an other exponential random variable, independent of the other two, with rate $|s_-|$ in order to obtain the system time of an accepted customer.

9 Appendix

Proposition 14. *The function $h(z) := \hat{S}(\delta + \alpha(z)) - z$ has a unique root z_δ in the open unit disk $|z| < 1$ of the complex plane.*

Proof. The system is stable for all values of the parameters due to the presence of disasters. Hence the power series that defines $P(0; z)$ converges uniformly on the closed unit disk $|z| \le 1$ and defines an analytic function there. Let $f(z) := -z$ and $g(z) := \hat{S}(\delta + \alpha(z))$ which are both analytic in $|z| \le 1$. Then

$$|g(z)| \le \int_0^\infty |\, e^{-(\delta+\alpha(z))x} \,|\, dS(x) = \int_0^\infty e^{-\delta x}\, e^{-\lambda x \Re(\alpha(z))} dS(x).$$

The real part of $\alpha(z)$ when $|z| = 1$, i.e. $z = e^{i\theta}$, $\theta \in [0, 2\pi)$, is

$$\Re\left(\lambda\left(1 - \sum_{k=1}^\infty \chi_k e^{ik\theta}\right)\right) = \lambda \sum_{k=1}^\infty \chi_k (1 - \cos k\theta) \ge 0, \qquad \theta \in [0, 2\pi)$$

and thus $|g(z)| \le \int_0^\infty e^{-\delta x} dS(x) < 1$. It follows by Rouché's theorem that $f(z)$ and $f(z) + g(z)$ will have the same number of zeros inside $|\, z \,| < 1$. Since $f(z)$ has only one zero inside this circle, $h(z)$ also has a single zero inside $|z| < 1$, denoted as z_δ. □

Since $h(0) = \hat{S}(\delta + \lambda) > 0$ and $h(1) = \hat{S}(\delta) - 1 < 0$, z_δ will in fact be real which makes its numerical determination particularly simple.

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