From the Stationary to the Palm Version of the $M/M/s$ Queue

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Abstract

Using the tools of Palm calculus together with the markovian structure of the $M/M/s$ queue and a reversibility argument we obtain the queue–length distribution observed by the kth customer who arrives after the time origin to a stationary queue, and similarly by the kth customer who arrives before the time origin.

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1 Introduction

Consider an $M/M/s$ system with arrival rate λ and service rate μ (with $\lambda < s\mu$) and let X_t denote the number of customers in the system at time t, $\{t_n\}$ the Poisson arrival process, and $\{d_n\}$ the departure process. We assume that $\{X_t; t \in \mathbb{R}\}$ is stationary under the probability measure $\mathbb{P}$ and we will denote its stationary distribution by

$$\pi_i := \mathbb{P}(X_t = i), \qquad i = 0, 1, 2, \ldots.$$

We also denote by $\mathbb{P}_A^0$ the Palm transformation of $\mathbb{P}$ under $\{t_n\}$ and by $\mathbb{P}_D^0$ the Palm transformation of $\mathbb{P}$ under $\{d_n\}$. Expectations with respect to these Palm probability measures will be denoted by $\mathbb{E}_A^0$ and $\mathbb{E}_D^0$ respectively. We will further assume that the sample paths of X_t are *right-continuous* *with probability 1*. We recall the standard numbering convention regarding point processes (see [1, p. 3]) according to which $\mathbb{P}$–a.s. $\cdots t_{-1} < t_0 < 0 < t_1 < t_2 \cdots$ whereas $\mathbb{P}^0_A(t_0 = 0) = 1$. As usual $\mathbb{N}$ denotes the natural numbers $\{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ the nonnegative integers.

Let us denote by P the transition probability matrix of the embedded Markov chain at departure epochs. Thus

$$P^k_{ij} = \mathbb{P}^0_D(X_{d_k} = j | X_{d_0} = i) \tag{1}$$

is the probability that the kth departure leaves behind j customers in the system, given that the 0th departure (which, under $\mathbb{P}^0_D$, occurs at time 0) leaves behind i customers.

As a result of PASTA, a customer who arrives at $t = 0$ under the Palm probability sees the system (excluding himself) in equilibrium, i.e. $\mathbb{P}^0_A(X_{0-} = i) = \pi_i$, and similarly the kth customer who arrives after $t = 0$ under the Palm probability also sees the system in equilibrium i.e. $\mathbb{P}^0_A(X_{t_k-} = i) = \pi_i$ for $k = 1, 2, \ldots$. However, the first customer who arrives after $t = 0$ under the stationary probability $\mathbb{P}$ *is not typical* and hence there is no reason to believe that he sees the system in equilibrium. This fact has been pointed out, among others, by Kelly [6, p.16]. Here we follow this line of investigation further.

2 Customer who arrives after a fixed time

We assume that the system is in stationarity and we will obtain an expression for the probability distribution of the number of customers in the system as observed by the kth customer who arrives after $t = 0$. We begin with the following

Lemma 1. *Suppose that $\{X_t; t \in \mathbb{R}\}$ is the queue-length process of an $M/M/s$ system with arrival rate λ and service rate μ, stationary under the probability measure $\mathbb{P}$. Then, with $\rho = \lambda/\mu$,*

$$\mu \mathbb{E}^0_D[d_1 - d_0 | X_0 = i] = \begin{cases} s^{-1} & \text{if } i \geq s, \\ \alpha_i + s^{-1} & \text{if } 0 \leq i < s, \end{cases} \tag{2}$$

where

$$\alpha_i = \begin{cases} \displaystyle\sum_{k=0}^{s-i-1} \frac{\rho^k}{(i+\rho)\cdots(i+k+\rho)} + \frac{1}{s}\left(\frac{\rho^{s-i}}{(i+\rho)\cdots(s-1+\rho)} - 1\right) & \textit{if } 0 < i < s, \\ \rho + \alpha_1 & \textit{if } i = 0. \end{cases} \tag{3}$$

Proof: The case $i \geq s$ is immediate. The case $i < s$ follows by a straight-forward conditioning argument and an elementary computation. ∎

We consider now the distribution of customers left behind by the kth departure after time 0.

Lemma 2. *Assuming that the queue is stationary, the kth departure after time 0 leaves behind the queue in stationarity i.e. for $k \geq 1$*

$$\mathbb{P}(X_{d_k} = i) = \pi_i. \tag{4}$$

Proof: Under $\mathbb{P}$ the queue-length process X_t is assumed to be stationary. As a result of the skip-free property of the system $\mathbb{P}_A^0(X_{0-} = j) = \mathbb{P}_D^0(X_0 = j)$. Also, due to the PASTA property $\mathbb{P}_A^0(X_{0-} = j) = \mathbb{P}(X_0 = j)$. Taken together, these two relations imply that

$$\mathbb{P}_D^0(X_0 = j) = \mathbb{P}(X_0 = j). \tag{5}$$

We have

$$P_D^0(X_{d_k} = i) = \sum_{j=0}^{\infty} \mathbb{P}_D^0(X_{d_k} = i | X_0 = j)\mathbb{P}_D^0(X_0 = j) \tag{6}$$

whereas, due to the Strong Markov property, we also have

$$\mathbb{P}_D^0(X_{d_k} = i | X_0 = j) = \mathbb{P}(X_{d_k} = i | X_0 = j) \tag{7}$$

From (5), (6), and (7) we have

$$P_D^0(X_{d_k} = i) = \sum_{j=0}^{\infty} \mathbb{P}(X_{d_k} = i | X_0 = j)\mathbb{P}(X_0 = j) = \mathbb{P}(X_{d_k} = i). \tag{8}$$

However, because of the invariance of $\mathbb{P}_D^0$ under the shifts θ_{d_k} we have

$$P_D^0(X_{d_k} = i) = \mathbb{P}_D^0(X_{d_k} = i).$$

Thus, putting all of the above together completes the proof of the lemma. ∎

The following theorem gives our main result.

Theorem 1. *In stationarity, the probability that the kth arriving customer* after time 0 *sees i customers in the system, $\mathbb{P}(X_{t_k-} = i)$, where $k \in \mathbb{N}$ is given by*

$$\begin{aligned} \mathbb{P}(X_{t_k-} = i) &= \lambda \pi_i \mathbb{E}_D^0[d_k - d_{k-1} | X_0 = i] & (9) \\ &= \lambda \pi_i \sum_{l=0}^{\infty} P_{il}^{k-1} \mathbb{E}_D^0[d_1 - d_0 | X_0 = l]. & (10) \end{aligned}$$

On the other hand for the customers that arrive before *the time origin in stationarity we have*

$$\mathbb{P}(X_{t_k-} = i) = \pi_i \qquad \textit{for } k = 0, -1, -2, -3, \ldots. \tag{11}$$

Proof: From the Palm inversion formula (see [1]) it follows that

$$\mathbb{P}(X_{t_k-} = i) = \lambda \mathbb{E}_A^0 \int_{t_0}^{t_1} \mathbf{1}(X_{t_k-} = i) ds = \lambda \mathbb{E}_A^0[(t_1 - t_0)\mathbf{1}(X_{t_k-} = i)]. \tag{12}$$

Let us now denote by $\{\widetilde{X}_t\}$ the process in reversed time by setting $\widetilde{X}_t = X_{-t}$ for all $t \in \mathbb{R}$. We also have $\widetilde{t}_n = -d_{-n+1}$ and $\widetilde{d}_n = -t_{-n+1}$ $\mathbb{P}$–a.s. Note however that $\mathbb{P}_A^0$–a.s. we have $\widetilde{d}_n = -t_{-n}$ and also $\mathbb{P}_D^0$–a.s. we have $\widetilde{t}_n = -d_{-n}$.

In view of the above, the expectation in the right hand side of (12) becomes

$$\begin{aligned} \mathbb{E}_A^0[(t_1 - t_0)\mathbf{1}(X_{t_k-} = i)] &= \mathbb{E}_A^0[(t_{-k} - t_{-k+1})\mathbf{1}(X_{t_0-} = i)] \\ &= \mathbb{E}_A^0[(\widetilde{d}_k - \widetilde{d}_{k-1})\mathbf{1}(\widetilde{X}_0 = i)]. & (13) \end{aligned}$$

Thus from (12), (13), and the reversibility of $\{X_t\}$ we have

$$\begin{aligned} \mathbb{P}(X_{t_k-} = i) &= \lambda \mathbb{E}_D^0[(d_k - d_{k-1})\mathbf{1}(X_0 = i)] \\ &= \lambda \mathbb{E}_D^0[d_k - d_{k-1} \mid X_0 = i] \mathbb{P}_D^0(X_0 = i). & (14) \end{aligned}$$

Taking into account (5) we see that (14) implies (9).

To obtain (10) use the strong Markov property to write the last expectation in equation (14) as

$$\mathbb{E}_D^0[d_k - d_{k-1} \mid X_0 = i] = \sum_{l=0}^{\infty} \mathbb{P}_D^0(X_{d_{k-1}} = l \mid X_{d_0} = i)\mathbb{E}_D^0[d_k - d_{k-1} \mid X_{d_{k-1}} = l]. \tag{15}$$

Using the shift-invariance property of $\mathbb{P}_D^0$ we have that

$$\mathbb{E}_D^0[d_k - d_{k-1} \mid X_{d_{k-1}} = l] \ = \ \mathbb{E}_D^0[d_1 - d_0 \mid X_{d_0} = l].$$

From the above and (1) we can write the right-hand-side of (15) as

$$\sum_{l=0}^{\infty} P_{il}^{k-1}\mathbb{E}_D^0[d_1 - d_0 \mid X_{d_0} = l]. \tag{16}$$

This completes the proof of (10).

We now turn to arrivals that occurred before the origin. Reversing time we realize that the nth arrival before time zero, denoted by t_{-n+1} due to the numbering convention corresponds to the nth departure after time zero, denoted by d_n. Thus an appeal to Lemma 2 establishes (11) and concludes the proof. ∎

Corollary 1. $\lim_{k\to\infty} \mathbb{P}(X_{t_k-} = i) = \pi_i$.

The corollary is intuitively obvious and is a consequence of a general theorem concerning stationary and ergodic random marked point processes (see [1, p. xx] and also [7]). The proof provided here is simple and direct and the argument used will also be useful in the sequel.

Proof: Because of the irreducibility, aperiodicity, and positive recurrence (under the stability condition) of the discrete time Markov chain with transition probability matrix P_{ij} we have

$$\lim_{k\to\infty} P_{il}^k = \mathbb{P}_D^0(X_0 = l) = \pi_l$$

where the last equality follows by the argument at the end of lemma 2. Also, because $\mathbb{E}_D^0[d_1 - d_0 \mid X_{d_0} = l] \leq \lambda^{-1} + \mu^{-1}$ for all $l \in \mathbb{N}$ and $\sum_{l=0}^{\infty} P_{il}^{k-1} = 1 < \infty$, we can use the Dominated

Convergence Theorem to conclude that

$$\begin{aligned}
\lim_{k\to\infty} \mathbb{P}(X_{t_k-} = i) &= \lambda\pi_i \lim_{k\to\infty} \sum_{l=0}^{\infty} P_{il}^{k-1} \mathbb{E}_D^0[d_1 - d_0 | X_0 = l] \\
&= \lambda\pi_i \sum_{l=0}^{\infty} \lim_{k\to\infty} P_{il}^{k-1} \mathbb{E}_D^0[d_1 - d_0 | X_0 = l] \\
&= \lambda\pi_i \sum_{l=0}^{\infty} \pi_l \mathbb{E}_D^0[d_1 - d_0 | X_0 = l] \\
&= \lambda\pi_i \mathbb{E}_D^0[d_1 - d_0] \\
&= \pi_i.
\end{aligned}$$

(In the last equation we have used the fact that $\mathbb{E}_D^0[d_1 - d_0] = \lambda^{-1}$.) ∎

We next use Lemma 1 in order to obtain an explicit expression for $\mathbb{P}(X_{t_k} = i)$. From (2) and (10) we obtain

$$\mathbb{P}(X_{t_k-} = i) \;=\; \lambda\pi_i \sum_{l=0}^{\infty} P_{il}^{k-1} \frac{1}{s\mu} + \lambda\pi_i \sum_{l=0}^{\infty} P_{il}^{k-1} \frac{\alpha_l}{\mu} \;=\; \frac{\rho}{s}\pi_i + \pi_i\rho \sum_{l=0}^{s-1} P_{il}^{k-1}\alpha_l.$$

Taking into account that

$$\pi_{i+1} = \frac{\rho}{s}\pi_i, \qquad i = s-1, s, s+1, \ldots$$

and that

$$P_{il}^{k-1} = 0 \qquad \text{for } i > l + k - 1$$

we obtain the following

Proposition 1. *The probability that the kth arrival after time 0 in a (time–) stationary $M/M/s$ queue sees i customers is given by the expression*

$$\mathbb{P}(X_{t_k-} = i) = \begin{cases} \pi_{i+1} & \text{if } i \geq s + k - 1, \\[2ex] \pi_{i+1} + \pi_i\rho \displaystyle\sum_{l=i-k+1}^{s-1} P_{il}^{k-1}\alpha_l & \text{if } k - 1 \leq i < s + k - 1, \\[2ex] \dfrac{\rho}{s}\pi_i + \pi_i\rho \displaystyle\sum_{l=0}^{s-1} P_{il}^{k-1}\alpha_l & \text{if } 0 \leq i < k - 1. \end{cases}$$

Specializing the above to the single server system and taking into account that, in that case,

$$\alpha_0 := \mu \mathbb{E}_D^0[d_1 - d_0 | X_0 = 0] - 1 = \frac{\mu}{\lambda} = \rho^{-1}$$

we obtain the following

Corollary 2. *For the $M/M/1$ queue we have*

$$\mathbb{P}(X_{t_k -} = i) = \begin{cases} \pi_{i+1} & \text{if } i \geq k, \\ \pi_{i+1} + \pi_i P_{i0}^{k-1} & \text{if } 0 \leq i \leq k-1. \end{cases} \tag{17}$$

Finally, time reversal allows us to state a proposition analogous to theorem 1 for the departures.

Corollary 3. *Consider an $M/M/s$ system which is stationary under $\mathbb{P}$. The probability that the kth departure leaves behind $i \in \mathbb{N}_0$ customers is given by*

$$\mathbb{P}(X_{d_k} = i) = \begin{cases} \pi_i, & k = 1, 2, 3, \ldots, \\ \lambda \pi_i \mathbb{E}_D^0[d_{-k+1} - d_{-k} | X_0 = i], & k = 0, -1, -2, -3, \ldots. \end{cases}$$

3 Absolute continuity and stochastic ordering

Fix $k \in \mathbb{N}$. The above results illustrate the fact that the two measures on $\mathbb{N}_0$, namely the stationary measure π and the shifted measure ν^k defined via $\nu^k(B) := \mathbb{P}(X_{t_k -} \in B)$, $B \subset \mathbb{N}_0$, are mutually absolutely continuous with Radon–Nikodỳm derivative given by

$$f_i := \frac{\nu_i^k}{\pi_i} = \lambda \mathbb{E}_D^0[d_k - d_{k-1} | X_{d_0} = i]. \tag{18}$$

This of course is a direct consequence of theorem 1. We will take advantage of this explicit representation for f_i to show that π stochastically dominates ν^k in the likelihood ordering sense (see Ross for a definition and some of the properties of this stochastic order).

Theorem 2. *For each k*

$$\pi \geq_{LR} \nu^k$$

or, equivalently, the sequence $\{f_i; i \in \mathbb{N}_0\}$ is increasing.

Proof: Indeed, with $\phi_l := \mathbb{E}_D^0[d_1 - d_0 | X_0 = l]$ can rewrite (10) as

$$f_i = \lambda \sum_{l=0}^{\infty} P_{il}^{k-1} \phi_l.$$

Clearly ϕ_l is an increasing sequence. Also, the family of probability measures on $\mathbb{N}_0$, $\{\mu^i; \in \mathbb{N}_0\}$ defined for each $B \subset \mathbb{N}_0$ via $\mu^i(B) := \sum_{l \in B} P_{il}^{k-1}$ is stochastically increasing, i.e. for $i \geq j$

$$\sum_{l=m}^{\infty} P_{il}^{k-1} \geq \sum_{l=m}^{\infty} P_{jl}^{k-1} \qquad \text{for all } m \in \mathbb{N}_0.$$

The simplest and most elegant way to prove this is via a coupling argument between a system starting with i customers at 0 and one starting with j customers at 0, as described in [8, p. 417]. Therefore $\sum_{l=0}^{\infty} P_{il}^{k-1} \phi_l \geq \sum_{l=0}^{\infty} P_{jl}^{k-1} \phi_l$ and thus $f_i \geq f_j$ for $i \geq j$. ∎

Since likelihood ratio ordering implies ordinary stochastic ordering (see [8]) we have the immediate

Corollary 4. *For any $k, i \in \mathbb{N}$,*

$$\mathbb{P}(X_{t_k-} \geq i) \leq \mathbb{P}(X_0 \geq i). \tag{19}$$

The above results are intuitively reasonable since, assuming the system to be stationary, we expect the first customer who arrives after time 0 to find the system less congested than usual, in view of the fact that the interarrival time between this customer and the one preceding him is the sum of *two* independent exponential random variables with rate λ.

4 The $M/M/1$ queue and the speed of convergence to stationarity

Since $P_{i0}^{k-1} = 0$ for $i \geq k$ equation (17) can be restated as

$$\nu_i^k = \pi_{i+1} + \pi_i P_{i0}^{k-1}. \tag{20}$$

Let $p = \frac{\mu}{\mu+\lambda}$ and $q = \frac{\lambda}{\mu+\lambda}$. We have the following

Proposition 2. *The generating function* $\Phi_i(z,w) := \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} w^k z^l P_{il}^k$ *is given by*

$$\Phi_i(z,w) = \frac{(1-z)wp\Phi_i(0,w) - (1-qz)z^{i+1}}{qz^2 - z + pw} \tag{21}$$

with

$$\Phi_i(0,w) = \frac{z_1(w)^i}{1-z_1(w)} \tag{22}$$

where $z_1(w) = \frac{1-\sqrt{1-4pqw}}{2q}$, $z_2(w) = \frac{1+\sqrt{1-4pqw}}{2q}$, *are the roots of the equation*

$$qz^2 - z + pw = 0. \tag{23}$$

Proof: Under the measure $\mathbb{P}_D^0$ we have

$$X_{d_n} = X_{d_{n-1}} + u_n - \mathbf{1}(X_{d_{n-1}} > 0) \tag{24}$$

where the random variables $\{u_n\}$ are independent geometric with distribution $\mathbb{P}_D^0(u_n = j) = pq^j$, $j = 0,1,2,\ldots$, with a corresponding probability generating function $\frac{p}{1-qz}$. If we set

$$\phi_{i,n}(z) := \mathbb{E}_D^0[z^{X_{d_n}} | X_{d_0} = i]$$

then from (24) we have

$$\mathbb{E}_D^0[z^{X_{d_n}} | X_{d_{n-1}}] = \frac{p}{1-qz} z^{X_{d_{n-1}} - \mathbf{1}(X_{d_{n-1}} > 0)}$$

whence, taking conditional expectation given that $X_{d_0} = i$, we obtain the recursive relation

$$\phi_{i,n}(z) = \frac{z^{-1}p}{1-qz}\phi_{i,n-1}(z) - (1-z)\frac{z^{-1}p}{1-qz}\phi_{i,n-1}(0), \qquad n = 1,2,\ldots. \tag{25}$$

(Note that $\phi_{i,n-1}(0) = \mathbb{P}_D^0(X_{d_{n-1}} = 0 | X_{d_0} = i) = P_{i0}^{n-1}$.) Multiplying both sides of the above recursion by w^n, summing for $n = 1,2,3,\ldots$, and taking into account that $\phi_{i,0}(z) = z^i$, we obtain after some elementary manipulations (21). The unknown function $\Phi_i(0,w) = \sum_{k=0}^{\infty} w^k \phi_{i,k}(0)$ can be determined by noting that one of the two roots of the denominator, z_1, satisfies the inequality $|z_1| < 1$ provided that $|w| < 1$. Thus, z_1 must also be a root of the numerator since for $|w| < 1$, $|z| < 1$, $\Phi_i(z,w)$ cannot have any singularities. We conclude that

$$\Phi_i(0,w) = \frac{(1-qz_1)z_1^{i+1}}{(1-z_1)wp}.$$

z_1 being a root of (23) we have that $z_1 - z_1^2 q = pw$ and hence we obtain (22). ∎

From the above we obtain readily the following

Corollary 5. *The generating function of the sequence* $\{\nu_i^k; k = 0, 1, 2, \ldots\}$ *is given by*

$$\sum_{k=0}^{\infty} \nu_i^k w^k = \frac{\pi_{i+1}}{1-w} + \pi_i \frac{z_1(w)^i}{1 - z_1(w)}. \tag{26}$$

The joint generating function $N_i(z,w) := \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \nu_i^k w^k z^i$ *is given by*

$$N_i(z,w) = \frac{\rho(1-\rho)}{(1-z\rho)(1-w)} + \frac{1}{(1-z)(1-wz_1(w))}. \tag{27}$$

In [9] the quantities

$$\int_0^{\infty} \left(\mathbb{P}(X_t = i) - \mathbb{P}(X_t = i | X_0 = 0)\right) dt,$$

$i = 0, 1, 2, \ldots$, and

$$\int_0^{\infty} \left(\mathbb{E}X_t - \mathbb{E}[X_t | X_0 = 0]\right) dt$$

have been proposed as natural measures of the speed of convergence of the $M/M/s$ queue to stationarity and their values have been computed in terms of the parameters of the process. In [3] this idea has been extended to general birth-and-death processes and in [4] the connection of these ideas to the fundamental matrix of the Markov process has been pointed out.

Theorem 3. *The quantity* $\sum_{k=1}^{\infty} \left(\mathbb{P}(X_0 = i) - \mathbb{P}(X_{t_k} = i)\right)$ *is given by*

$$\sum_{k=1}^{\infty} (\pi_i - \nu_i^k) = i\pi_i \qquad \textit{for } k = 1, 2, \ldots \textit{ and } i = 0, 1, 2, \ldots. \tag{28}$$

Also,

$$\sum_{k=1}^{\infty} \mathbb{E}X_0 - \mathbb{E}X_{t_k} = \left(\frac{\rho}{1+\rho}\right)^2. \tag{29}$$

Proof: Define the sequence $a_k := \pi_i - \nu_i^k$ and the generating function $A(w) := \sum_{k=1}^{\infty} a_k w^k$. From Corollary 3 and the fact that, for the $M/M/1$ queue $\pi_i = (1-\rho)\rho^i$, we have

$$A(w) = \frac{\pi_i}{1-w}\left(1 - \rho - z_1(w)^i \frac{1-w}{1-z_1(w)}\right).$$

A standard argument involving de l'Hôpital's theorem shows that $\lim_{w\uparrow 1} A(w) = i\pi_i$. Since the power series $\sum_{k=1}^{\infty} a_k w^k$ converges for every $w \in [0,1]$ by Abel's theorem (e.g. see [5, p.259]) we have $\sum_{k=1}^{\infty} a_k = \lim_{w\uparrow 1} A(w) = i\pi_i$.

Similarly $\sum_{k=1}^{\infty} [\mathbb{E}X_0 - \mathbb{E}X_{t_k}] = \sum_{i=0}^{\infty} i\pi_i^2$ and thence follows (29). ∎

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