Synchronized Queues with Deterministic Arrivals

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Abstract

We consider m independent exponential servers in parallel, driven by the same deterministic input. This is a modification of the Flatto-Hahn-Wright model which turns out to be easily tractable. We focus on the time-stationary distribution of the number of customers which is obtained using the Palm inversion formula.

KEYWORDS: FORK-JOIN QUEUES, FLATTO-HAHN-WRIGHT MODEL, D/M/1 QUEUE, PALM PROBABILITIES
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1 Introduction

Synchronized (or fork-join) queues have been an object of study over the last three decades as models of parallel processing in computer systems and assembly operations in manufacturing. In the model we examine here, the service facility consists of m single servers in parallel, each with its own queue. The m buffers have infinite capacity and individually operate according to the FIFO discipline. Upon arrival to the service facility, each customer splits in m parts, each part joining the corresponding queue. While each station viewed separately is an ordinary single server queue, the determination of the *joint statistics* of the m queues is in general hard to obtain.

The above system when customers arrive according to a Poisson process and the service requirements for the parts are independent, exponential random variables with rate depending on the type of part, is known as the FHW (Flatto–Hahn–Wright) model (see [8], [9], [16]). Flatto and Hahn [9], and Flatto [8], have studied the system (for the case $m = 2$) using complex analysis techniques. The FHW model is of course a special case of a two–dimensional random walk on the positive quadrant. There is a rich theory connecting this problem to boundary value problems and the multidimensional extension of the

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Wiener–Hopf factorization. The reader is referred to Fayolle, Iasnogorodski, and Malyshev [10] both for an overview and for a state-of-the-art treatment of these issues.

Fork-join systems consisting of two queues with Poisson arrivals and service requirements which are i.i.d. sequences of exchangeable pairs of random variables have been studied in Baccelli [3]. Ayhan and Baccelli [1] use a Taylor series approach to analyze fork-join systems with general service requirements. Baccelli, Makowski, and Swhartz [5] obtain bounds for the performance of more general fork-join queues by means of stochastic ordering arguments (in the same vein see also Li and Xu [13]). Ko and Serfozo [14] obtain an approximate expression for the sojourn times in $M/M/s$ fork-join queues. Because of the intractability of the FHW model most of the explicit results are asymptotic in nature. These include both asymptotics based on generating functions obtained by complex analysis techniques (e.g. [8], [9]) and results obtained using large deviation techniques [15]. We also mention the diffusion approximation of [12] and the related problem of fork-join fluid queues studied in [11]. A related line or research that studies queueing networks with signals and concurrent movements examines the FHW model in the framework of markovian queueing networks. We refer the interested reader to [7] and [6].

The model examined in this paper, unlike the classical FHW model, is tractable by means of elementary tools. In fact, due to the deterministic nature of arrivals and the independence of the service processes in the m queues, the *customer-stationary* (Palm with respect to the arrival processes) queue lengths are independent and thus the system (under the Palm probability measure) can be viewed as m independent $D/M/1$ queues. The situation becomes more complicated when we turn our attention to the *(time-) stationary version of the process* and this is the main focus of this paper.

Section 2 gives a more detailed description of the model while in section 3 the Palm inversion formula in conjunction with an argument based on generating functions is used in order to derive the joint distribution of the stationary number of customers in the system. Section 4 provides an illustration of the above results by examining in more detail the system with two stations ($m = 2$). An expression is obtained for the stationary distribution of the workload, and the deterministic model is compared to the classical FHW model with Poisson arrivals in terms of the correlation coefficient of the stationary queue sizes.

2 Synchronized queues with deterministic arrivals

In the system considered here, customers, each consisting of m parts, arrive to the service facility according to a deterministic process with constant interarrival times, equal to a. Upon arrival to the system, each customer splits into its constituent parts which join the corresponding queues. From that point on the parts move independently even though, for some applications, it may be useful to think that, after service completion, the parts of a customer that finish first wait in a “staging area” for their counterparts and once all parts have completed their processing they are assembled into a finished unit. In a manufacturing context this could describe an assembly operation. The point process of arrival epochs to the system will be denoted by $\{T_n; n \in \mathbb{Z}\}$ where $T_{n+1} = T_n + a$. Service requirements for each queue are independent, exponential random variables with rate μ_k for the kth station. Clearly the system is stable iff

$a\min\{\mu_1,\dots,\mu_m\} > 1$. Denote by $\{X_t^k; t \in \mathbb{R}\}$ the number of customers in station k, ($k = 1,\dots,m$) and let $\mathbf{X}_t := (X_t^1,\dots,X_t^m)$ denote the number of customers in the m queues. We will assume that this process has *right-continuous sample paths* with probability 1. In particular $(X_{T_n-}^1,\dots,X_{T_n-}^m)$ is the number of customers in the m queues as seen by an arrival, right before the arrival epoch. Suppose now that a stationary version of this process has been constructed on the probability space $(\Omega, \mathcal{F}, P)$ and let P^0 denote the Palm transformation of P under the point process $\{T_n; n \in \mathbb{Z}\}$. We will denote by E^0 the expectation with respect to P^0. Intuitively, P^0 is the probability measure conditioned on the event that the origin coincides with a typical arrival point, which by convention is denoted by T_0. Thus $P^0(T_0 = 0) = 1$. We refer the reader to Baccelli and Brémaud [4] for formal definitions and the mathematical framework. Since arrivals are deterministic and service times are independent in the m queues it is easy to see that, under P^0, the m queue-length processes $\{X_t^k; t \in \mathbb{R}\}$, $k = 1,2,\dots,m$, are independent. Thus the Palm version of the process can be readily analyzed by studying m independent $D/M/1$ systems. In particular

$$P^0(X_{0-}^1 = n_1,\dots,X_{0-}^m = n_m) = \prod_{k=1}^m (1-\sigma_k)\sigma_k^{n_k}, \qquad n_k = 0,1,2,\dots. \tag{1}$$

where σ_k is the unique solution of the equation

$$x = e^{-a\mu_k(1-x)}, \qquad k = 1,\dots,m, \tag{2}$$

that is less than one. Indeed, besides the obvious solution, $x = 1$, it is clear from a convexity argument that the above equation has one more solution which, as is well known (see [2]), belongs to the interval $(0,1)$ provided that the stability condition $a\mu_k > 1$ holds.

3 The stationary number of customers in the system

We now turn to the stationary version of the process. It is clear that the m queue–length processes are no longer independent. From standard results concerning the $GI/M/1$ queue [2, p. 280] it follows that the marginal distribution for the stationary number of customers in each queue is a modified geometric distribution given by

$$\begin{aligned} P(X_0^k = n) &= \rho_k(1-\sigma_k)\sigma_k^{n-1}, \quad n = 1,2,\dots \\ P(X_0^k = 0) &= 1-\rho_k, \end{aligned} \tag{3}$$

with $\rho_k = (a\mu_k)^{-1}$ for $k = 1,\dots,m$. The corresponding p.g.f. (probability generating function) is given by

$$\varphi(z) := 1 - \rho_k + z\rho_k \frac{1-\sigma_k}{1-z\sigma_k}. \tag{4}$$

On the other hand, the joint distribution of the queue–lengths under the stationary probability measure P is harder to find. As we will see next it can be obtained from the Palm inversion formula using a conditioning argument. We start with the following elementary lemma where, as usual, x^+ denotes the positive part of the real number x.

Lemma 1. *Let Y be a geometric random variable with p.g.f. $Ez^Y = \frac{(1-\sigma)z}{1-z\sigma}$, where $\sigma \in (0,1)$, and N a Poisson random variable, independent of Y, with mean β. Then*

$$Ez^{(Y-N)^+} = 1 - \frac{1-z}{1-z\sigma} e^{-\beta(1-\sigma)}.$$

Proof: Condition on N to obtain

$$E[z^{(Y-N)^+}|N] = \sum_{k=1}^{N} (1-\sigma)\sigma^{k-1} + \sum_{k=N+1}^{\infty} (1-\sigma)\sigma^{k-1} z^{k-N} = 1 - \sigma^N \frac{1-z}{1-z\sigma}.$$

Taking expectation with respect to N completes the proof. □

Denote by

$$\varphi(z_1, \ldots, z_m) = E \prod_{k=1}^{m} z_k^{X_0^k}$$

the probability generating function of the stationary number of customers in the system. Let us also denote by $\mathcal{A}_r$ the class of all subsets of the set $S_m := \{1, 2, \ldots, m\}$ containing exactly r elements. In particular we have of course that $|\mathcal{A}_r| = \binom{m}{r}$ where, as usual, $|B|$ denotes the cardinality of the set B. Also, for any $\vec{n} = (n_1, \ldots, n_m) \in \mathbb{N}_0^m$ define $\Phi_{\vec{n}} := \{k : n_k \geq 1\} \subseteq S_m$, the set of all indices corresponding to non-zero components of the vector $\vec{n}$. We are ready to state our main result.

Theorem 1. *The probability generating function of the stationary number of customers in the system is given by*

$$\varphi(z_1, \ldots, z_m) = 1 + \sum_{r=1}^{m} (-1)^r \sum_{A \in \mathcal{A}_r} C_A \prod_{k \in A} \frac{1 - z_k}{1 - z_k \sigma_k} \tag{5}$$

where

$$C_A := \frac{1 - \prod_{k \in A} \sigma_k}{\sum_{k \in A} \rho_k^{-1} (1 - \sigma_k)}, \tag{6}$$

the constants being indexed by the subsets $A \subseteq \{1, 2, \ldots, m\}$. The corresponding probability distribution is given by

$$P(X_0^1 = n_1, \ldots, X_0^m = n_m) = \Gamma_{\vec{n}} \prod_{k \in \Phi_{\vec{n}}} (1 - \sigma_k) \sigma_k^{n_k - 1} \tag{7}$$

where

$$\Gamma_{\vec{n}} := \sum_{\{A : A \supseteq \Phi_{\vec{n}}\}} (-1)^{|A| - |\Phi_{\vec{n}}|} C_A \;=\; \sum_{r=|\Phi_{\vec{n}}|}^{m} (-1)^{r - |\Phi_{\vec{n}}|} \sum_{A \in \mathcal{A}_r} C_A \mathbf{1}(\Phi_{\vec{n}} \subseteq A). \tag{8}$$

Remark: The expression (7) for $\Phi_{\vec{n}} = \varnothing$ (i.e. for $\vec{n} = (0, 0, \ldots, 0)$) becomes

$$P(X_0^1 = 0, \ldots, X_0^m = 0) = 1 + \sum_{r=1}^{m} (-1)^r \sum_{A \in \mathcal{A}_r} C_A. \tag{9}$$

Also, for $\Phi_{\vec{n}} = S_m$, i.e. when $n_k \geq 1$ for all k, $\Gamma_{\vec{n}} = C_{S_m} = \frac{1 - \prod_{k=1}^m \sigma_k}{\sum_{k=1}^m \rho_k^{-1}(1-\sigma_k)}$.

Proof: A straight-forward application of the Palm inversion formula (see [4]) gives

$$\varphi(z_1, \ldots, z_m) = a^{-1} E^0 \int_0^a \left(\prod_{k=1}^m z_k^{X_t^k} \right) dt = a^{-1} E^0 \int_0^a \left(\prod_{k=1}^m z_k^{(X_0^k - N_t^k)^+} \right) dt. \tag{10}$$

In the above expression $\{(N_t^1, \ldots, N_t^m); t \geq 0\}$ are m independent Poisson processes with rates μ_k, $k = 1, \ldots, m$, representing the service processes in the m exponential servers. Furthermore, these Poisson processes are independent of the vector of queue lengths at time 0, $(X_0^1, \ldots, X_0^m)$. Finally, under the probability measure P^0, and since the sample paths are right-continuous,

$$P^0(X_0^1 = n_1, \ldots, X_0^m = n_m) = \prod_{k=1}^m (1 - \sigma_k) \sigma_k^{n_k - 1}, \quad n_k = 1, 2, \ldots.$$

(In the above expression the customer arriving at $t = 0$ has been taken into account–cf. (1)). Thus, appealing to the Fubini theorem, and using the independence of the X_0^k under P^0 and Lemma 1, we can write the right-hand side of (10) as

$$a^{-1} \int_0^a \prod_{k=1}^m E^0 z_k^{(X_0^k - N_t^k)^+} dt = a^{-1} \int_0^a \prod_{k=1}^m \left(1 - \frac{1 - z_k}{1 - z_k \sigma_k} e^{-\mu_k t (1 - \sigma_k)} \right) dt. \tag{11}$$

The product inside the integral on the right hand side of the above expression can be written as

$$\prod_{k=1}^m \left(1 - \frac{1 - z_k}{1 - z_k \sigma_k} e^{-\mu_k t (1 - \sigma_k)} \right) = 1 + \sum_{r=1}^m (-1)^r \sum_{A \in \mathcal{A}_r} e^{-t \sum_{k \in A} \mu_k (1 - \sigma_k)} \prod_{k \in A} \frac{1 - z_k}{1 - z_k \sigma_k}$$

and thus the right hand side of (11) becomes

$$1 + \sum_{r=1}^m (-1)^r \sum_{A \in \mathcal{A}_r} \left(\prod_{k \in A} \frac{1 - z_k}{1 - z_k \sigma_k} \right) a^{-1} \int_0^a e^{-t \sum_{k \in A} \mu_k (1 - \sigma_k)} dt. \tag{12}$$

However,

$$a^{-1} \int_0^a e^{-t \sum_{k \in A} \mu_k (1 - \sigma_k)} dt = \frac{1 - e^{-a \sum_{k \in A} \mu_k (1 - \sigma_k)}}{a \sum_{k \in A} \mu_k (1 - \sigma_k)} = \frac{1 - \prod_{k \in A} \sigma_k}{\sum_{k \in A} \rho_k^{-1} (1 - \sigma_k)}$$

where, in the last equation we have made use of the defining relation for the σ_k, the definition of ρ_k, and (2). From the above the p.g.f. of the stationary number of customers in the m queues becomes

$$\varphi(z_1,\ldots,z_m) = 1 + \sum_{r=1}^{m}(-1)^r \sum_{A\in\mathcal{A}_r} \frac{1-\prod_{k\in A}\sigma_k}{\sum_{k\in A}\rho_k^{-1}(1-\sigma_k)} \prod_{k\in A} \frac{1-z_k}{1-z_k\sigma_k}.$$

This establishes (5). Since

$$\frac{1-z}{1-z\sigma} = 1 - \frac{z(1-\sigma)}{1-z\sigma} = 1 - \sum_{n=1}^{\infty}(1-\sigma)\sigma^{n-1}z^n$$

we have

$$\varphi(z_1,\ldots,z_m) = 1 + \sum_{r=1}^{m}(-1)^r \sum_{A\in\mathcal{A}_r} C_A \prod_{k\in A}\left(1 - \frac{z_k(1-\sigma_k)}{1-z_k\sigma_k}\right)$$

or

$$\varphi(z_1,\ldots,z_m) = 1 + \sum_{r=1}^{m}(-1)^r \sum_{A\in\mathcal{A}_r} C_A \prod_{k\in A}\left(1 - \sum_{n_k=1}^{\infty}(1-\sigma_k)\sigma_k^{n_k-1}z_k^{n_k}\right). \tag{13}$$

We can now imagine the process of collecting terms from the above expression. We begin with an example: When $\Phi_{\vec{n}} = S_m$, i.e. when $n_k \geq 1$ for all $k = 1, 2, \ldots, m$, only the product

$$\prod_{k\in S_m}\left(1 - \sum_{n_k=1}^{\infty}(1-\sigma_k)\sigma_k^{n_k-1}z_k^{n_k}\right)$$

in (13) contains the term $z_1^{n_1}z_2^{n_2}\cdots z_m^{n_m}$ and the corresponding coefficient is $(-1)^m\prod_{k=1}^{m}(1-\sigma_k)\sigma_k^{n_k-1}$. From (13) we see that this term is multiplied by $(-1)^m C_{S_m}$ and thus the coefficient of the term $z_1^{n_1}z_2^{n_2}\cdots z_m^{n_m}$ in the expansion of $\varphi(z_1,\ldots,z_m)$ is equal to

$$(-1)^{m+m}C_{S_m}\prod_{k=1}^{m}(1-\sigma_k)\sigma_k^{n_k-1} = \frac{1-\prod_{k=1}^{m}\sigma_k}{\sum_{k=1}^{m}\rho_k^{-1}(1-\sigma_k)}\prod_{k=1}^{m}(1-\sigma_k)\sigma_k^{n_k-1}.$$

(cf. Remark 1.) In the general case, the product $\prod_{k\in A}\left(1-\sum_{n_k=1}^{\infty}(1-\sigma_k)\sigma_k^{n_k-1}z_k^{n_k}\right)$ indexed by the set A contains the term $z_1^{n_1}z_2^{n_2}\cdots z_m^{n_m}$ if and only if $\Phi_{\vec{n}} \subseteq A$. The coefficient of this term when we expand this product is

$$1^{|A|-|\Phi_{\vec{n}}|}\cdot(-1)^{|\Phi_{\vec{n}}|}\prod_{k\in\Phi_{\vec{n}}}(1-\sigma_k)\sigma_k^{n_k-1}.$$

In order to find the coefficient of $z_1^{n_1}z_2^{n_2}\cdots z_m^{n_m}$ in the expansion of $\varphi(z_1,\ldots,z_m)$ it suffices to multiply this term by $(-1)^{|A|}C_A$ and then to sum over all sets $A \supseteq \Phi_{\vec{n}}$. We thus have

$$P(X_0^1 = n_1,\ldots,X_0^m = n_m) = \sum_{\Phi_{\vec{n}}\subseteq A}(-1)^{|A|-|\Phi_{\vec{n}}|}C_A\prod_{k\in\Phi_{\vec{n}}}(1-\sigma_k)\sigma_k^{n_k-1}. \tag{14}$$

The expression (7) is a restatement of the above. In the second expression for $\Gamma_{\vec{n}}$ in (8) we have split the sum according to the cardinality of the index set A. Finally (9) is the special case where $\Phi_{\vec{n}} = \varnothing$ and this completes the proof. □

Corollary 1. *In the symmetric case, where the service rates in all stations are equal to μ, the p.g.f. of the stationary number of customers in the system is given by*

$$\varphi(z_1, \ldots, z_m) = 1 + \sum_{r=1}^{m} (-1)^r \rho \frac{1-\sigma^r}{r(1-\sigma)} \sum_{A \in \mathcal{A}_r} \prod_{k \in A} \frac{1-z_k}{1-z_k\sigma}. \tag{15}$$

The corresponding probability distribution is given by

$$P(X_0^1 = n_1, \ldots, X_0^m = n_m) = (1-\sigma)^{|\Phi_{\vec{n}}|} \sigma^{\left(\sum_{k=1}^m n_k\right) - |\Phi_{\vec{n}}|} \sum_{r=|\Phi_{\vec{n}}|}^{m} (-1)^{r-|\Phi_{\vec{n}}|} \rho \frac{1-\sigma^r}{r(1-\sigma)} \binom{m-|\Phi_{\vec{n}}|}{r-|\Phi_{\vec{n}}|} \tag{16}$$

for $\Phi_{\vec{n}} \neq \varnothing$, and

$$P(X_0^1 = 0, \ldots, X_0^m = 0) = 1 + \sum_{r=1}^{m} (-1)^r \binom{m}{r} \rho \frac{1-\sigma^r}{r(1-\sigma)}. \tag{17}$$

Proof: Since all the service rates are the same we also have $\rho_k = \rho$ and $\sigma_k = \sigma$ for $k = 1, 2, \ldots, m$. Also note that

$$C_A = \rho \frac{1-\sigma^r}{r(1-\sigma)} \quad \text{for all} \quad A \in \mathcal{A}_r. \tag{18}$$

The p.g.f. (15) follows by using (18) in (5). In order to derive (16) it suffices to use (18) in (7) to obtain

$$P(X_0^1 = n_1, \ldots, X_0^m = n_m) = \sum_{r=1}^{m} (-1)^{r+|\Phi_{\vec{n}}|} \rho \frac{1-\sigma^r}{r(1-\sigma)} \sum_{A \in \mathcal{A}_r} \mathbf{1}(\Phi_{\vec{n}} \subseteq A) \prod_{k \in \Phi_{\vec{n}}} (1-\sigma)\sigma^{n_k - 1}.$$

An elementary combinatorial argument gives $\sum_{A \in \mathcal{A}_r} \mathbf{1}(\Phi_{\vec{n}} \subseteq A) = \binom{m-|\Phi_{\vec{n}}|}{r-|\Phi_{\vec{n}}|}$ and hence (16) follows. From these considerations, and the fact that $\Phi_{\vec{n}} = \varnothing$ for $\vec{n} = (0, 0, \ldots, 0)$, (17) is also obtained. □

4 The two-server system and further performance measures

To illustrate the above results we will apply them to a system with two servers ($m = 2$).

Proposition 1. *The stationary number of customers in a synchronized D/M/1 system with two stations is given by*

$$
\begin{aligned}
P(X_0^1 = n_1, X_0^2 = n_2) &= C_{\{1,2\}}(1-\sigma_1)\sigma_1^{n_1-1}(1-\sigma_2)\sigma_2^{n_2-1} \qquad n_1 \geq 1, n_2 \geq 1,\\
P(X_0^1 = 0, X_0^2 = n_2) &= (1-\sigma_2)\sigma_2^{n_2-1}\left(\rho_2 - C_{\{1,2\}}\right) \qquad n_2 \geq 1,\\
P(X_0^1 = n_1, X_0^2 = 0) &= (1-\sigma_1)\sigma_1^{n_1-1}\left(\rho_1 - C_{\{1,2\}}\right) \qquad n_1 \geq 1,\\
P(X_0^1 = 0, X_0^2 = 0) &= 1 - \rho_1 - \rho_2 + C_{\{1,2\}},
\end{aligned}
$$

where

$$C_{\{1,2\}} := \frac{1-\sigma_1\sigma_2}{\rho_1^{-1}(1-\sigma_1) + \rho_2^{-1}(1-\sigma_2)}. \tag{19}$$

Proof: We apply theorem 1 noting that $C_{\{1\}} = \frac{1-\sigma_1}{\rho_1^{-1}(1-\sigma_1)} = \rho_1$ and, similarly $C_{\{2\}} = \rho_2$. □

The correlation coefficient for the stationary number of customers in the two queues can be computed easily from the above stationary distribution and is given by

$$r = \sqrt{\frac{\rho_1\rho_2}{(1+\sigma_1-\rho_1)(1+\sigma_2-\rho_2)}}\left(\frac{(\rho_1\rho_2)^{-1}(1-\sigma_1\sigma_2)}{(1-\sigma_1)\rho_1^{-1} + (1-\sigma_2)\rho_2^{-1}} - 1\right).$$

For the symmetric case, i.e. when $\mu_1 = \mu_2$ and hence $\rho_1 = \rho_2 = \rho$ and $\sigma_1 = \sigma_2 = \sigma$, we have

$$r = \frac{1}{2}\left(1 - \frac{\rho}{1+\sigma-\rho}\right). \tag{20}$$

A plot of the correlation coefficient r as a function of ρ is given in figure 1.

It is of some interest to compare the correlation coefficient (20) to that of the symmetric, classic FHW model (with Poisson arrivals) which is given by

$$r = \frac{1}{2} - \frac{\rho}{8} \tag{21}$$

(see theorem 6.2 of [9]). As expected, the case with Poisson arrivals exhibits higher correlation between the two queues. More interesting perhaps is the heavy traffic behavior. In the case of deterministic arrivals examined in this paper, the correlation between the two queues goes to zero as $\rho \to 1$. On the other hand, in the classic FHW model with Poisson arrivals the correlation coefficient goes to $3/8$ as $\rho \to 1$. (Despite the fact that $\sigma = \rho$ in the case of Poisson arrivals, it would be mistaken to expect (20) to reduce to (21) in that case since the whole analysis leading to (20) is based on the assumption that arrivals are deterministic.)

Finally we can use the results of theorem 1 together with the memoryless property of the exponential distribution in order to obtain the statistics for the workload process. For the sake of simplicity we present it for the case of a two–server station. The extension to the general m server model is obvious. If (W_t^1, W_t^2) is the workload vector at time t then we have the following

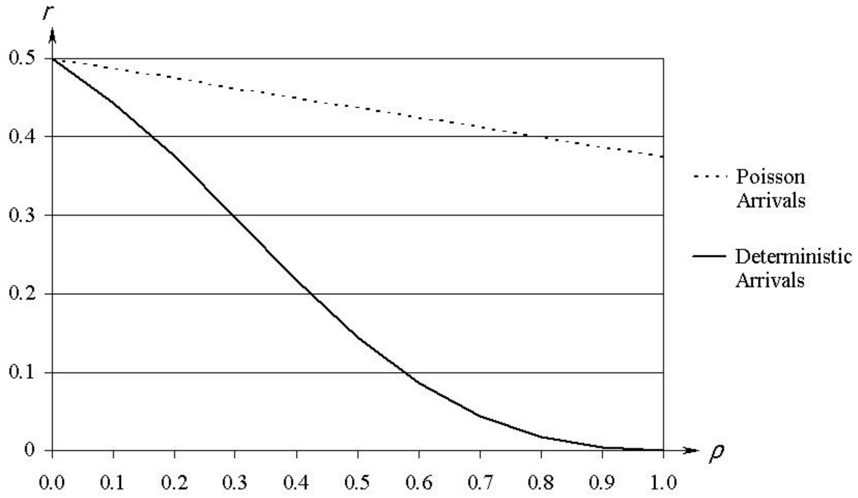

Figure 1: The correlation coefficient, r, between the queue sizes in the two queues in the symmetric case as a function of the utilization ρ. Two plots are given, one for the model with deterministic arrivals and the other for the classical model with Poisson arrivals.

Proposition 2. *The stationary joint distribution of the workload in the two queues,* $F(x_1, x_2) := P(W_0^1 \leq x_1, W_0^2 \leq x_2)$ *is given by*

$$F(x_1,x_2) \;=\; 1 - \rho_1 e^{-\mu_1(1-\sigma_1)x_1} - \rho_2 e^{-\mu_2(1-\sigma_2)x_2} + \frac{1-\sigma_1\sigma_2}{(1-\sigma_1)\rho_1^{-1}+(1-\sigma_2)\rho_2^{-1}}\, e^{-\mu_1(1-\sigma_1)x_1-\mu_2(1-\sigma_2)x_2}.$$

Proof: Start by conditioning on the number of customers present in the system at time 0, under the stationary probability measure P. Then $E\left[e^{-s_1 W_0^1 - s_2 W_0^2}\middle| X_0^1 = n_1, X_0^2 = n_2\right] = \left(\frac{\mu_1}{s_1+\mu_1}\right)^{n_1}\left(\frac{\mu_2}{s_2+\mu_2}\right)^{n_2}$ for all $n_1, n_2 = 0, 1, 2, \ldots$. Taking into account the expression for the stationary distribution of the number of customers in the two queues we obtain, after some simplifications, the following expression for the joint Laplace transform of the stationary workload

$$\begin{aligned} Ee^{-s_1 W_0^1 - s_2 W_0^2} \;&=\; C_{\{1,2\}} \frac{\mu_1(1-\sigma_1)}{s_1+\mu_1(1-\sigma_1)} \frac{\mu_2(1-\sigma_2)}{s_2+\mu_2(1-\sigma_2)} + \frac{\mu_1(1-\sigma_1)}{s_1+\mu_1(1-\sigma_1)}\left(\rho_1 - C_{\{1,2\}}\right) \\ &\quad + \frac{\mu_2(1-\sigma_2)}{s_2+\mu_2(1-\sigma_2)}\left(\rho_2 - C_{\{1,2\}}\right) + 1 - \rho_1 - \rho_2 + C_{\{1,2\}} \end{aligned}$$

where $C_{\{1,2\}}$ is the constant given in (19). Straight-forward inversion of this transform completes the proof. □

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