A discrete time proof of Neveu's exchange formula

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Abstract

Neveu's exchange formula relates the Palm probabilities with respect to two jointly stationary simple point processes. We give a new proof of the exchange formula by using a simple result from discrete time stationary stochastic processes.

1 Introduction

Consider two simple point processes A, B, defined on a common probability space $(\Omega, \mathcal{F}, P)$ endowed with a P-preserving measurable flow $\{\theta_t,\ t \in \mathbf{R}\}$. Assume that the point processes are jointly stationary, i.e., $A(\theta_t\omega, X) = A(\omega, X+t)$, $B(\theta_t\omega, X) = B(\omega, X+t)$, for all $\omega \in \Omega$, all Borel sets $X \subseteq \mathbf{R}$ and all $t \in \mathbf{R}$. The Palm probabilities of P with respect to A, B are denoted by P^A, P^B, respectively. Since, for any such point process A, the Palm transformation $P \mapsto P^A$ (see eq. (5)) is invertible, there is a relation between P^A and P^B which is given explicitly by the rule known as Neveu's exchange formula (c.f. Neveu [7, 8] and eq. (6)).

The original demonstration of the exchange formula appears in Neveu [7, 8]. Franken and Lisek [4] proved a similar result aiming at a generalization of Wald's identity. Brémaud [1] noted that the formula can be derived as a consequence of Miyazawa's rate conservation principle by means of a suitably chosen stationary stochastic process. The same author [2] derived a general-purpose Palm calculus formula (a consequence of which is the exchange formula) from "first principles", i.e., with the use of the definition of Palm probability via Mecke's formula (see eq. (5) below).

What makes the exchange formula useful is that it encompasses a number of interesting relations for stationary systems driven by point processes. Such systems arise, for

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instance, in queueing theory. It is not the purpose of this article to show the applicability of the exchange formula (see for instance our previous work [6], in which the exchange formula has been used to construct sensitivity analysis estimators in a non-regenerative context). Rather, our aim is to show that a direct demonstration of the exchange formula is possible. In fact, it will be seen that the exchange formula is a rewriting of the discrete-time Palm inversion formula. It seems that this point of view, albeit simple and straightforward, has not been explicitly noticed.

Palm probabilities with respect to discrete-time point processes can be defined by straightforward conditioning on the occurrence of a point at the origin of time. The inverse relation can be found by a simple ergodic theory type of argument, essentially due to Kac [5], which we present in Section 2, for the sake of completeness. In Section 3 we apply the discrete time result for appropriately defined measures and flows, thus demonstrating the exchange formula.

2 Discrete time Palm theory

The basic ingredients are a simple definition of a conditional probability and a formula for unraveling this conditioning. Consider a probability space $(\Omega, \mathcal{F}, Q)$ endowed with a Q-preserving measurable shift $\varphi : \Omega \to \Omega$ with measurable inverse. Let ξ be a 0-1 valued random variable with $Q(\xi = 1) > 0$ and let $\xi_n = \xi \circ \varphi^n$, $n \in \mathbf{Z}$ be the stationary process obtained by shifting ξ. A stationary point process can be defined by letting $\{S_j\ , j \in \mathbf{Z}\}$ be the set of times n such that $\xi_n = 1$. We adopt the ordering convention $\cdots < S_{-1} < S_0 \leq 0 < S_1 < \cdots$.

The Palm transformation Q^0 of Q with respect to the above point process is just Q conditional on the event that there is a point at zero:

$$Q^0(Y) := \frac{Q(Y, \xi)}{Q(\xi)}\ , \tag{1}$$

where Y is any bounded random variable. The reader should note that throughout this paper we use the same letter for a probability measure and the expectation with respect to it. Also note that $Q^0(S_0 = 0) = 1$. The following theorem asserts that the transformation $Q \mapsto Q^0$ can be inverted.

Theorem 1 (discrete time inversion formula) *With the above notation,*

$$Q(Y) = \frac{1}{Q^0(S_1)}\ Q^0 \sum_{S_0 \leq n < S_1} Y \circ \varphi^n\ . \tag{2}$$

Proof. Clearly, it suffices to prove (2) for a 0-1 valued random variable Y. Let $Y_n = Y \circ \varphi^n$, $n \in \mathbf{Z}$. We start by writing

$$Q^0 \left[\sum_{n=0}^{S_1 - 1} Y_n \right] = \frac{1}{Q(\xi)} Q \left[\sum_{n=0}^{S_1 - 1} Y_n,\ \xi \right] = \frac{1}{Q(\xi)} \sum_{n=0}^{\infty} Q[S_1 - 1 \geq n, Y_n, \xi]\ . \tag{3}$$

Let E_n be the event appearing in the last summation:

$$E_n = \{\xi_0 = 1, S_1 \geq n+1, Y_n = 1\} = \{\xi_0 = 1, \xi_1 = \cdots = \xi_{n-1} = 0, Y_n = 1\} \ .$$

On noting that $Q(E_n) = Q(\varphi^n E_n)$, by stationarity, that the events

$$\varphi^n E_n = \{\xi_{-n} = 1, \xi_{-n+1} = \cdots = \xi_{-1} = 0, Y_0 = 1\}$$

are disjoint, and that their union is the event

$$\{Y_0 = 1, \text{ and there is } n < 0 \text{ such that } \xi_n = 1\} \ ,$$

we conclude that (3) can be written as

$$Q^0 \left[\sum_{n=0}^{S_1 - 1} Y_n \right] = \frac{1}{Q(\xi)} \sum_{n=0}^{\infty} Q(\varphi^n E_n) = \frac{1}{Q(\xi)} Q(\cup_{n=0}^{\infty} \varphi^n E_n) = \frac{Q(Y)}{Q(\xi)} \ . \tag{4}$$

We used the fact the the event $\{$there is $n < 0$ such that $\xi_n = 1\}$ has Q-probability 1, due to stationarity and the assumption $Q(\xi = 1) > 0$. Setting $Y = 1$ in (4) we deduce that $Q^0(S_1) = 1/Q(\xi)$, thus establishing formula (2). □

3 The general case

In this section we apply the results of Section 2 in order to prove the general exchange formula (6). Let A, B be two jointly stationary point processes as in Section 1. Denote by $T_n^A, n \in \mathbf{Z}$ the points of A with the convention $\ldots < T_{-1}^A < T_0^A \leq 0 < T_1^A < \ldots$, and by λ_A the rate of A, assumed to be nonzero and finite. Similar notation and conventions are used for B and all point processes below. The definition of the Palm transformation $P \mapsto P^A$ is given via Mecke's formula for P^A:

$$P^A(Y) = \frac{1}{\lambda_A t} P \int_{[0,t)} Y \circ \theta_t A(dt) \ , \tag{5}$$

where Y is any bounded random variable as before. In what follows we use (5) and Theorem 1 in order to demonstrate the exchange formula for P^A and P^B:

Theorem 2 (exchange formula) *For any two point process A, B as above,*

$$\lambda_A P^A(Y) = \lambda_B P^B \int_{[T_0^B, \ T_1^B)} Y \circ \theta_t A(dt) \ . \tag{6}$$

Proof. Let C be the simple point process whose points are the union of points of A and the points of B. Let B' be the point process that contains the points of B that are *not* points of A.

We first use the fact that $B \le C$ (i.e., the points of B are points of C). We claim that the relation of P^B to P^C is the same as the relation of Q^0 to Q with $\varphi = \theta_{T_1^C - T_0^C}$ and $\xi = 1\{T_0^C \in B\}$, where $\{T_0^C \in B\}$ is the event that T_0^C is also a point of B (we use the same letter for a point process and the set of its points).

To prove the claim we need to check that (1) holds with $Q = P^C$ and $Q^0 = P^B$. The numerator of the right hand side of (1) is, directly from Mecke's formula (5),

$$P^C(Y,\xi) = \frac{1}{\lambda_C t} P \sum_{0 \le T_n^C < t} Y \circ \theta_{T_n^C} 1\{T_n^C \in B\} = \frac{1}{\lambda_C t} \sum_{0 \le T_m^B < t} Y \circ \theta_{T_m^B} = \frac{\lambda_B t}{\lambda_C t} P^B(Y) \ .$$

The denominator of the right hand side of (1) is

$$P^C(\xi) = \lambda_B / \lambda_C$$

(let $Y = 1$ in the previous display). Thus (1) holds and the claim is true. Hence the inversion formula (2) also holds. Noting that the set $\{S_0 \le n < S_1\}$ is the set $\{T_0^B \le T_n^C < T_1^B\}$ (by the definition of ξ_n as the indicator that the n-th point of C is also a point of B), the inversion formula (2) can be written as

$$P^C(Y) = \frac{1}{P^B(S_1)} P^B \sum_{S_0 \le n < S_1} Y \circ \varphi^n = \frac{\lambda_B}{\lambda_C} P^B \sum_{T_0^B \le T_n^C < T_1^B} Y \circ \theta_{T_n^C - T_0^C} \ .$$

Note that $T_0^C = 0$, P^B-a.s., and write this equation in integral form:

$$\lambda_C P^C(Y) = \lambda_B P^B \int_{[T_0^B, T_1^B)} Y \circ \theta_t C(dt) \ . \tag{7}$$

But this is just the exchange formula between the processes B and C. So if one process is dominated by another the exchange formula is a direct consequence of the discrete inversion (see also Franken et al [3]).

The general case follows just as easily. Recall that (the set of points of) C is the disjoint union of A and B'. Write the left hand side of (7) as

$$\lambda_C P^C(Y) = \lambda_A P^A(Y) + \lambda_{B'} P^{B'}(Y) \ . \tag{8}$$

This follows from Mecke's formula: use eq. (5) for P^C, write the integral with respect to C as an integral with respect to A and an integral with respect to B' and re-use the same formula for P^A and $P^{B'}$. On the other hand, the right hand side of (7) can, for the same reason, be written as

$$\begin{aligned} \lambda_B P^B \int_{[T_0^B, T_1^B)} Y \circ \theta_t C(dt) &= \lambda_B P^B \int_{[T_0^B, T_1^B)} Y \circ \theta_t A(dt) + \lambda_B P^B \int_{[T_0^B, T_1^B)} Y \circ \theta_t B'(dt) \\ &= \lambda_B P^B \int_{[T_0^B, T_1^B)} Y \circ \theta_t A(dt) + \lambda_B P^B(Y) \ . \end{aligned} \tag{9}$$

Indeed, only the point $t = T_0^B$ can contribute to the integral with respect to to B'. We claim that the last terms of (8) and (9) are equal. The reason is this: since $B' \le B$ (points of B' are also points of B) the relation between $P^{B'}$ and P^B is the same as the relation between Q^0 and Q with $\varphi = \theta_{T_1^B - T_0^B}$ and $\xi = 1\{T_0^B \in B'\}$, just like in the first paragraph. Thus the direct relation (1) between Q^0 and Q yields the equality $\lambda_{B'} P^{B'}(Y) = \lambda_B P^B(Y)$. Using this and (7), (8), (9), we conclude that

$$\lambda_A P^A(Y) = \lambda_B P^B \int_{[T_0^B, T_1^B)} Y \circ \theta_t A(dt) \ ,$$

which is the exchange formula (6) between the general point processes A and B. □

To recapitulate, we used the discrete time inversion formula (2) for the case $B \leq C$ and the direct conditioning (1) for the case $B' \leq B$.

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References

[1] Brémaud, P. (1991). An elementary proof of Sengupta's invariance relation and a remark on Miyazawa's conservation principle. *J. Appl. Prob.*, **28**, No. 4, 950-954.

[2] Brémaud, P. (1993). A Swiss army formula for Palm calculus. *J. Appl. Prob.*, **30**, 40-51.

[3] Franken, P., König, D., Arndt, U. and Schmidt, V. (1981). *Queues and Point Processes.* Wiley, New York.

[4] Franken, P., and B. Lisek (1982). On Wald's identity for dependent variables. *Zeit. für Wahrsch.*, **60**, 143-150.

[5] Kac, M. (1947). On the notion of recurrence in discrete stochastic processes. *Bull. AMS,* **53,** 1002-1010.

[6] Konstantopoulos, T. and M. Zazanis, (1992). Sensitivity analysis for stationary and ergodic queues. *Adv. Appl. Prob.*, **24**, 738-750.

[7] Neveu, J. (1976). Sur les mesures de Palm de deux processus ponctuels stationnaires. *Zeit. für Wahrsch.*, **34**, 199-203.

[8] Neveu, J. (1976). Processus ponctuels. In: *Ecole d'été de Probabilités de St. Flour VI, Lect. Notes Math.,* **598**, 249-447, Springer-Verlag.