The distribution of age-of-information performance measures for message processing systems

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Abstract

The idea behind the recently introduced "age of information" performance measure of a networked message processing system is that it indicates our knowledge regarding the "freshness" of the most recent piece of information that can be used as a criterion for real-time control. In this paper, we examine two such measures, one that has been extensively studied in the recent literature and a new one that could be more relevant from the point of view of the processor. Considering these measures as stochastic processes in a stationary environment (defined by the arrival processes, message processing times and admission controls in bufferless systems) we characterize their distributions using the Palm inversion formula. Under renewal assumptions we derive explicit solutions for their Laplace transforms and show some interesting decomposition properties. Previous work has mostly focused on computation of expectations. We conclude with a discussion of future work, including assessment of enqueueing policies that may have smaller age of information in some cases.

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AMS 2010 *subject classification.* Primary 60G55,60K05; secondary 60G50,60K30

1 Introduction

The Internet is now commonly used to transmit latency-sensitive information that is part of a real-time control or decision process. As an example, consider a temperature or pressure sensor which could periodically transmit a reading to a latency-critical remote control. Other examples include decision systems for an airplane, driverless vehicles, financial transactions, power systems, sensor/actuator systems or other "cyber physical" systems. In the power system case, a high temperature reading of a transmission line could indicate reduced capacity or predict near-term failure. In the sensor system example, the sensor could indicate an alarm such as a motion detector which needs to be manually reset once tripped; any alarm message would render stale any queued or in-transmission "heartbeat" message that is periodically sent to indicate no intruder is present and that the sensor is properly functioning. In the actuator system case, messages may embody commands to a remote actuator of a time-critical control system.

Systems such as the ones described above naturally depend on the *age* of the most recently received reading from a remote sensor. This is a quantity that takes into account the time since the reading was generated. In view of the speeds involved a decision must be taken upon arrival of a new information packet: to read or not read it. The choice is crucial and depends on the packet length and the frequency of information packet arrivals, quantities that may not be completely known. If current time is denoted by t and if the arrival time of the most recent completely read message before t is denoted by A_t^* then the quantity $\alpha(t) := t - A_t^*$ has been introduced in the literature and has been termed *"age of information*

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(AoI)". This has been introduced as a measure of freshness and its expectation has been studied in, e.g., [8, 6, 11, 2, 9]. From a performance point of view, we are interested not only in its expectation but also in its probability distribution. We derive results about the latter in this paper.

One can argue that the above measure may have limited usefulness for applications that cannot control the arrivals of messages. And thus, one may assert that the freshness of information should be gauged not against the current time t but against the last arrival time A_t of a message before t. By definition, $A_t^* \leq A_t$ with equality if and only if the message arriving at A_t is completely read. We thus introduce the measure $\beta(t) := A_t - A_t^*$. Since there is no terminology for this quantity, we are free to choose one: we call it *"new age of information (NAoI)"*. Notice that $\beta(t) \leq \alpha(t)$ provided that the same acceptance policy is used on both sides. But, as seen in the paper, there could be a class of policies such that the supremum of the left-hand side over this class is not smaller than the infimum of the right-hand side over the same class. To better understand the difference between the two quantities, consider the case where messages arrive randomly (according to, say, a Poisson process) but have very small duration. Then it is unlikely that a message will arrive while another is being read. Notice then that $\beta(t)$ is most of the time zero, indicating that the information possessed by the server is most of the time fresh. Upon arrival of a message however, β is set to the interarrival time between the current and the previous message. This indicates that the information is old. On the other hand, $\alpha(t)$ increases linearly between two messages and this is due to the age $t - A_t$ of the arrival process. From a system point of view, one may wish to keep α low. However, from a server point of view, one may wish to keep β low.

We consider $\alpha(t)$ and $\beta(t)$ as (random) functions of time t and are interested in their steady-state characteristics. The simplest such characteristic is the expectation. However, the expectation of α may be arbitrarily large (potentially infinite) if, say, the arrival process is renewal with large-variance interarrival times. Another characteristic that we may wish to keep low is the complemetary probability $\mathbb{P}(\alpha(t) > u)$ for some $u > 0$, and, similarly, $\mathbb{P}(\beta(t) > u)$.

We study policies in two extreme cases: the fully push-out policy (every new message immediately obsoletes the current one, if any) and the fully blocking policy (the system ignores all messages arriving during the time that a message is being read). We do so in order to obtain concrete formulas and explain the methods. However, in principle, our methods will work on any policy. In addition, we work only with buffereless systems (but see the last section for a discussion). The reason we do so is that, from the point of view of keeping any of the two age of information measures small it makes no sense to store more than one message. To see why, suppose a new (freshest) message arrives to a system having queued messages together with a message that is currently being read. Processing any of the queued messages will simply delay that of the freshest message thus increasing the age of information.

The paper is organized as follows. In Section 2 we present the setup and the definition of the models and all relevant stochastic processes. Section 3 is a brief outline of some of the results. Formulas for distributions and moments of both the AoI and the NAoI for the fully push-out system are derived in Section 4. This is done by carefully applying Palm theory, first in a stationary context and then by specializing to the case involving independence assumptions. The stronger the assumptions, the more explicit the results. For the queueing theorist, it is not a surprise that the formulas become quite explicit when the arrival process is Poisson. Similarly pleasing and explicit is the case when the message lengths are independent exponentially distributed random variables. If both Poissonian assumptions hold then we are in the best of all worlds. The NAoI is the subject of Section 5. The action plan is the same as in the fully push-out system case, but, here, all calculations are more involved. This is due to the fact that the fully blocking system has more complicated dynamics than the push-out system. Nevertheless, closed-form formulas are also possible. In Section 6, we discuss variations of the AoI problem to be considered in future work; in particular, we discuss other enqueueing policies that may have smaller Age of Information in some cases. Last but not least, one might wonder why we only study bufferless systems. The reason is that storing any message at all will not improve the age of information. This is heuristically true but also supported by numerical simulations presented in the last section.

2 System definitions

The goal of this section is to define the two measures of the age of information for a general bufferless processing system. We are careful to include the possibility that some of the quantities below may be restricted on a lattice. We first define such a system, allowing the possibility to accept or reject messages. We then give the definitions of the age of information measures as functions of time. Lastly, we introduce stochastic assumptions which make the age of information processes random functions of time. Some notation/terminology used throughout is as follows: The set of integers is denoted by $\mathbb{Z}$. The indicator function of a set A is denoted by $\mathbf{1}_A$. The notation $\mathbb{E}[X; A]$ stands for $\mathbb{E}[X\mathbf{1}_A]$. If S is a set and $s \in S$, then δ_s denotes the Dirac measure $\delta_s(B) = \mathbf{1}_{s\in B}$, $B \subset S$. By point measure on $\mathbb{R}$ (or $\mathbb{R}^2$) we mean a measure assuming nonnegative integer values; necessarily, it is a finite or countable sum of Dirac measures. A point process is a random point measure. If X is a positive random variable with finite expectation, we say that $\overline{X}$ is the stationary version of X if it has density $\mathbb{P}(X > x)/\mathbb{E}X$:

$$\mathbb{P}(\overline{X} \in dx) = \frac{\mathbb{P}(X > x)}{\mathbb{E}X} dx.$$

We then have

$$\mathbb{E}e^{-u\overline{X}} = \frac{1 - \mathbb{E}e^{-uX}}{u\mathbb{E}X}, \quad \mathbb{E}\overline{X} = \frac{\mathbb{E}X^2}{2\mathbb{E}X}.$$

When X and Y are random variables (on, possibly, different probability spaces) $X \overset{(d)}{=} Y$ denotes equality of their laws (distributions). The symbol $\widetilde{\mathbb{P}}$ denotes the probability governing a time-stationary system, whereas $\mathbb{P}$ denots the Palm probability of $\widetilde{\mathbb{P}}$ with respect to the arrival process. (We choose this unconventional notation because the former symbol is used less frequently than the latter.)

2.1 Bufferless message processing systems

Messages arrive in a bufferless server which can read one message at a time. Denote by T_n, $n \in \mathbb{Z}$, the message arrival times. We assume that

$$T_n < T_{n+1},\ n \in \mathbb{Z}, \qquad \sup_{n\in\mathbb{Z}} T_n = +\infty, \qquad \inf_{n\in\mathbb{Z}} T_n = -\infty.$$

We shall fix an ordering by letting T_0 be such that $T_0 \le 0 < T_1$. We denote by

$$\mathfrak{a} := \sum_{n\in\mathbb{Z}} \delta_{T_n}$$

the arrival process, considered as a point measure. We shall also let, for all $n \in \mathbb{Z}$,

$$\tau_n := T_{n+1} - T_n.$$

We introduce, for each $n \in \mathbb{Z}$, the *acceptance index* χ_n, setting

$$\chi_n = \begin{cases} 1, & \text{if the message arriving at } T_n \text{ is accepted} \\ 0, & \text{otherwise.} \end{cases}$$

The χ_n is a decision variable that depends on the *acceptance policy*. See below for some example. In this paper we shall only consider specific policies leaving optimization/control problems for future work. The length of message n (the message arriving at time T_n) is denoted by σ_n and its departure time by T'_n. The latter given by

$$T'_n := \begin{cases} T_n, & \text{if } \chi_n = 0 \\ (T_n + \sigma_n) \wedge \inf\{T_r : r > n, \chi_r = 1\}, & \text{if } \chi_n = 1 \end{cases}. \tag{1}$$

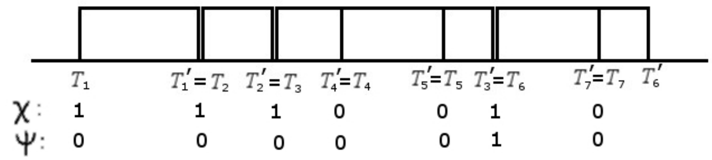

Figure 1: *A message arrives at time T_1 at an idle server and is immediately accepted. A double line indicates that a message pushes out the previous one, while a single line indicates that the message is blocked. Thus, messages 1, 2, 3 and 6 are accepted, while 4, 5 and 7 are rejected. Only message 6 is successful. The server started reading message 1 at time T_1 and finishes reading message 6 in its entirety at time $T_6' = T_6 + \sigma_6$.*

This means that an arriving message will either be immediately rejected (and thus depart immediatly) or accepted, in which case it will either be read in its entirety or pushed out by another accepted message. Note that the sets $\{T_n, n \in \mathbb{Z}\}$ and $\{T_n', n \in \mathbb{Z}\}$ may have common elements (e.g., if we allow all variables take vales that are integer multiples of a common unit). It is easy to see from (1) that the intervals $[T_n, T_n')$ and $[T_m, T_m')$ are disjoint if $m \neq n$. Thus, for all t, the quantity

$$q(t) := \sum_{n \in \mathbb{Z}} \chi_n \, \mathbf{1}_{T_n \le t < T_n'} \tag{2}$$

is either 0 or 1. The $q(t)$ is the state of the server at time t: $q(t) = 1$ if the server is busy or 0 if not. Notice that $q(\cdot)$ is right-continuous (by choice rather than by necessity).

We call message n *successful* if it departs immediately after having being read in its entirety. The *success index* is the binary variable

$$\psi_n := \mathbf{1}_{T_n' = T_n + \sigma_n}. \tag{3}$$

By definition, for all n,

$$\psi_n \le \chi_n.$$

See Figure 1 for an illustrative example of an arbitrary policy.

Consider $n \in \mathbb{Z}$ and the statement

$$\mathcal{Z}_n := \text{“}q(T_n-) = 0 \text{ or } T_m' = T_n \text{ for some } m < n\text{”} \tag{4}$$

which expresses the event that the server is idle at the arrival time T_n either because it was idle on some interval $(T_n - \varepsilon, T_n)$ or because a message just departed at time T_n. We shall throughout assume that the *non-idling condition*

$$\text{for all } n \in \mathbb{Z} \text{ if } \mathcal{Z}_n \text{ then } \chi_n = 1 \tag{NI}$$

holds. For those n for which $\mathcal{Z}_n$ is violated the determination of χ_n is a matter of the acceptance policy.

Here are four examples of acceptance policies. Let ℓ be a nonnegative integer.

Example 1. The *fully push-out* ($\mathcal{P}$) policy. All messages are accepted:

$$\chi_n = 1, \quad n \in \mathbb{Z}.$$

From (1) and (3) it is easy to see that

$$\psi_n = \mathbf{1}_{T_n + \sigma_n \le T_{n+1}} = \mathbf{1}_{\tau_n \ge \sigma_n}, \quad n \in \mathbb{Z}.$$

Example 2. The *fully blocking* ($\mathcal{B}$) policy. No message other than those satisfying the non-idling condition (NI) are accepted:

$$\chi_n = 1 \iff \mathcal{Z}_n \text{ holds.}$$

Note that, here, $\psi_n = \chi_n$ for all n, that is, every accepted message is successful.

Example 3. The $\mathcal{BP}(\ell)$ policy. If T_n is the arrival of a message at an idle server, and if there are at most ℓ arrivals on $(T_n, T_n + \sigma_n]$ then reject them all; otherwise, reject the first ℓ of them and accept every arrival until the next time that a message arrives at an idle server.

Example 4. The $\mathcal{PB}(\ell)$ policy. If T_n is the arrival of a message at an idle server, and if there are at most ℓ arrivals on $(T_n, T_n + \sigma_n]$ then accept them all; otherwise, accept the first ℓ of them and reject every arrival until the next time that a message arrives at an idle server.

We shall only study the first two policies in this paper, leaving the study of the others, as well as optimal policies, for future work.

2.2 Age of information processes

To define the age of information functions (of time) we need to introduce the following functions on $\mathbb{R}$. The *last arrival* before $t \in \mathbb{R}$ is defined by

$$A_t := \sup\{T_n : n \in \mathbb{Z}, T_n \le t\}.$$

The *last successful arrival* before t is defined by

$$S_t := \sup\{T_n : n \in \mathbb{Z}, T_n \le t, \psi_n = 1\};$$

The *last successful departure* before t is defined by

$$D_t := \sup\{T_n + \sigma_n : n \in \mathbb{Z}, T_n + \sigma_n \le t, \psi_n = 1\}.$$

Note that, under our assumptions on the sequence T_n, the sup in the definition of A_t is actually a max. Assuming further that

$$\inf\{n : \psi_n = 1\} = -\infty \tag{A1}$$

we have that the sup in S_t and D_t is replaced by a max. If, in addition,

$$\sup\{n : \psi_n = 1\} = \infty \tag{A2}$$

then $S_t, D_t < \infty$ for all t.

Definition 1. *Under assumptions* (A1) *and* (A2), *the* age of information (AoI) *function is defined by*

$$\alpha(t) := t - S_{D_t}, \quad t \in \mathbb{R}, \tag{5}$$

and the new age of information (NAoI) function *is defined by*

$$\beta(t) := A_t - S_{D_t}, \quad t \in \mathbb{R}. \tag{6}$$

Note that the functions A, S, D above are right-continuous and increasing ($s < t \Rightarrow A_s \le A_t, S_s \le S_t, D_s \le D_t$). It follows that α and β are also right-continuous. Moreover,

$$\Delta\alpha(t) := \alpha(t) - \alpha(t-) = -\Delta S_{D_t} = -\lim_{\varepsilon \downarrow 0}(S_{D_t} - S_{D_{t-\varepsilon}}) \le 0.$$

So jumps of α can only be negative. Notice that

$$\Delta\alpha(t) = S_{D_t} - S_{(D_{t-})-}.$$

On the other hand, β can have both positive and negative jumps.

We shall also use the following notations and terminology. Consider the arrival times T_n of messages arriving at a idle server:

$$\{B_k : k \in \mathbb{Z}\} := \{T_n : \mathscr{Z}_n \text{ holds}\}.$$

By convention, we enumerate these points as

$$\cdots < B_{-1} < B_0 \le 0 < B_1 < \cdots$$

They form the beginnings of reading intervals. An interval with endpoints B_k and B_{k+1} will be referred to as *cycle*. Define also

$$\{B'_k : k \in \mathbb{Z}\} := \{T_n + \sigma_n : n \in \mathbb{Z}, \psi_n = 1\}$$

and again assume that

$$\cdots < B'_{-1} < B'_0 \le 0 < B'_1 < \cdots$$

These are the ends of reading intervals. The two sequences, $\{B_k\}$ and $\{B'_k\}$, are interlaced: between two successive elements of one sequence there is exactly one element of the other. See Figure 2. An interval

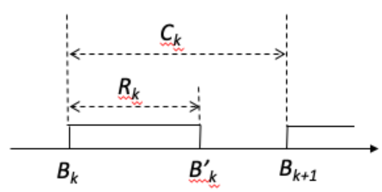

Figure 2: *The interval $[B_k, B_{k+1})$ is a cycle and the subinterval $[B_k, B'_k)$ is a reading interval.*

with endpoints B_k and B_{k+1} is called a *cycle*. We set

$$\mathbf{C}_k := B_{k+1} - B_k$$

for the cycle length. The subinterval with endpoints B_k and B'_k is called a *reading interval*. We set

$$\mathbf{R}_k := B'_k - B_k$$

for the reading length.

2.3 The stationary framework

Let $(\Omega, \mathscr{F}, \widetilde{\mathbb{P}})$ be a probability space endowed with a flow, i.e., a family of invertible measurable functions $\theta_t : \Omega \to \Omega$, $t \in \mathbb{R}$, such that θ_t^{-1} are also measurable and such that

$$\theta_{t+s} = \theta_t \circ \theta_s, \quad s, t \in \mathbb{R}. \tag{7}$$

Assume further that the flow preserves $\widetilde{\mathbb{P}}$, that is,

$$\widetilde{\mathbb{P}} \circ \theta_t = \widetilde{\mathbb{P}}, \quad t \in \mathbb{R}.$$

Let T_n, σ_n be random variables such that the marked[1] point process $\sum_n \delta_{(T_n, \sigma_n)}$ is stationary, that is,

$$\Big(\sum_n \delta_{(T_n,\sigma_n)}\Big) \circ \theta_t = \sum_n \delta_{(T_n - t, \sigma_n)}, \quad t \in \mathbb{R}. \tag{8}$$

[1]A point process φ on a product space $S \times M$ is called M-marked (or just marked) if $\varphi(\{s\} \times M) \in \{0, 1\}$ for all $s \in S$.

Note then that

$$A_t \circ \theta_s = A_{t+s} - s, \quad s, t \in \mathbb{R}.$$

It follows that the arrival rate

$$\lambda := \widetilde{\mathbb{E}} \sum_n \mathbf{1}_{0 \le T_n \le 1}$$

is positive and finite. Consider next a acceptance policy as specified by the acceptance random variables χ_n, $n \in \mathbb{Z}$, defined on $(\Omega, \mathscr{F})$. We say that the system is in *steady-state* if, in addition to (8),

$$\Big(\sum_n \delta_{(T_n, \sigma_n, \chi_n)} \Big) \circ \theta_t = \sum_n \delta_{(T_n - t, \sigma_n, \chi_n)}, \quad t \in \mathbb{R}. \tag{9}$$

If the system is in steady-state then it follows from (9), (7) (3) and (1) that

$$\Big(\sum_n \delta_{(T_n, \sigma_n, \chi_n, \psi_n)} \Big) \circ \theta_t = \sum_n \delta_{(T_n - t, \sigma_n, \chi_n, \psi_n)}, \quad t \in \mathbb{R}, \tag{10}$$

and, for all $s, t \in \mathbb{R}$,

$$S_t \circ \theta_s = S_{t+s} - s, \quad D_t \circ \theta_s = D_{t+s} - s,$$
$$\alpha(s) \circ \theta_t = \alpha(t+s), \quad \beta(s) \circ \theta_t = \beta(t+s), \quad q(s) \circ \theta_t = q(t+s).$$

In general, it is not obvious that (9) holds. Of the four acceptance policies mentioned above, the push-out $\mathcal{P}$ immediately satisfies (9) owing to that $\chi_n = 1$ and $\psi_n = \mathbf{1}_{T_{n+1} - T_n \ge \sigma_n}$ for all n. For the fully blocking $\mathcal{B}$ policy note that the system is identical to the so called $G/G/1/1$ queue. That (9) holds is proved in [1, Section 5.3] and may require enlarging the probability space $(\Omega, \mathscr{F}, \widetilde{\mathbb{P}})$.

Definition 2. *We shall denote by $\mathbb{P}$ the Palm probability of $\widetilde{\mathbb{P}}$ with respect to the point process $\mathfrak{a} = \sum_{n \in \mathbb{Z}} \delta_{T_n}$. If (9) holds we shall denote by $\mathbb{P}^*$ the Palm probabilityof $\widetilde{\mathbb{P}}$ with respect to the point process $\sum_{k \in \mathbb{Z}} \delta_{B_k}$.*

For the notion of Palm probability see, e.g., Daley and Vere-Jones [4, Chapter 13], Kallenberg [7] and Baccelli and Brémaud [1]. Formally, with $\mathscr{B}$ denoting the class of Borel sets on $\mathbb{R}$, the measure $\mathscr{B} \ni C \mapsto \widetilde{\mathbb{E}}(\mathbf{1}_A \sum_n \mathbf{1}_{T_n \in C})$ is absolutely continuous, and hence differentiable, with respect to the measure $\mathscr{B} \ni C \mapsto \widetilde{\mathbb{E}}(\sum_n \mathbf{1}_{T_n \in C})$. The value of the derivative at 0 is precisely $\mathbb{P}(A)$. The Palm probability $\mathbb{P}^*(A)$ can be obtained in exactly the same manner. However, since $\{B_k\}$ is precisely the set of T_n for which $\mathcal{Z}_n$ holds, it follows that $\mathbb{P}^*$ is obtained from $\mathbb{P}$ via elementary conditioning:

$$\mathbb{P}^* = \mathbb{P}(\cdot | \mathcal{Z}_0 = 1).$$

Integrals with respect to $\mathbb{P}$, $\mathbb{P}^*$ and $\widetilde{\mathbb{P}}$ are denoted by $\mathbb{E}$, $\mathbb{E}^*$ and $\widetilde{\mathbb{E}}$ respectively. Moreover, $\mathbb{P}(T_0 = 0) = 1$ and $\mathbb{P}^*(B_0 = T_0 = 0) = 1$. We denote by θ_{T_n} the map defined by $\theta_{T_n}(\omega) = \theta_{T_n(\omega)}(\omega)$. Then θ_{T_n}, $n \in \mathbb{Z}$, forms a discrete time flow that preserves $\mathbb{P}$. In other words, $\mathbb{P}$-a.s., $\theta_{T_n} \circ \theta_{T_m} = \theta_{T_{n+m}}$ for all $m, n \in \mathbb{Z}$ and $\mathbb{P} \circ \theta_{T_n} = \mathbb{P}$ for all $n \in \mathbb{Z}$. Similarly, $\mathbb{P}^*$-a.s., $\theta_{B_k} \circ \theta_{B_\ell} = \theta_{B_{k+\ell}}$ for all $k, \ell \in \mathbb{Z}$ and $\mathbb{P}^* \circ \theta_{B_k} = \mathbb{P}^*$ for all $k \in \mathbb{Z}$.

The $\mathbb{P}$-law of (τ_n, σ_n) does not depend on n. In what follows, we let (τ, σ) be a generic random element whose law is the same as the $\mathbb{P}$-law of (τ_0, σ_0). The definition of Palm probability and the fact $\lambda > 0$ implies that

$$\mathbb{E}\tau = 1/\lambda < \infty.$$

This is the minimal condition imposed by stationarity and thus it cannot be avoided. It is important to note however that we shall make no assumptions about finiteness of higher $\mathbb{P}$-moments of τ.

Referring to Figure 2, note that, under $\mathbb{P}^*$, all cycles have identical law and so do all reading intervals. We denote by $\mathbf{C}$ a *typical cycle length*, that is, a random variable whose law is the $\mathbb{P}^*$-law of the length any cycle. Similarly, $\mathbf{R}$ denotes a *typical reading interval length.*

3 Outline of some of the results

All results concern stationary processes. Denote by $\alpha_{\mathcal{P}}$, $\alpha_{\mathcal{B}}$ the AoI processes for the fully push-out and fully blocking systems, respectively. Similarly, we let $\beta_{\mathcal{P}}$, $\beta_{\mathcal{B}}$ be the NAoI processes for the two systems.

Under stationary assumptions only, the main results for $\alpha_{\mathcal{P}}$ and $\alpha_{\mathcal{B}}$ are Theorems 1 and 5, respectively. They give relations for the marginal distributions of the two processes that can be solved provided further assumptions are made. In particular, under i.i.d. assumptions, we find that (see Theorem 2), in steady-state,

$$\alpha_{\mathcal{P}}(t) \overset{(\mathrm{d})}{=} \overline{\tau} + \mathbf{R}_{\mathcal{P}},$$

where $\overline{\tau}$ is a random variable whose law is the law of the stationary version of the interarrival time and $\mathbf{R}_{\mathcal{P}}$ is an independent random variable distributed as the typical reading interval of the fully push-out system. We also find that (Theorem 6), in steady-state,

$$\alpha_{\mathcal{B}}(t) \overset{(\mathrm{d})}{=} \sigma + \overline{\mathbf{C}}_{\mathcal{B}},$$

where σ is a random variable distributed as the typical message length and $\overline{\mathbf{C}}_{\mathcal{B}}$ is an independent random variable distributed as the stationary version of the typical cycle.

Under stationary assumptions only, the main results for processes $\beta_{\mathcal{P}}$ and $\beta_{\mathcal{B}}$ are Theorems 3 and 7 respectively. Regarding $\beta_{\mathcal{P}}$, $\beta_{\mathcal{B}}$, under i.i.d. assumptions, we find that they have atoms at 0 and that (see Theorem 4)

$$(\beta_{\mathcal{P}}(t) | \beta_{\mathcal{P}}(t) > 0) \overset{(\mathrm{d})}{=} \mathbf{C}_{\mathcal{P}},$$

where $\mathbf{C}_{\mathcal{P}}$ is distributed as the typical cycle. Under stationarity assumptions only, we find (Theorem 7 and Remark 4) that

$$\beta_{\mathcal{B}}(t) \mathbf{1}_{\beta_{\mathcal{B}}(t)>0} \overset{(\mathrm{d})}{=} \beta_{+}(t),$$

where $\beta_{+}(t)$ is the NAoI process for an appropriately defined variant of the fully-blocking system: remove from the system all undisturbed messages, that is, all messages that arrive at an idle system and are such that no other messages arrice while they are being processed. Specializing to iid assumptions, we find (Theorem 8) that $(\beta_{\mathcal{B}}(t) | \beta_{\mathcal{B}}(t) > 0)$ has density and Laplace transforms depending on functions that satisfy renewal equations. The dependence on these renewal functions is complicated but quite explicit: see equation (66).

Further assuming that one of the variables τ, σ is exponential results into explicit formulas both for Laplace transforms and expectations. These results are expressed as corollaries following each of the theorems.

4 The fully push-out system

The dynamics of the push-out system is quite simple: every arriving message is admitted: $\chi_n = 1$ for all $n \in \mathbb{Z}$. The message arriving at T_n is successful if and only of $T_n + \sigma_n \le T_{n+1}$. Thus

$$\psi_n = \mathbf{1}_{\tau_n \ge \sigma_n}, \quad n \in \mathbb{Z}.$$

Since, for all n, $\chi_n = 1$ and $\psi_n = \mathbf{1}_{\tau_n \ge \sigma_n}$, it follows from (1) that the state process q of (2) is alternatively given by

$$q(t) = \begin{cases} 0, & T_n + \sigma_n \le t < T_{n+1} \quad \text{for some } n \\ 1, & \text{otherwise} \end{cases}.$$

If $\mathbb{P}(\tau_0 < \sigma_0) = 1$ then $\mathbb{P}(\tau_n < \sigma_n \text{ for all } n) = 1$ and so q is identically equal to 1. This is an uninteresting case resulting in infinite AoI and NAoI. We thus assume that

$$\mathbb{P}(\tau_0 \ge \sigma_0) > 0. \tag{11}$$

By the Poincaré recurrence theorem [5, Theorem 7.3.4], $\inf\{n : \psi_n = 1\} = -\infty$, $\sup\{n : \psi_n = 1\} = +\infty$, $\mathbb{P}$-a.s., and hence $\widetilde{\mathbb{P}}$-a.s. This implies that α, β are well-defined and finitely-valued processes.

It is easy to see that, for the fully push-out system, the beginnings of cycles satisfy

$$\{B_k : k \in \mathbb{Z}\} = \{T_n : n \in \mathbb{Z}, \psi_{n-1} = 1\}.$$

We therefore have:

Lemma 1. *The Palm probability $\mathbb{P}^*$ of Definition 2 is the Palm probability of $\widetilde{\mathbb{P}}$ with respect to the (stationary) point process*

$$\sum_{n \in \mathbb{Z}} \psi_{n-1} \delta_{T_n}$$

and

$$\mathbb{P}^* = \mathbb{P}(\cdot | \psi_{-1} = 1) = \mathbb{P}(\cdot | \tau_0 \geq \sigma_0). \tag{12}$$

In particular,

$$B_1 = \inf\{T_n : n \in \mathbb{Z}, T_n > 0, \psi_{n-1} = 1\}, \quad B_0 = \sup\{T_n : n \in \mathbb{Z}, T_n \leq 0, \psi_{n-1} = 1\}. \tag{13}$$

4.1 The age of information for the fully push-out system

To compute the law of $\alpha(0)$ we shall use the Palm inversion formula

$$\widetilde{\mathbb{E}} f(\alpha(0)) = \frac{\mathbb{E}^* \int_{B_0}^{B_1} f(\alpha(t))\, dt}{\mathbb{E}^*(B_1 - B_0)}, \tag{14}$$

where $f : \mathbb{R} \to \mathbb{R}$ is bounded and measurable or of constant sign and measurable. The denominator is easy to compute:

$$\mathbb{E}^*(B_1 - B_0) = \left(\widetilde{\mathbb{E}} \sum_n \psi_{n-1} \mathbf{1}_{0 < T_n < 1} \right)^{-1} = \left(\lambda \mathbb{E} \int_{\mathbb{R}} \psi_{-1} \mathbf{1}_{0<t<1}\, dt \right)^{-1} = \frac{1}{\lambda \mathbb{P}(\tau_0 \geq \sigma_0)}, \tag{15}$$

where we used Campbell's formula. By the non-triviality assumption (11), $\mathbb{E}^*(B_1 - B_0) < \infty$.

Theorem 1. *Consider the fully push-out system under stationarity assumptions and assume that (11) holds. Let $F : \mathbb{R}_+ \to \mathbb{R}$ be a bounded absolutely continuous function with a.e. derivative F'. Then*

$$\widetilde{\mathbb{E}} F'(\alpha(0)) = \lambda \mathbb{E} \left[F\left(\tau_{-1} + \sum_{i=0}^{N-1} \tau_i + \sigma_N \right) - F(\sigma_N);\, \tau_{-1} > \sigma_{-1} \right], \tag{16}$$

where $N := \inf\{\ell \geq 0 : \tau_\ell \geq \sigma_\ell\} = \min\{\ell \geq 0 : \tau_\ell \geq \sigma_\ell\}$.

Proof. We have $N < \infty$ because of stationarity and hence the expression in the brackets of (16) makes sense. Message N is successful ($\psi_N = 1$) and, by the first of (13),

$$B_1 = T_{N+1}, \quad \mathbb{P}^*\text{-a.s.}$$

To compute the integral in the numerator of (14) we take a close look at the function α restricted on the interval $[B_0, B_1)$. If $B_0 \leq t < T_N + \sigma_N$ then $D_t = B_{-1}$ and $S_{D_t} = T_{-1}$. If $T_N + \sigma_N \leq t < T_{N+1}$ then $D_t = T_N$ and $S_{D_t} = T_0$. Thus,

$$\alpha(t) = \begin{cases} t - T_{-1}, & T_0 \leq t < T_N + \sigma_N \\ t - T_N, & T_N + \sigma_N \leq t < T_{N+1} \end{cases}, \quad \mathbb{P}^*\text{-a.s.}$$

Then, $\mathbb{P}^*$-a.s.,

$$\int_{B_0}^{B_1} f(\alpha(t))\,dt = \int_{T_0}^{T_{N+1}} f(\alpha(t))\,dt = \int_{T_0}^{T_N+\sigma_N} f(t-T_{-1})\,dt + \int_{T_N+\sigma_N}^{T_{N+1}} f(t-T_N)\,dt$$
$$= F(T_N+\sigma_N-T_{-1}) - F(T_0-T_{-1}) + F(T_{N+1}-T_N) - F(\sigma_N),$$

and thus, since $\mathbb{E}^*F(T_0-T_{-1}) = \mathbb{E}^*F(T_{N+1}-T_N)$,

$$\mathbb{E}^* \int_{B_0}^{B_1} f(\alpha(t))\,dt = \mathbb{E}^*\left[F\left(\tau_{-1} + \sum_{i=0}^{N-1}\tau_i + \sigma_N\right) - F(\sigma_N)\right].$$

We can rewrite (15) as $\mathbb{E}^*(B_1-B_0) = 1/\lambda\mathbb{P}(\tau_{-1} \geq \sigma_{-1})$. Dividing the last display by this expression and using the relation (12) between $\mathbb{P}^*$ and $\mathbb{P}$ we arrive at (16). □

At this level of generality it is not possible to have a more explicit formula. However, given information about the law of the sequence (τ_n, σ_n), $n \in \mathbb{Z}$, we can proceed further. For example, assuming that the τ_n, $n \in \mathbb{Z}$, is independent of σ_n, $n \in \mathbb{Z}$, and both sequences have known laws then a further simplification is possible. If, in addition, the $\mathbb{P}$-law of one of the sequences is that of i.i.d. exponential random variables then it is possible to elaborate further and derive an almost closed-form formula.

Theorem 2. *Consider the fully push-out system and assume that (τ_n, σ_n), $n \in \mathbb{Z}$, is i.i.d. under $\mathbb{P}$ and such that $\mathbb{E}\tau_0 < \infty$ and $\mathbb{P}(\tau_0 \geq \sigma_0) > 0$. Assume further that τ_n is independent of σ_n for all n. Then, for $u > 0$,*

$$\widetilde{\mathbb{E}}e^{-u\alpha(0)} = \frac{1-\mathbb{E}e^{-u\tau}}{u\mathbb{E}\tau}\,\frac{\mathbb{E}[e^{-u\sigma};\tau \geq \sigma]}{1-\mathbb{E}[e^{-u\tau};\tau < \sigma]} \tag{17}$$

In particular, under $\widetilde{\mathbb{P}}$, $\alpha(0)$ is the sum of two independent random variables:

$$\alpha(0) \overset{(\mathrm{d})}{=} \overline{\tau} + \mathbf{R}, \tag{18}$$

where $\overline{\tau}$ is the stationary version of τ and $\mathbf{R}$ is a typical reading interval length.

Corollary 1. *The $\widetilde{\mathbb{P}}$-distribution of $\alpha(0)$ is absolutely continuous.*

To prove Theorem 2, we shall make use of the following elementary fact, often known under the name "découpage de Lévy".

Lemma 2. *Let $X_1, X_2, \ldots$ be i.i.d. random elements in an arbitrary measurable space $(S, \mathscr{S})$ with common law μ and let $B \in \mathscr{S}$ have $\mu(B) > 0$. Let $N = \inf\{n \geq 1 : X_n \in B\}$. Then*

(i) $(X_1, \ldots, X_{N-1})$ *is independent of* X_N*;*

(ii) X_N *has law* $\mu(\cdot|B)$*;*

(iii) $\mathbb{P}(N = n) = \mu(S-B)^{n-1}\mu(B)$, $n \geq 1$.

Moreover, the distribution of $(X_1, \ldots, X_N)$ can be expressed neatly as follows. Let $X'', X_1', X_2', \ldots$ be independent random elements, and independent of N, such that

$$P(X'' \in \cdot) = \mu(\cdot|B), \quad \mathbb{P}(X_i' \in \cdot) = \mu(\cdot|S-B), \quad i = 1, 2, \ldots$$

Then

$$(X_1, \ldots, X_N) \overset{(\mathrm{d})}{=} (X_1', \ldots, X_{N-1}', X''),$$

where, by definition, $(X_1', \ldots, X_{N-1}', X'') = X''$ if $N = 1$.

The proof is trivial and is thus omitted.

Proof of Theorem 2. For fixed $u > 0$, let $F(x) = e^{-ux}$, $x \geq 0$. Then $F'(x) = -ue^{-ux}$ and $F(x_1 + x_2) = F(x_1)F(x_2)$ for all $x_1, x_2 \geq 0$. With a view towards applying Lemma 2 to the sequence (τ_n, σ_n), $n \geq 0$, let $B := \{(t, s) \in \mathbb{R}^2 : t \geq s \geq 0\}$. For simplicity, let

$$p := \mathbb{P}(\tau \geq \sigma), \quad q = 1 - p.$$

By (16),

$$\widetilde{\mathbb{E}}F'(\alpha(0)) = \lambda p\, \mathbb{E}^* \left[F\left(\tau_{-1} + \sum_{i=0}^{N-1} \tau_i + \sigma_N \right) - F(\sigma_N) \right] = \lambda p\, \mathbb{E} \left[F\left(\tau'' + \sum_{i=0}^{N-1} \tau_i' + \sigma'' \right) - F(\sigma'') \right],$$

where $N, \tau'', \tau_1', \tau_2', \dots, \sigma''$ are independent random variables such that

$$\mathbb{P}(N = n) = q^n p, \quad \tau'' \stackrel{\text{(d)}}{=} (\tau | \tau > \sigma), \quad \sigma'' \stackrel{\text{(d)}}{=} (\sigma | \tau > \sigma), \quad \tau' \stackrel{\text{(d)}}{=} (\tau | \tau \leq \sigma). \tag{19}$$

Hence, letting $F(x) = e^{-ux}$ for some fixed $u > 0$ we have

$$\begin{aligned}
\widetilde{\mathbb{E}}F'(\alpha(0)) &= \lambda p\, \mathbb{E} \left\{ F(\tau'')F(\sigma'') \prod_{i=0}^{N-1} F(\tau_i') - F(\sigma'') \right\} = \lambda p\, \mathbb{E}F(\sigma'')\, \left\{ \mathbb{E}F(\tau'')\, \mathbb{E}[(\mathbb{E}F(\tau'))^N] - 1 \right\} \\
&= \lambda p\, \mathbb{E}F(\sigma'')\, \left\{ \mathbb{E}F(\tau'')\, \frac{p}{1 - q\mathbb{E}F(\tau')} - 1 \right\} = \lambda p\, \frac{\mathbb{E}F(\sigma'')\, (\mathbb{E}F(\tau) - 1)}{1 - q\mathbb{E}F(\tau')},
\end{aligned}$$

whence, after a little algebra, we obtain (17):

$$-u\widetilde{\mathbb{E}}e^{-u\alpha(0)} = \lambda(\mathbb{E}e^{-u\tau} - 1)\, \frac{p\mathbb{E}e^{-u\sigma''}}{1 - q\mathbb{E}e^{-u\tau'}} = \lambda(\mathbb{E}e^{-u\tau} - 1)\, \frac{\mathbb{E}[e^{-u\sigma}; \tau \geq \sigma]}{1 - \mathbb{E}[e^{-u\tau}; \tau < \sigma]}.$$

To prove (18) note that the first term in (17) equals $\frac{1 - \mathbb{E}e^{-u\tau}}{u\mathbb{E}\tau}$ is equal to $\mathbb{E}e^{-u\overline{\tau}}$. So $\alpha(0) \stackrel{\text{(d)}}{=} \tau + Y$ where Y is an independent random variable whose Laplace transform is the second term in (17):

$$\mathbb{E}e^{-uY} = \frac{\mathbb{E}[e^{-u\sigma}; \tau \geq \sigma]}{1 - \mathbb{E}[e^{-u\tau}; \tau < \sigma]}. \tag{20}$$

Recalling that N is the index of the first successful arrival after the origin, we see that, again after a little algebra involving a geometric series,

$$\mathbb{E}e^{-u(T_N + \sigma_N)} = \mathbb{E} \sum_{n=0}^{\infty} e^{-u(\tau_0 + \cdots + \tau_{n-1} + \sigma_n)}\, \mathbf{1}_{\tau_0 < \sigma_0, \dots, \tau_{n-1} < \sigma_{n-1}, \tau_n \geq \sigma_n} = \frac{\mathbb{E}[e^{-u\sigma}; \tau \geq \sigma]}{1 - \mathbb{E}[e^{-u\tau}; \tau < \sigma]}. \tag{21}$$

This shows that $\mathbb{E}e^{-uY} = \mathbb{E}e^{-u(T_N + \sigma_N)}$ for all $u > 0$, and thus

$$Y \stackrel{\text{(d)}}{=} T_N + \sigma_N.$$

But $T_N + \sigma_N = B_1 - B_0$, $\mathbb{P}^*$–a.s. $\square$

***Remark* 1.** We may decompose $\alpha(0)$ in a different way. Rearranging terms in the $\widetilde{\mathbb{P}}$-Laplace transform of $\alpha(0)$ we have

$$\widetilde{\mathbb{E}}e^{-u\alpha(0)} = \mathbb{E}e^{-u\sigma''}\, \frac{\lambda p}{u}\, \frac{1 - \mathbb{E}e^{-u\tau}}{1 - q\mathbb{E}e^{-u\tau'}},$$

which implies that there is a second decomposition for the law of $\alpha(0)$:

$$\alpha(0) \stackrel{\text{(d)}}{=} \sigma'' + Z,$$

where σ'' and Z are independent random variables, with σ'' having the law of σ conditional on $\tau \geq \sigma$ and Z having Laplace transform $(\lambda p/u)(1 - \mathbb{E}e^{-u\tau})/(1 - q\mathbb{E}e^{-u\tau'})$.

Corollary 2. *Under the assumptions of Theorem 2, we have*

$$\widetilde{\mathbb{E}}\alpha(0) = \frac{\mathbb{E}\tau^2}{2\mathbb{E}\tau} + \frac{\mathbb{E}\tau \wedge \sigma}{\mathbb{P}(\tau \geq \sigma)}. \tag{22}$$

Proof. Look at (18). We have $\mathbb{E}\overline{\tau} = \mathbb{E}\tau^2/2\mathbb{E}\tau$ and

$$\mathbb{E}\mathbf{R} = \mathbb{E}(T_N + \sigma_N) = \frac{\mathbb{E}\tau \wedge \sigma}{p}.$$

□

Corollary 3. *Under the assumptions of Theorem 2, and if, in addition, the variables σ_n are exponential with rate μ, then, under $\widetilde{\mathbb{P}}$,*

$$\alpha(0) \overset{(\mathrm{d})}{=} \overline{\tau} + \frac{\mathbf{e}}{\mu},$$

where $\mathbf{e}$ is a rate-1 exponential random variable, independent of $\overline{\tau}$ and so

$$\widetilde{\mathbb{E}}\alpha(0) = \frac{\mathbb{E}\tau^2}{2\mathbb{E}\tau} + \frac{1}{\mu}.$$

Proof. We use (18). We just have to show that the reading interval length $\mathbf{R}$ is exponential with rate μ. Since

$$\mathbb{E}[e^{-u\sigma}; \tau \geq \sigma] = \mathbb{E}\int_0^\tau e^{-us}\mu e^{-\mu s} ds = \mu\mathbb{E}\int_0^\tau e^{-(u+\mu)s} ds = \frac{\mu}{u+\mu}[1 - \mathbb{E}e^{-(u+\mu)\tau}],$$
$$\mathbb{E}[e^{-u\tau}; \tau < \sigma] = \mathbb{E}e^{-u\tau}\mathbb{P}(\sigma \geq \tau|\tau) = \mathbb{E}e^{-u\tau}e^{-\mu\tau} = \mathbb{E}e^{-(u+\mu)\tau},$$

we have, from (20), that the Laplace transform of $\mathbf{R}$ is

$$\mathbb{E}e^{-u\mathbf{R}} = \frac{\mathbb{E}[e^{-u\sigma}; \tau \geq \sigma]}{1 - \mathbb{E}[e^{-u\tau}; \tau < \sigma]} = \frac{\frac{\mu}{u+\mu}[1 - \mathbb{E}e^{-(u+\mu)\tau}]}{1 - \mathbb{E}e^{-(u+\mu)\tau}} = \frac{\mu}{u+\mu}.$$

□

Corollary 4. *Under the assumptions of Theorem 2, and if, in addition, the variables τ_n are exponential with rate λ, then*

$$\widetilde{\mathbb{E}}e^{-u\alpha(0)} = \frac{\lambda\mathbb{E}e^{-(\lambda+u)\sigma}}{u + \lambda\mathbb{E}e^{-(\lambda+u)\sigma}}, \quad \widetilde{\mathbb{E}}\alpha(0) = \frac{1}{\lambda\,\mathbb{E}e^{-\lambda\sigma}}.$$

Proof. Since τ is exponential we have $\overline{\tau} \overset{(\mathrm{d})}{=} \tau$ and so

$$\mathbb{E}e^{-u\overline{\tau}} = \mathbb{E}e^{-u\tau} = \frac{\lambda}{u+\lambda}.$$

Using (20), we have

$$\mathbb{E}e^{-u\mathbf{R}} = \frac{(u+\lambda)\mathbb{E}e^{-(u+\lambda)\sigma}}{u + \lambda\mathbb{E}e^{-(u+\lambda)\sigma}}.$$

Equation (17) says that the Laplace transform of $\alpha(0)$ is the product of the last two displays and so this derives the first formula. Next use (22). Since

$$\mathbb{E}\tau \wedge \sigma = \frac{1}{\lambda}(1 - \mathbb{E}e^{-\lambda\sigma}), \quad \mathbb{P}(\tau > \sigma) = \mathbb{E}e^{-\lambda\sigma},$$

we have

$$\widetilde{\mathbb{E}}\alpha(0) = \frac{1}{\lambda} + \frac{1}{\lambda} \cdot \frac{1 - \mathbb{E}e^{-\lambda\sigma}}{\mathbb{E}e^{-\lambda\sigma}} = \frac{1}{\lambda \mathbb{E}e^{-\lambda\sigma}}.$$

□

Finally, a direct consequence of either of the above corollaries is:

Corollary 5. *If the τ_n are i.i.d. exponential with rate λ, if the σ_n are i.i.d. exponential with rate μ, and if the two sequences are independent, then, under $\widetilde{\mathbb{P}}$,*

$$\alpha(0) \overset{(\mathrm{d})}{=} \frac{\mathbf{e}_1}{\lambda} + \frac{\mathbf{e}_2}{\mu},$$

where $\mathbf{e}_1, \mathbf{e}_2$ are two independent unit-rate exponential random variables.

4.2 The new age of information for the fully push-out system

Recall that $\beta(t) = A_t - S_{D_t}$. Under $\widetilde{\mathbb{P}}$, the law of $\beta(t)$ is independent of t.

Lemma 3. *The $\widetilde{\mathbb{P}}$-law of $\beta(t)$ has a nontrivial atom at 0.*

Proof. Indeed,

$$\widetilde{\mathbb{P}}(\beta(t) = 0) = \widetilde{\mathbb{P}}(A_t = S_{D_t}) = \widetilde{\mathbb{P}}(q(t) = 0) > 0.$$

The latter is positive because of the non-triviality assumption (11). □

Theorem 3. *Consider the fully push-out system under stationarity assumptions and assume that (11) holds. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a measurable function that is bounded or nonnegative. Then*

$$\widetilde{\mathbb{E}}f(\beta(0)) = \lambda \mathbb{E}\left[\sum_{i=0}^{N-1} \tau_i f\left(\sum_{j=-1}^{i-1} \tau_j\right) + \sigma_N f\left(\sum_{j=-1}^{N-1} \tau_j\right) + (\tau_N - \sigma_N) f(0);\, \tau_{-1} > \sigma_{-1}\right], \tag{23}$$

where N is as in Theorem 1.

Proof. We use again the Palm inversion formula

$$\widetilde{\mathbb{E}}f(\beta(0)) = \frac{\mathbb{E}^* \int_{B_0}^{B_1} f(\beta(t))\, dt}{\mathbb{E}^*(B_1 - B_0)}, \tag{24}$$

where the notation is as before. We now have

$$\beta(t) = A_t - S_{D_t} = \begin{cases} T_i - T_{-1}, & T_0 \le T_i \le t < T_{i+1} \le T_N + \sigma_N,\, i \ge 0, \\ 0, & T_N + \sigma_N \le t < T_{N+1} \end{cases}, \quad \mathbb{P}^*\text{-a.s.}$$

Hence the integral in (24) is

$$\begin{aligned} \int_{B_0}^{B_1} f(\beta(t))\, dt &= \int_{T_0}^{T_{N+1}} f(\beta(t))\, dt \\ &= \sum_{i: T_0 \le T_i < T_{i+1} \le T_N} \int_{T_i}^{T_{i+1}} f(T_i - T_{-1}) dt + \int_{T_N}^{T_N + \sigma_N} f(T_N - T_{-1}) dt + \int_{T_N + \sigma_N}^{T_{N+1}} f(0) dt \\ &= \sum_{i=0}^{N-1} \tau_i f(T_i - T_{-1}) + \sigma_N f(T_N - T_{-1}) + (\tau_N - \sigma_N) f(0). \end{aligned}$$

Substitute this into (24) and use $\mathbb{E}^*(B_1 - B_0) = 1/\lambda \mathbb{P}(\tau_{-1} \ge \sigma_{-1})$ to obtain (23). □

Corollary 6 (Continuation of Lemma 3)**.** *The atom of $\beta(0)$ at 0 has value*

$$\widetilde{\mathbb{P}}(\beta(0)=0)=\lambda\,\mathbb{E}[(\tau_N-\sigma_N);\,\tau_{-1}>\sigma_{-1}]. \tag{25}$$

Proof. Let, in (23), $f(x):=\mathbf{1}_{x=0}$. Since all the τ_n and σ_n are nonzero with probability 1, (25) follows. $\square$

Theorem 4. *Consider the fully push-out system and assume that (τ_n,σ_n), $n\in\mathbb{Z}$, is i.i.d. under $\mathbb{P}$ and such that $\mathbb{E}\tau_0<\infty$ and $\mathbb{P}(\tau_0\ge\sigma_0)>0$. Assume further that τ_n is independent of σ_n for all n. Then the $\widetilde{\mathbb{P}}$-law of $\beta(0)$ can be described as*

$$\beta(0)\overset{(\mathrm{d})}{=}\begin{cases}0, & \textit{with probability } \dfrac{\mathbb{E}(\tau-\sigma)^+}{\mathbb{E}\tau}\\ \mathbf{C}, & \textit{with probability } \dfrac{\mathbb{E}\tau\wedge\sigma}{\mathbb{E}\tau}\end{cases}, \tag{26}$$

where $\mathbf{C}$ has the distribution of a typical cycle length;

$$\mathbb{E}e^{-u\mathbf{C}}=\mathbb{E}^*e^{-u(B_1-B_0)}=\frac{\mathbb{E}[e^{-u\tau};\tau\ge\sigma]}{1-\mathbb{E}[e^{-u\tau};\sigma<\tau]}, \tag{27}$$

In particular,

$$\widetilde{\mathbb{E}}\beta(0)=\frac{\mathbb{E}\tau\wedge\sigma}{\mathbb{P}(\tau\ge\sigma)}. \tag{28}$$

Proof. Using (25) and independence,

$$\widetilde{\mathbb{P}}(\beta(0)=0)=\lambda\,\mathbb{E}(\tau_N-\sigma_N)\,\mathbb{P}(\tau_{-1}>\sigma_{-1}).$$

By Lemma 2 and (19), we further have

$$\begin{aligned}\widetilde{\mathbb{P}}(\beta(0)=0)&=\lambda\,\mathbb{E}(\tau''-\sigma'')\,\mathbb{P}(\tau>\sigma)\\&=\lambda\,\mathbb{E}(\tau-\sigma|\tau>\sigma)\,\mathbb{P}(\tau>\sigma)\\&=\lambda\,\mathbb{E}(\tau-\sigma)^+.\end{aligned}$$

This proves the upper part of (26). To prove the lower part notice, from (23),

$$\begin{aligned}\widetilde{\mathbb{E}}[f(\beta(0));\beta(0)>0]&=\lambda\,\mathbb{P}(\tau_{-1}>\sigma_{-1})\,\mathbb{E}\left[\sum_{i=0}^{N-1}\tau_i f\left(\sum_{j=-1}^{i-1}\tau_j\right)+\sigma_N f\left(\sum_{j=-1}^{N-1}\tau_j\right)\Bigg|\tau_{-1}>\sigma_{-1}\right]\\&=\lambda p\,\mathbb{E}\left[\sum_{i=0}^{N-1}\tau_i' f\left(\tau_{-1}''+\sum_{j=-1}^{i-1}\tau_j'\right)+\sigma'' f\left(\tau_{-1}''+\sum_{j=-1}^{N-1}\tau_j'\right)\right],\end{aligned}$$

where we used Lemma 2 and the definitions (19). Next, let $f(x)=e^{-ux}$ and write the above as

$$\begin{aligned}\widetilde{\mathbb{E}}[f(\beta(0));\beta(0)>0]&=\lambda p\,(\mathbb{E}f(\tau''))\,\mathbb{E}\left[\sum_{i=0}^{N-1}(\mathbb{E}\tau')(\mathbb{E}f(\tau'))^i+(\mathbb{E}\sigma'')(\mathbb{E}f(\tau'))^N\right]\\&=\lambda p\,(\mathbb{E}f(\tau''))\left[(\mathbb{E}\tau')\,\mathbb{E}\left(\frac{1-(\mathbb{E}f(\tau'))^N}{1-\mathbb{E}f(\tau')}\right)+(\mathbb{E}\sigma'')(\mathbb{E}f(\tau'))^N\right]\\&=\lambda p\,(\mathbb{E}f(\tau''))\left[\frac{\mathbb{E}\tau'}{1-\mathbb{E}f(\tau')}\left(1-\frac{p}{1-q\mathbb{E}f(\tau')}\right)+(\mathbb{E}\sigma'')\frac{p}{1-q\mathbb{E}f(\tau')}\right]\\&=\lambda p\,(\mathbb{E}f(\tau''))\,\frac{q\mathbb{E}\tau'+p\mathbb{E}\sigma''}{1-q\mathbb{E}f(\tau')}=\lambda p\,(\mathbb{E}f(\tau''))\,\frac{\mathbb{E}\tau\wedge\sigma}{1-q\mathbb{E}f(\tau')},\end{aligned}$$

that is precisely the lower part of (26). The last equality in (27) is easily verified along the same lines. To finally show (28) just note that

$$\widetilde{\mathbb{E}}\beta(0) = \frac{\mathbb{E}\tau\wedge\sigma}{\mathbb{E}\tau}\,\mathbb{E}Z = \frac{\mathbb{E}\tau\wedge\sigma}{\mathbb{E}\tau}\,\frac{\mathbb{E}\tau}{\mathbb{P}(\tau>\sigma)}.$$

□

***Remark* 2.** Notice that β does not suffer from the same drawback as α when τ^2 is not integrable. Indeed, here, under the condition $\mathbb{E}\tau<\infty$ we have $\widetilde{\mathbb{E}}\beta(0)\le 1$, regardless of the variance of τ.

Corollary 7. *Let the assumptions of Theorem 4 hold true.*
(i) If the variables τ_n are exponential with rate λ, then

$$\widetilde{\mathbb{E}}e^{-u\beta(0)} = 1-\frac{u(1-\mathbb{E}e^{-\lambda\sigma})}{u+\lambda\mathbb{E}e^{-(\lambda+u)\sigma}}, \qquad \widetilde{\mathbb{E}}\beta(0) = \frac{1}{\lambda\mathbb{E}e^{-\lambda\sigma}}-\frac{1}{\lambda}.$$

(ii) If the variables σ_n are exponential with rate μ, then

$$\widetilde{\mathbb{E}}e^{-u\beta(0)} = 1-\frac{1-\mathbb{E}e^{-\mu\tau}}{\mu\mathbb{E}\tau}\,\frac{1-\mathbb{E}e^{-u\tau}}{1-\mathbb{E}e^{-(u+\mu)\tau}}, \qquad \widetilde{\mathbb{E}}\beta(0) = \frac{1}{\mu}.$$

(iii) If the τ_n are with rate λ, and the σ_n are exponential with rate μ then, under $\widetilde{\mathbb{P}}$,

$$\beta(0) \overset{(\mathrm{d})}{=} \begin{cases} 0, & \textit{with probability } \frac{\mu}{\lambda+\mu} \\ \frac{\mathbf{e}_1}{\lambda}+\frac{\mathbf{e}_2}{\mu}, & \textit{with probability } \frac{\lambda}{\lambda+\mu} \end{cases}, \qquad \widetilde{\mathbb{E}}\beta(0) = \frac{1}{\mu},$$

where $\mathbf{e}_1$, $\mathbf{e}_2$ are two independent unit-rate exponential random variables.

5 The fully blocking system

The fully blocking system is defined by the requirement that only those messages for which $\mathcal{Z}_n$ holds are admited. The remaining ones are immediately rejected (blocked). The system has the dynamics of the G/G/1/1 queue. It is well-known that if

$$\mathbb{P}(\sup_{i\le -1}(\sigma_i - T_i)\le 0) > 0 \tag{29}$$

then the system admits a unique steady-state, see [1, Section 5.2]. Under this condition, (10) holds.

We have $\psi_n=\chi_n$ for all $n\in\mathbb{Z}$ (a message is successful if and only if it is admitted) and

$$\psi_n \text{ is a measurable function of } (\tau_m,\sigma_m : m\le n-1). \tag{30}$$

Recall that we use letters B_k, B'_k for the beginings and ends of reading periods, respectively. In other words,

$$\{B_k : k\in\mathbb{Z}\} = \{T_n : n\in\mathbb{Z}, \psi_n=1\},$$
$$\{B'_k : k\in\mathbb{Z}\} = \{T_n+\sigma_n : n\in\mathbb{Z}, \psi_n=1\}.$$

Therefore the Palm probability $\mathbb{P}^*$ of $\mathbb{P}$ with respect to $\{B_k\}$ admits a simpler representation:

Lemma 4. *$\mathbb{P}^*$ is the Palm probability of $\widetilde{\mathbb{P}}$ with respect to the (stationary) point process*

$$\sum_{n\in\mathbb{Z}}\psi_n\delta_{T_n}$$

and

$$\mathbb{P}^* = \mathbb{P}(\cdot|\psi_0=1). \tag{31}$$

Recalling that $\{B_k\}$ and $\{B'_k\}$ are interlaced sequences let us compute the quantities S_t (last successful arrival before t), D_t (last successful departure before t), and S_{D_t} (last successful arrival before D_t) depending whether t falls in a reading interval (that is, between B_k and B'_k for some k) or not (that is, between B'_k and B_{k+1} for some k). Since $\{B_k\}$ is the totality of successful arrivals, we have that, for all $k \in \mathbb{Z}$,

$$B_k \le t < B_{k+1} \Rightarrow S_t = B_k.$$

Since $\{B'_k\}$ is the totality of successful departures, we have that, for all $k \in \mathbb{Z}$,

$$B'_k \le t < B'_{k+1} \Rightarrow D_t = B'_k.$$

It then follows that, for all $k \in \mathbb{Z}$,

$$S_{D_t} = \begin{cases} B_{k-1}, & \text{if } B_k \le t < B'_k \\ B_k, & \text{if } B'_k \le t < B_{k+1} \end{cases}. \tag{32}$$

5.1 The age of information for the fully blocking system

We shall use the Palm inversion formula (14) for the process $\alpha(t) = t - S_{D_t}$, $t \in \mathbb{R}$, for the fully blocking system. By Campbell's formula we have that the denominator of (14) is

$$\mathbb{E}^*(B_1 - B_0) = \frac{1}{\lambda \mathbb{P}(\psi_0 = 1)}, \tag{33}$$

however, unlike in the push-out system, the probability in the denominator depends on the full distribution and the dynamics of the system and so it does not admit an explicit form without further assumptions.

Theorem 5. *Consider the fully blocking system under stationarity assumptions and assume that (29) holds. Let f be bounded and measurable or locally integrable and nonnegative function and let F be such that $F' = f$. Then*

$$\widetilde{\mathbb{E}} f(\alpha(0)) = \lambda\, \mathbb{E}[F(T_N + \sigma_N) - F(\sigma_N);\, \psi_0 = 1] = \frac{\mathbb{E}[F(T_N + \sigma_N) - F(\sigma_N) | \psi_0 = 1]}{\mathbb{E}[T_N | \psi_0 = 1]}, \tag{34}$$

where

$$N := \inf\{\ell \ge 1 :\, \tau_0 + \cdots + \tau_{\ell-1} \ge \sigma_0\}. \tag{35}$$

Proof. Under $\mathbb{P}^*$, message 0 is successful (admitted) and N is the first successful (admitted) message after that. Note that $N < \infty$. Thus,

$$B_1 = T_N, \quad \mathbb{P}^*\text{-a.s.} \tag{36}$$

Note also that, with $\mathfrak{a} = \sum_{n \in \mathbb{Z}} \delta_{T_n}$,

$$N = \mathfrak{a}([0, \sigma_0]) = \sum_{n=0}^{\infty} \mathbf{1}_{T_n \le \sigma_0}, \quad \mathbb{P}\text{-a.s. and (hence) } \mathbb{P}^*\text{-a.s.} \tag{37}$$

By (32), and since $B'_0 = T_0 + \sigma_0$, $\mathbb{P}^*$-a.s., the function α on $[B_0, B_1)$ is given by

$$\alpha(t) = t - S_{D_t} = \begin{cases} t - B_{-1}, & T_0 \le t < T_0 + \sigma_0 \\ t - B_0, & T_0 + \sigma_0 \le t < T_N \end{cases}, \quad \mathbb{P}^*\text{-a.s.} \tag{38}$$

Hence, for functions f, F as in the theorem statement, with $F' = f$,

$$\begin{aligned} \int_{B_0}^{B_1} f(\alpha(t))\, dt = \int_{T_0}^{T_N} f(\alpha(t))\, dt &= \int_{T_0}^{T_0+\sigma_0} f(t - B_{-1})\, dt + \int_{T_0+\sigma_0}^{T_N} f(t - B_0)\, dt \\ &= F(B_0 - B_{-1} + \sigma_0) - F(B_0 - B_{-1}) + F(B_1 - B_0) - F(\sigma_0), \quad \mathbb{P}^*\text{-a.s.,} \end{aligned}$$

and thus, since $\mathbb{E}^* F(B_0 - B_{-1}) = \mathbb{E}^* F(B_1 - B_0)$,

$$\mathbb{E}^* \int_{B_0}^{B_1} f(\alpha(t))\, dt = \mathbb{E}^* F(B_0 - B_{-1} + \sigma_0) - \mathbb{E}^* F(\sigma_0)$$
$$= \mathbb{E}^* F(B_1 - B_0 + \sigma_N) - \mathbb{E}^* F(\sigma_N).$$

Here we used the fact that $\mathbb{P}^*$ is preserved by θ_{B_k} for all $k \in \mathbb{Z}$. Taking into account (14), (33) and (36), we can conclude. □

***Remark* 3.** Note that, since there is no ready-made expression for $\mathbb{P}(\psi_0 = 1)$, the second formula in (34) turns out to be more useful for further computations.

We now introduce

$$N_t := \inf\{\ell \geq 0 : T_\ell \geq t\}, \quad t \geq 0, \tag{39}$$

so that the variable N defined by (35) is simply the value of N_t for $t = \sigma_0$:

$$N_{\sigma_0} = N.$$

Note that N is left-continuous on $[0, \infty)$ with $N_0 = 0$ and $N_{0+} = 1$. Since $\mathfrak{a} = \sum_{n \in \mathbb{Z}} \delta_{T_n}$, we have

$$N_t = \mathfrak{a}([0, t)) = 1 + \mathfrak{a}((0, t)), \quad t \geq 0.$$

Remembering that $\mathbb{P}$ is a Palm probability and $\mathbb{P}(T_0 = 0) = 1$, define

$$U(t) := \mathbb{E} N_t = \mathbb{E}\mathfrak{a}([0, t)) = \sum_{n=0}^{\infty} \mathbb{P}(T_n < t), \quad t \geq 0. \tag{40}$$

If the τ_n are i.i.d., then U is known as 0-potential function (if $T_0, T_1, T_2, \ldots$ is thought of as a random walk) or renewal function (if $T_0, T_1, T_2, \ldots$ are thought of as the points of a renewal process). We have that U is left-continuous on $[0, \infty)$ with $U(0) = 0$, $U(0+) = 1$. We shall deal with the renewal case next. We will also need the definition

$$W(f, t) := \mathbb{E} f(T_{N_t}), \quad t \geq 0, \tag{41}$$

where f is an appropriate function for which the expectation exists. In particular, with $f(x) = e^{-ux}$ for some $u > 0$, we let

$$W_u(t) = \mathbb{E} e^{-u T_{N_t}}, \tag{42}$$

and with $f(x) = x^p$ for some $p > 0$, we let

$$M_p(t) = \mathbb{E} T_{N_t}^p.$$

The following result gives the Laplace transform of the $\widetilde{\mathbb{P}}$-marginal of $\alpha(t)$ in terms of functions that can be computed as unique solutions to fixed-point equations.

Theorem 6. *Consider the fully blocking system and assume that (τ_n, σ_n), $n \in \mathbb{Z}$, is i.i.d. under $\mathbb{P}$ and such that $\mathbb{E}\tau_0 < \infty$ and $\mathbb{P}(\tau_0 \geq \sigma_0) > 0$. Assume further that τ_n is independent of σ_n for all n. Then, for $u > 0$,*

$$\widetilde{\mathbb{E}} e^{-u\alpha(0)} = \mathbb{E} e^{-u\sigma} \cdot \frac{1 - \mathbb{E} e^{-u T_N}}{u \mathbb{E} T_N} = \mathbb{E} e^{-u\sigma} \cdot \frac{1 - \mathbb{E} W_u(\sigma)}{u\, \mathbb{E}\tau\, \mathbb{E} U(\sigma)}, \tag{43}$$

where U and W_u are the unique solutions to the fixed-point equations

$$U(t) = 1 + \int_{(0,t]} U(t - x) \mathbb{P}(\tau \in dx) \tag{44}$$

$$W_u(t) = \int_{(t,\infty)} e^{-ux} \mathbb{P}(\tau \in dx) + \int_{(0,t]} W_u(t - x) e^{-ux} \mathbb{P}(\tau \in dx). \tag{45}$$

In particular, under $\widetilde{\mathbb{P}}$, $\alpha(0)$ *is the sum of two independent random variables:*

$$\alpha(0) \stackrel{(\mathrm{d})}{=} \sigma + \overline{T_N}, \tag{46}$$

where $\overline{T_N}$ *is the stationary version of* T_N.

Proof. Observe first that $\mathbb{P}(\tau_0 \geq \sigma_0) > 0$ implies (by the ergodic theorem) (29) and hence a unique steady-state version exists. Using the fact that ψ_n is a measurable function of the variables τ_m, σ_m with $m \leq n$ [see (30)] we write (34) as

$$\widetilde{\mathbb{E}} F'(\alpha(0)) = \frac{\mathbb{E}[F(T_N + \sigma_N) - F(\sigma_N)]}{\mathbb{E} T_N}, \tag{47}$$

with $N = \inf\{\ell \geq 1 : \tau_0 + \cdots + \tau_{\ell-1} \geq \sigma_0\}$, $\mathbb{P}$-a.s. Since $N - 1 = \inf\{i \geq 0 : \tau_0 + \cdots + \tau_i \geq \sigma_0\}$, it follows that $N-1$ is a stopping time with respect to $\mathscr{A}_i$, $i \geq 0$, where $\mathscr{A}_i$ is the σ-algebra generated by $(\sigma_0, \tau_0, \ldots, \tau_i)$. Let $F(x) = e^{-ux}$. Then

$$\mathbb{E}[F(T_N + \sigma_N) - F(\sigma_N)] = \mathbb{E}[F(T_N)F(\sigma_N) - F(\sigma_N)] = [\mathbb{E}F(T_N) - 1]\, \mathbb{E}F(\sigma_N),$$

where the last equality needs that $N-1$ is a stopping time. Noting that $\mathbb{E}F(\sigma_N) = \mathbb{E}F(\sigma)$ we obtain the first equality in (43) from which decomposition (46) follows at once.
For the last equality of (43) we have

$$\mathbb{E}\; T_N = \mathbb{E} \sum_{i=0}^{N-1} \tau_i = \mathbb{E} \sum_{i=0}^{\infty} \tau_i \, \mathbf{1}_{T_i \leq \sigma_0} = \sum_{i=0}^{\infty} (\mathbb{E}\tau_i) \mathbb{P}(T_i \leq \sigma_0) = (\mathbb{E}\tau) \sum_{i=0}^{\infty} \mathbb{P}(T_i \leq \sigma_0) = (\mathbb{E}\tau)\, \mathbb{E}U(\sigma), \tag{48}$$

and,

$$\mathbb{E}e^{-uT_N} = \mathbb{E}e^{-uT_{N_{\sigma_0}}} = \mathbb{E}\,\mathbb{E}[e^{-uT_{N_{\sigma_0}}} | \sigma_0] = \mathbb{E}\, W_{\mathcal{E}_u}(\sigma_0). \tag{49}$$

Equation (44) is the renewal equation from standard renewal theory. To obtain(45) we write

$$W(f,t) = \mathbb{E}f(T_{N_t}) = \mathbb{E}[f(T_{N_t}); t < \tau_0] + \mathbb{E}[f(T_{N_t}); t \geq \tau_0].$$

If $t < \tau_0$ then $N_t = 1$, $T_{N_t} = T_1 = \tau_0$, $\mathbb{P}$-a.s., If $t \geq \tau_0$ and $\tau_0 = x$ then $T_{N_t} \stackrel{(\mathrm{d})}{=} x + T_{N_{t-x}}$, under $\mathbb{P}$. Set $\Phi_t := T_{N_t}$. If τ is independent of (Φ_t) we have $T_{N_t} \stackrel{(\mathrm{d})}{=} \tau + \Phi_{t-\tau}$ and so $f(T_{N_t})\mathbf{1}_{\tau_0 \leq t} \stackrel{(\mathrm{d})}{=} f(\tau + \Phi_{t-\tau})\mathbf{1}_{\tau \leq t}$. Hence

$$W(f,t) = \mathbb{E}[f(\tau);\, \tau > t] + \mathbb{E}[f(\tau + \Phi_{t-\tau});\, \tau \leq t]. \tag{50}$$

Letting $f(x) = e^{-ux}$ we further have $F(\tau + \Phi_{t-\tau}) = e^{-u\tau}e^{-u\Phi_{t-\tau}}$ and so

$$\mathbb{E}[e^{-u(\tau + \Phi_{t-\tau})};\, \tau \leq t] = \mathbb{E}[e^{-u\tau}\mathbb{E}(e^{-u\Phi_{t-\tau}}|\tau)\mathbf{1}_{\tau \leq t}] = \mathbb{E}[e^{-u\tau} W_u(t-\tau)\mathbf{1}_{\tau \leq t}],$$

and this establishes (45). □

To compute the first moment of the AoI we need to know the second moment of T_{N_t}. Recall that $M_p(t) = \mathbb{E}T_{N_t}^p$ is the p-th moment of T_{N_t}.

Lemma 5. *If* p *is a positive integer we have*

$$\begin{aligned} M_p(t) &= \mathbb{E}M_p(t-\tau) + \mathbb{E}\tau^p + \sum_{k=1}^{p} \binom{p}{k} \mathbb{E}[\tau^k W_{p-k}(t-\tau)] \\ &= \int_{(0,t]} W_{p-k}(t-x)\mathbb{P}(\tau \in dx) + \mathbb{E}\tau^p + \sum_{k=1}^{p} \binom{p}{k} \int_{(0,t]} x^k W_{p-k}(t-x)\mathbb{P}(\tau \in dx). \end{aligned} \tag{51}$$

Proof. Proceed as in the proof of Theorem 6 but let $F(x) = x^p$ in (50); then use the binomial theorem. $\square$

Let $(U * U)(t) := \int_0^t U(t-x)\,U(dx)$.

Corollary 8. *Under the assumptions of Theorem 6,*

$$\widetilde{\mathbb{E}}\alpha(0) = \mathbb{E}\sigma + \frac{\mathbb{E}T_N^2}{2\mathbb{E}T_N} = \mathbb{E}\sigma + \frac{\mathbb{E}M_2(\sigma)}{2\mathbb{E}M_1(\sigma)} = \mathbb{E}\sigma + \frac{\mathbb{E}\tau^2}{2\mathbb{E}\tau} + \frac{\mathbb{E}(\tau(U*U)(\sigma-\tau))}{\mathbb{E}U(\sigma)}. \tag{52}$$

Proof. The first equality in (52) follows from the decomposition (46). The second equality follows from $\mathbb{E}T_N^p = \mathbb{E}\mathbb{E}[T_{N_{\sigma_0}}^p|\sigma_0] = \mathbb{E}M_p(\sigma)$. We next have

$$M_1(t) = \mathbb{E}\tau\, U(t) \tag{53}$$

and, from (51) with $p=2$,

$$M_2(t) = \mathbb{E}\tau^2 + 2\mathbb{E}[\tau\, M_1(t-\tau)] + \mathbb{E}M_2(t-\tau) = \mathbb{E}\tau^2 + 2\mathbb{E}\tau\,\mathbb{E}[\tau\, U(t-\tau)] + \mathbb{E}M_2(t-\tau).$$

With the help of (44) we can solve this explicitly and express M_2 as a function of U:

$$M_2(t) = \mathbb{E}\tau^2 \cdot U(t) + 2\,\mathbb{E}\tau\,\mathbb{E}[\tau\,(U*U)(t-\tau)]. \tag{54}$$

Using (53) and (54) in the second equality of (52) we arrive at the third one. $\square$

The Laplace transforms of U, W_u and M_2 are easy to obtain explicitly in terms of the Laplace transform of τ:

Lemma 6.

$$\widehat{U}(\xi) := \int_0^\infty e^{-\xi t} U(t)dt = \frac{1/\xi}{1-\mathbb{E}e^{-\xi\tau}}, \tag{55}$$

$$\widehat{W_u}(\xi) := \int_0^\infty e^{-\xi t} W_u(t)dt = \frac{1}{\xi}\cdot\frac{\mathbb{E}[e^{-u\tau} - e^{-(u+\xi)\tau}]}{1-\mathbb{E}e^{-(u+\xi)\tau}}. \tag{56}$$

$$\widehat{M_2}(\xi) := \int_0^\infty e^{-\xi t} M_2(t)dt = \frac{\mathbb{E}\tau^2}{\xi(1-\mathbb{E}e^{-\xi\tau})} + 2(\mathbb{E}\tau)\frac{\mathbb{E}(\tau e^{-\xi\tau})}{\xi(1-\mathbb{E}e^{-\xi\tau})^2}. \tag{57}$$

Proof. Equation (44) then gives

$$\widehat{U}(\xi) = \frac{1}{\xi} + \widehat{U}(\xi)\,\mathbb{E}e^{-\xi\tau},$$

and hence (55) follows. Equation (45) gives

$$\begin{aligned}
\widehat{W_u}(\xi) &= \int_0^\infty e^{-\xi t}\mathbb{E}[e^{-u\tau}\mathbf{1}_{\tau>t}]\,dt + \int_0^\infty e^{-\xi t}\mathbb{E}[W_u(t-\tau)e^{-u\tau}\mathbf{1}_{\tau\le t}]\,dt \\
&= \mathbb{E}\left[e^{-u\tau}\frac{1-e^{-\xi\tau}}{\xi}\right] + \mathbb{E}\left[\mathbf{e}^{-u\tau}e^{-\xi\tau}\int_\tau^\infty e^{-\xi(t-\tau)}W_u(t-\tau)dt\right] \\
&= \frac{1}{\xi}\mathbb{E}\left[e^{-u\tau}(1-e^{-\xi\tau})\right] + \mathbb{E}\left[e^{-u\tau}e^{-\xi\tau}\right]\widehat{W_u}(\xi),
\end{aligned}$$

from which (56) follows. Finally, (57) follows from (54) and (55). $\square$

Corollary 9. *Let the assumptions of Theorem 4 hold true.*
(i) If the variables τ_n are exponential with rate λ, then

$$\widetilde{\mathbb{E}}e^{-u\alpha(0)} = \frac{\lambda}{1+\lambda\mathbb{E}\sigma}\cdot\frac{(u+\lambda-\lambda\mathbb{E}e^{-u\sigma})\mathbb{E}e^{-u\sigma}}{u(u+\lambda)}, \qquad \widetilde{\mathbb{E}}\alpha(0) = \mathbb{E}\sigma + \frac{1}{\lambda} + \frac{\lambda}{2}\cdot\frac{\mathbb{E}\sigma^2}{1+\lambda\mathbb{E}\sigma}.$$

(ii) If the variables σ_n are exponential with rate μ, then

$$\widetilde{\mathbb{E}}e^{-u\alpha(0)} = \frac{1}{\mathbb{E}\tau}\cdot\frac{\mu}{(\mu+u)u}\cdot\frac{(1-\mathbb{E}e^{-\mu\tau})(1-\mathbb{E}e^{-u\tau})}{1-\mathbb{E}e^{-(\mu+u)\tau}} = \frac{\mu^2}{(\mu+u)^2}\cdot\frac{\mathbb{E}e^{-\mu\overline{\tau}}\,\mathbb{E}e^{-u\overline{\tau}}}{\mathbb{E}e^{-(\mu+u)\overline{\tau}}},$$

$$\widetilde{\mathbb{E}}\alpha(0) = \frac{1}{\mu} + \frac{\mathbb{E}\tau^2}{2\mathbb{E}\tau} + \frac{\mathbb{E}(\tau e^{-\mu\tau})}{1-\mathbb{E}e^{-\mu\tau}}.$$

(iii) If the τ_n are exponential with rate λ, and the σ_n are exponential with rate μ then, under $\widetilde{\mathbb{P}}$,

$$\widetilde{\mathbb{E}}e^{-u\alpha(0)} = \frac{\mu^2\lambda(\lambda+\mu+u)}{(\lambda+\mu)(\lambda+u)(\mu+u)^2}, \qquad \widetilde{\mathbb{E}}\alpha(0) = \frac{1}{\mu} + \frac{1}{\lambda} + \frac{\lambda}{\mu(\lambda+\mu)}.$$

Proof. (ia) We compute the functions $U(t)$ and $W_u(t)$ that enter formula (43). Since, under $\mathbb{P}$, $\mathfrak{a} = \sum_n \delta_{T_n}$ is a Poisson process with a point at 0 we have, directly from (40), $U(t) = 1+\lambda t$. Since τ is exponential, (56) explicitly gives the Laplace transform of W_u:

$$\widehat{W}_u(\xi) = \frac{1}{\xi}\cdot\frac{\mathbb{E}[e^{-u\tau}(1-e^{-\xi\tau})]}{1-\mathbb{E}e^{-u\tau}e^{-\xi\tau}} = \frac{1}{\xi}\cdot\frac{\frac{\lambda}{\lambda+u}-\frac{\lambda}{\lambda+u+\xi}}{1-\frac{\lambda}{\lambda+u+\xi}} = \frac{\lambda}{\lambda+u}\cdot\frac{1}{u+\xi},$$

and hence

$$W_u(t) = \frac{\lambda}{\lambda+u}e^{-ut}.$$

Substituting into (43) we obtain the announced formula for $\widetilde{\mathbb{E}}e^{-u\alpha(0)}$.
(ib) Equations (55) and (57) give

$$\widehat{M}_2(\xi) = \mathbb{E}\tau^2\,\widehat{U}(\xi) + 2(\mathbb{E}\tau)\frac{\mathbb{E}(\tau e^{-\xi\tau})}{\xi(1-\mathbb{E}e^{-\xi\tau})^2} = \mathbb{E}\tau^2\,\widehat{U}(\xi) + \frac{2\lambda\mathbb{E}\tau}{\xi^3}.$$

Hence

$$M_2(t) = \mathbb{E}\tau^2\,U(t) + \lambda(\mathbb{E}\tau)t^2.$$

Using this and $M_1(t) = \mathbb{E}\tau\,U(t)$ in (52) we obtain the announced formula for $\widetilde{\mathbb{E}}\alpha(0)$.
(iia) If σ is exponential with rate μ then $\mathbb{E}U(\sigma) = \mu\widehat{U}(\mu)$ and $\mathbb{E}W_u(\sigma) = \mu\widehat{W}_u(\mu)$. Hence (43) gives

$$\widetilde{\mathbb{E}}e^{-u\alpha(0)} = \mathbb{E}e^{-u\sigma}\,\frac{1-\mu\widehat{W}_u(\mu)}{u\,\mathbb{E}\tau\,\mu\,\widehat{U}(\mu)}$$

But the Laplace transforms $\widehat{U}$ and $\widehat{W}_u$ are known from Lemma 6. Substituting in the last display we obtain the first announced equality for $\widetilde{\mathbb{E}}e^{-u\alpha(0)}$. For the second equality, simply replace the three terms of the form $1-\mathbb{E}e^{-\xi\tau}$ by $\xi(\mathbb{E}\tau)\mathbb{E}e^{-\xi\overline{\tau}}$.
(iib) From the middle of (52) we have

$$\widetilde{\mathbb{E}}\alpha(0) = \frac{1}{\mu} + \frac{\widehat{W}_2(\mu)}{2\widehat{W}_1(\mu)}$$

and the formula follows from the previously derived formulas for $\widehat{W}_2$ and $\widehat{W}_1$.
(iiia) Consider the second equality in (ii). Since $\overline{\tau} \overset{(d)}{=} \tau$ we have $\mathbb{E}e^{-\xi\overline{\tau}} = \lambda/(\xi+\lambda)$. Replacing the three terms in the second equality in (ii) by such ratios we arrive at the announced formula. Alternatively, letting $\mathbb{E}e^{-\mu\sigma} = \mu/(\mu+u)$ and $\mathbb{E}\sigma = 1/\mu$ in (i) we arrive at the same formula.
(iiib) Set $\mathbb{E}\sigma = 1/\mu$, $\mathbb{E}\sigma^2 = 2/\mu^2$ in the last formula of (i). $\square$

5.2 The new age of information for the fully blocking system

Things are a bit more delicate here. Recall that the NAoI process is given by $\beta(t) = A_t - S_{D_t}$, where A_t is the last arrival (accepted or not) before t and S_{D_t} is the last successful departure before the last successful arrival before t; this quantity is given by (32).

Theorem 7. *Consider the fully blocking system under stationarity assumptions and assume that* (29) *holds. Then the* $\widetilde{\mathbb{P}}$*-law of* $\beta(0)$ *has an atom at* 0 *satisfying*

$$\widetilde{\mathbb{P}}(\beta(0) = 0) = \frac{\mathbb{E}^*(\tau_0 - \sigma_0)^+}{\mathbb{E}^* T_N}, \tag{58}$$

while, for f *bounded and measurable function,*

$$\widetilde{\mathbb{E}}[f(\beta(0)); \beta(0) > 0] = \frac{1}{\mathbb{E}^* T_N}\mathbb{E}^*\left\{\sum_{i=0}^{N-1} \tau_i f(T_i - T_M) - (T_N - \sigma_0)\, f(T_{N-1} - T_M)\right\} + \frac{1}{\mathbb{E}^* T_N}\mathbb{E}^*\left\{(T_N - \sigma_0)\, f(T_{N-1})\mathbf{1}_{T_{N-1}>0}\right\}, \tag{59}$$

where $N = \inf\{\ell \geq 1 : \psi_\ell = 1\}$ *and* $M = \sup\{\ell \leq -1 : \psi_\ell = 1\}$.

Proof. Notice that $N = \inf\{\ell \geq 1 : T_\ell \geq \sigma_0\}$, $\mathbb{P}^*$-a.s. We use the Palm inversion formula:

$$\widetilde{\mathbb{E}} f(\beta(0)) = \frac{\mathbb{E}^* \int_{B_0}^{B_1} f(\beta(t))\, dt}{\mathbb{E}^*(B_1 - B_0)}. \tag{60}$$

Since M, N are the indices of the admitted messages nearest to 0,

$$B_{-1} = T_M \leq T_{-1} < T_0 = B_0 = 0 < T_1 < \cdots < T_{N-1} < \sigma_0 \leq T_N = B_1, \quad \mathbb{P}^*\text{-a.s.}$$

In particular, $B_1 - B_0 = T_N$, $\mathbb{P}^*$-a.s. Since $\beta(t) = A_t - S_{D_t}$, using (32) we have

$$\beta(t) = \begin{cases} T_i - T_M, & \text{if } T_0 \leq T_i \leq t < T_{i+1} \leq T_{N-1} \\ T_{N-1} - T_M, & \text{if } T_{N-1} \leq t < T_0 + \sigma_0 \\ T_{N-1} - T_0, & \text{if } T_0 + \sigma_0 \leq t < T_N \end{cases}.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and measurable. We write the integral in the numerator of (60) as:

$$\begin{aligned}
\int_{T_0}^{T_N} f(\beta(t))\, dt &= \int_{T_0}^{T_{N-1}} f(\beta(t))\, dt + \int_{T_{N-1}}^{T_0+\sigma_0} f(\beta(t))\, dt + \int_{T_0+\sigma_0}^{T_N} f(\beta(t))\, dt \\
&= \sum_{i=0}^{N-2} \int_{T_i}^{T_{i+1}} f(T_i - T_M) dt + \int_{T_{N-1}}^{T_0+\sigma_0} f(T_{N-1} - T_M) dt + \int_{T_0+\sigma_0}^{T_N} f(T_{N-1} - T_0) dt \\
&= \sum_{i=0}^{N-2} \tau_i f(T_i - T_M) + (\sigma_0 - T_{N-1})\, f(T_{N-1} - T_M) + (T_N - \sigma_0)\, f(T_{N-1}).
\end{aligned} \tag{61}$$

Add and subtract the term corresponding to $i = N-1$ to write the last line as

$$\begin{aligned}
&= \sum_{i=0}^{N-1} \tau_i f(T_i - T_M) - \tau_{N-1} f(T_{N-1} - T_M) + (\sigma_0 - T_{N-1})\, f(T_{N-1} - T_M) + (T_N - \sigma_0)\, f(T_{N-1}) \\
&= \sum_{i=0}^{N-1} \tau_i f(T_i - T_M) + (\sigma_0 - T_{N-1} - \tau_{N-1})\, f(T_{N-1} - T_M) + (T_N - \sigma_0)\, f(T_{N-1}) \\
&= \left\{\sum_{i=0}^{N-1} \tau_i f(T_i - T_M) - (T_N - \sigma_0)\, f(T_{N-1} - T_M)\right\} + (T_N - \sigma_0)\, f(T_{N-1}).
\end{aligned} \tag{62}$$

(For $f \geq 0$, the term in the bracket is positive because the last term of the sum is $\tau_{N-1} f(T_{N-1} - T_M)$ is bigger than $(T_N - \sigma_0)\, f(T_{N-1} - T_M)$ and this is because $\tau_{N-1} - (T_N - \sigma_0) = \sigma_0 - T_{N-1} > 0$.) By (60),

$$\mathbb{E}^* T_N\, \widetilde{\mathbb{E}} f(\beta(0)) = \mathbb{E}^* \left\{ \sum_{i=0}^{N-1} \tau_i f(T_i - T_M) - (T_N - \sigma_0)\, f(T_{N-1} - T_M) \right\} + \mathbb{E}^* \left\{ (T_N - \sigma_0)\, f(T_{N-1}) \right\}. \quad (63)$$

To reveal the atom of the $\widetilde{\mathbb{P}}$-law of $\beta(0)$ at 0, let

$$f(x) = \mathbf{1}_{x=0}.$$

Then $f(T_i - T_M) = 0$ because $T_M > 0$. Also, $f(T_{N-1}) = \mathbf{1}_{T_{N-1}=0} = \mathbf{1}_{N=1} = \mathbf{1}_{\tau_0 \geq \sigma_0}$. Hence

$$\mathbb{E}^* T_N\, \widetilde{\mathbb{P}}(\beta(0) = 0) = \mathbb{E}^* \left\{ (T_N - \sigma_0)\, \mathbf{1}_{N=1} \right\} = \mathbb{E}^* \left\{ (\tau_0 - \sigma_0)\, \mathbf{1}_{\tau_0 \geq \sigma_0} \right\} = \mathbb{E}^* (\tau_0 - \sigma_0)^+.$$

On the other hand,

$$\begin{aligned} \widetilde{\mathbb{E}}[f(\beta(0)); \beta(0) > 0] &= \widetilde{\mathbb{E}} f(\beta) - \widetilde{\mathbb{E}}[f(\beta(0)); \beta(0) = 0] \\ &= \widetilde{\mathbb{E}} f(\beta) - f(0) \widetilde{\mathbb{P}}(\beta(0) = 0) \\ &= \widetilde{\mathbb{E}}[f(\beta) - f(0) \mathbf{1}_{\beta(0)=0}] \equiv \widetilde{\mathbb{E}} g(\beta(0)), \end{aligned}$$

where

$$g(x) = f(x) - f(0) \mathbf{1}_{x=0}.$$

We use g in place of f in (63) after noting that $g(T_i - T_M) = f(T_i - T_M) - f(0) \mathbf{1}(T_i = T_M) = f(T_i - T_M)$ for $i \geq 0$, and $g(T_{N-1}) = f(T_{N-1}) - f(0) \mathbf{1}_{T_{N-1}=0} = f(T_{N-1}) - f(0) \mathbf{1}_{N=1}$. So

$$\begin{aligned} &\mathbb{E}^* T_N\, \widetilde{\mathbb{E}}[f(\beta(0)); \beta(0) > 0] = \\ &\quad = \mathbb{E}^* \left\{ \sum_{i=0}^{N-1} \tau_i f(T_i - T_M) - (T_N - \sigma_0)\, f(T_{N-1} - T_M) \right\} + \mathbb{E}^* \left\{ (T_N - \sigma_0)\, (f(T_{N-1}) - f(0) \mathbf{1}_{N=1}) \right\}. \end{aligned}$$

Notice that

$$f(T_{N-1}) - f(0) \mathbf{1}_{N=1} = f(T_{N-1}) - f(T_{N-1}) \mathbf{1}_{N=1} = f(T_{N-1})\, \mathbf{1}_{N>1} = f(T_{N-1})\, \mathbf{1}_{T_{N-1}>0}$$

and substitute into the last display to obtain the announced formula. $\square$

By Palm theory and stationarity, we have that $|M|$ and N have the same $\mathbb{P}^*$-law and so do $|T_M|$ and T_N. This simple fact is stated as an stand-alone lemma because it holds only under stationary assumptions and because it is needed when we explicitly compute distributions under independence assumptions.

Remark 4. We now give a physical meaning to the $\widetilde{\mathbb{P}}$-law of $\beta(0)$ conditional on $\beta(0) > 0$. Say that the message arriving at time T_n is *undisturbed* if it is admitted (and hence successful) and no other messages arrive during the time it is being processed; i.e., $\psi_n = 0$ and $T_n + \sigma_n \leq T_{n+1}$. Therefore, for $T_n \leq t < T_{n+1}$ we have $\beta(t) = 0$: undisturbed messages provided the freshest possible information; this is what contributes to the atom at 0 for $\beta(0)$. Define then an auxiliary system, pathwise, by removing all undisturbed messages. If $\beta_+(t)$ denotes the NAoI process for the auxiliary system then we have that, under $\widetilde{\mathbb{P}}$, $\beta(0)$ equals 0 with probability $\frac{\mathbb{E}^*(\tau_0 - \sigma_0)^+}{\mathbb{E}^* T_N}$ or $\beta_+(0)$ with the remaining probability. In particular,

$$\widetilde{\mathbb{E}}[f(\beta(0));\ \beta(0) > 0] = \widetilde{\mathbb{E}} f(\beta_+(0)).$$

Lemma 7. *Assume that* (τ_n, σ_n), $n \in \mathbb{Z}$, *is stationary under* $\mathbb{P}$. *Let* $N = \inf\{\ell \geq 1 : \psi_\ell = 1\}$ *and* $M = \sup\{\ell \leq -1 : \psi_\ell = 1\}$. *Then*

$$\mathbb{E}(g(-T_M)|\psi_0 = 1) = \mathbb{E}(g(T_N)|\psi_0 = 1),$$

for any bounded and measurable function g.

Proof. The point process $\sum_n \psi_n \delta_n$ is stationary under $\widetilde{\mathbb{P}}$ and the Palm probability of the latter with respect to this point process is denoted by $\mathbb{P}^*$. If $\cdots < T^*_{-1} < T^*_0 \leq 0 < T^*_1 < T^*_2 < \cdots$ is an enumeration of the points of $\sum_n \psi_n \delta_n$ in their natural order then $\mathbb{E}^* g(-T^*_{-1}) = \mathbb{E}^* g(T^*_1)$ for any bounded measurable function g. But $T^*_1 = T_N$ and $T^*_{-1} = T_M$ and $\mathbb{P} = \mathbb{P}^*(\cdot|\psi_0 = 1)$. □

Under i.i.d. assumptions, and because the decision on whether to admit a message or not is past-dependent, the ensued regeneration results into further simplification and the vanishing of the M from the formula. We explain this below. First fix $u \geq 0$ and consider the function $W_u(t)$ introduced in (42) as well as

$$V_u(t) := \mathbb{E} \sum_{i=0}^{N_t - 1} e^{-uT_i}, \quad t \geq 0 \tag{64}$$

$$Q_u(t) := \mathbb{E}\big\{(T_{N_t} - t)\, e^{-uT_{N_t-1}}\big\}, \quad t \geq 0. \tag{65}$$

Theorem 8. *Consider the fully blocking system and assume that* (τ_n, σ_n), $n \in \mathbb{Z}$, *is i.i.d. under* $\mathbb{P}$ *and such that* $\mathbb{E}\tau_0 < \infty$ *and* $\mathbb{P}(\tau_0 \geq \sigma_0) > 0$. *Assume further that* τ_n *is independent of* σ_n *for all* n. *Then* $\widetilde{\mathbb{P}}(\beta(0) = 0) = \frac{\mathbb{E}(\tau_0 - \sigma_0)^+}{\mathbb{E}T_N}$ *and*

$$\begin{aligned}\widetilde{\mathbb{E}}[e^{-u\beta(0)}; \beta(0) > 0] &= \frac{1}{\mathbb{E}T_N} \mathbb{E}e^{-uT_N} \left\{ \mathbb{E}\tau\, \mathbb{E} \sum_{i=0}^{N-1} e^{-uT_i} - \mathbb{E}(T_N - \sigma_0)\, e^{-uT_{N-1}} \right\} \\ &\qquad\qquad + \frac{1}{\mathbb{E}T_N} \mathbb{E}\left\{(T_N - \sigma_0)\, e^{-uT_{N-1}} \mathbf{1}_{T_{N-1} > 0}\right\} \\ &= \frac{\mathbb{E}W_u(\sigma)\,[\mathbb{E}\tau\, \mathbb{E}V_u(\sigma) - \mathbb{E}Q_u(\sigma)] + \mathbb{E}[e^{-u\tau} Q_u(\sigma - \tau)]}{\mathbb{E}\tau\, \mathbb{E}U(\sigma)},\end{aligned} \tag{66}$$

where U, W_u *are uique solutions to the fixed point equations* (44), (45), *respectively, while* V_u, Q_u *are unique solutions to*

$$V_u(t) = 1 + \int_{(0,t]} V_u(t - x)\, e^{-ux}\, \mathbb{P}(\tau \in dx), \tag{67}$$

$$Q_u(t) = \mathbb{E}(\tau - t)^+ + \int_{(0,t]} Q_u(t - x) e^{-ux} \mathbb{P}(\tau \in dx). \tag{68}$$

In particular, under $\widetilde{\mathbb{P}}$, *and conditional on* $\beta(0) > 0$, *the random variable* $\beta(0)$ *is absolutely continuous.*

***Remark* 5.** The term $\mathbb{E}\tau\, \mathbb{E}V_u(\sigma) - \mathbb{E}Q_u(\sigma)] + \mathbb{E}[e^{-u\tau} Q_u(\sigma - \tau)$ in (66) is nonnegative and this is due to the remark made below (62) about the nonnegativity of the bracketed term in (66).

Proof. The value of $\widetilde{\mathbb{P}}(\beta(0) = 0)$ follows from (58) and (30) that allows us to replace $\mathbb{E}^*$ by $\mathbb{E}$. To show the rest, we look at the various terms in (59) with $f(x) = e^{-ux}$. Using (30) we obtain

$$\mathbb{E}^* \sum_{i=0}^{N-1} \tau_i e^{-u(T_i - T_M)} = \mathbb{E}(e^{uT_M}|\psi_0 = 1)\, \mathbb{E} \sum_{i=0}^{N-1} \tau_i e^{-uT_i} \tag{69}$$

Due to Lemma 7, the first term of the product is further written as:

$$\mathbb{E}(e^{uT_M}|\psi_0 = 1) = \mathbb{E}(e^{-uT_N}|\psi_0 = 1) = \mathbb{E}e^{-uT_N}.$$

The second term in the last product of (69) is computed as follows.

$$\begin{aligned}\mathbb{E}\sum_{i=0}^{N-1}\tau_i e^{-uT_i} = \mathbb{E}\sum_{i=0}^{\infty}\tau_i e^{-uT_i}\mathbf{1}_{T_i<\sigma_0} &= \sum_{i=0}^{\infty}\mathbb{E}\{\mathbb{E}[\tau_i e^{-uT_i}\mathbf{1}_{T_i<\sigma_0}|\sigma_0,\tau_0,\dots,\tau_{i-1}]\} \\ &= \sum_{i=0}^{\infty}\mathbb{E}\{e^{-uT_i}\mathbf{1}_{T_i<\sigma_0}\mathbb{E}[\tau_i|\sigma_0,\tau_0,\dots,\tau_{i-1}]\} = (\mathbb{E}\tau)\,\mathbb{E}\sum_{i=0}^{N-1}e^{-uT_i}. \end{aligned} \tag{70}$$

Using the same logic,

$$\mathbb{E}^*(T_N-\sigma_0)e^{-u(T_{N-1}-T_M)} = \mathbb{E}^* e^{uT_M}\,\mathbb{E}(T_N-\sigma_0)\,e^{-uT_{N-1}} = \mathbb{E}e^{uT_M}\,\mathbb{E}(T_N-\sigma_0)\,e^{-uT_{N-1}} \tag{71}$$

$$\mathbb{E}^*\left\{(T_N-\sigma_0)\,f(T_{N-1})\mathbf{1}_{T_{N-1}>0}\right\} = \mathbb{E}\left\{(T_N-\sigma_0)\,f(T_{N-1})\mathbf{1}_{T_{N-1}>0}\right\} \tag{72}$$

Substituting (70) into (69) and then this, together with (71) and (72), into (59) we arrive at the first equality for (66). For the second equality, use (48), (49) and (42), (64), (65) and observe that

$$\mathbb{E}\big\{(T_{N_t}-t)\,e^{-uT_{N_t-1}}\mathbf{1}_{T_{N_t}-1>0}\big\} = \mathbb{E}[e^{-u\tau}\,Q_u(t-\tau)].$$

To see that V_u satisfies (67), notice that

$$\begin{aligned} V_u(t) &= \mathbb{E}\left[\sum_{i=0}^{N_t-1}e^{-uT_i};\, T_1>t\right] + \mathbb{E}\left[\sum_{i=0}^{N_t-1}e^{-uT_i};\, T_1\le t\right] \\ &= \mathbb{E}[e^{-uT_0};\, T_1>t] + \mathbb{E}\left[e^{-uT_0}+e^{-uT_1}V_u(t-T_1);\, t-T_1\ge 0\right] \\ &= e^{-u0} + \mathbb{E}[e^{-u\tau}V_u(t-\tau)\mathbf{1}_{\tau\le t}]. \end{aligned}$$

To see that Q_u satisfies (68), notice that

$$\begin{aligned} Q_u(t) &= \mathbb{E}\left[(T_{N_t}-t)\,e^{-uT_{N_t-1}};\, T_1>t\right] + \mathbb{E}\left[(T_{N_t}-t)\,e^{-uT_{N_t-1}};\, T_1\le t\right] \\ &= \mathbb{E}\left[(T_1-t)\,e^{-uT_0};\, T_1>t\right] + \int \mathbb{E}\left[(x+T_{N_{t-x}}-t)\,e^{-u(x+T_{N_{t-x}-1})}\right]\mathbf{1}_{x\le t}\,\mathbb{P}(T_1\in dx) \\ &= \mathbb{E}\left[(\tau-t)\,e^{-u0};\, \tau>t\right] + \int_{(0,t]} e^{-ux}\,\mathbb{E}\left[(T_{N_{t-x}}-(t-x))\,e^{-u(T_{N_{t-x}-1})}\right]\mathbb{P}(\tau\in dx) \\ &= \mathbb{E}(\tau-t)^+ + \int_{(0,t]} e^{-ux}\,Q_u(t-x)\,\mathbb{P}(\tau\in dx). \end{aligned}$$

□

Continuing in the same manner as Lemma 6, we obtain the Laplace transforms of V_u and Q_u.

Lemma 8.

$$\widehat{V}_u(\xi) = \frac{1/\xi}{1-\mathbb{E}e^{-(u+\xi)\tau}} \tag{73}$$

$$\widehat{Q}_u(\xi) = \frac{1}{\xi^2}\frac{\xi\mathbb{E}\tau - 1 + \mathbb{E}e^{-\xi\tau}}{1-\mathbb{E}e^{-(u+\xi)\tau}} \tag{74}$$

Proof. Directly from (67) and (68). $\square$

Corollary 10. *Let the assumptions of Theorem 4 hold true.*
(i) If the variables τ_n are exponential with rate λ, then

$$\widetilde{\mathbb{P}}(\beta(0)=0) = \frac{\mathbb{E}e^{-\lambda\sigma}}{1+\lambda\mathbb{E}\sigma}$$

and, with $L_\sigma(u) = \mathbb{E}e^{-u\sigma}$,

$$\widetilde{\mathbb{E}}[e^{-u\beta(0)};\beta(0)>0] = \frac{\lambda}{1+\lambda\mathbb{E}\sigma}\left[\frac{L_\sigma(u)}{\lambda+u}\,\frac{\lambda^2(1-L_\sigma(u)) - u^2(1-L_\sigma(\lambda))}{u(\lambda-u)} + \frac{L_\sigma(u)-L_\sigma(\lambda)}{\lambda-u}\right]$$

(ii) If the variables σ_n are exponential with rate μ, then, with $L_\tau(u) = \mathbb{E}e^{-u\tau}$,

$$\widetilde{\mathbb{P}}(\beta(0)=0) = \frac{1}{\mu\mathbb{E}\tau}(1-\mathbb{E}e^{-\mu\tau})(\mu\mathbb{E}\tau - 1 - \mathbb{E}e^{-\mu\tau}),$$

$$\widetilde{\mathbb{E}}[e^{-u\beta(0)};\beta(0)>0] = \frac{1-L_\tau(\mu)}{\mu\mathbb{E}\tau(1-L_\tau(u+\mu))}\left[\frac{L_\tau(u)-L_\tau(u+\mu)}{1-L_\tau(u+\mu)}(1-L_\tau(\mu)) + L_\tau(u+\mu)\,(\mu\mathbb{E}\tau - 1 - L_\tau(\mu))\right]$$

(iii) If the τ_n are exponential with rate λ, and the σ_n are exponential with rate μ then, under $\widetilde{\mathbb{P}}$,

$$\beta(0) \overset{\text{(d)}}{=} \begin{cases} 0, & \textit{with probability } \frac{\mu^2}{(\lambda+\mu)^2} \\ \zeta, & \textit{with probability } \frac{\lambda(\lambda+2\mu)}{(\lambda+\mu)^2} \end{cases},$$

where ζ is an absolutely continuous random variable with

$$\mathbb{E}e^{-u\zeta} = \frac{\mu^2}{\lambda+2\mu}\,\frac{u^2+(2\lambda+\mu)u+\lambda(\lambda+2\mu)}{(u+\lambda)(u+\mu)^2}.$$

Proof. From Theorem (8), we have $\widetilde{\mathbb{P}}(\beta(0)=0) = \lambda\mathbb{E}(\tau-\sigma)^+/\mathbb{E}U(\sigma)$ and the expressions of this are obtained by elementary integrals in all cases. We rewrite (66) as

$$\widetilde{\mathbb{E}}[e^{-u\beta(0)};\beta(0)>0] = \frac{\mathbb{E}W_u(\sigma)\,\mathbb{E}H_u(\sigma) + \mathbb{E}Q_u^+(\sigma)}{\mathbb{E}\tau\,\mathbb{E}U(\sigma)}, \tag{75}$$

where

$$H_u(t) = \mathbb{E}\tau\, V_u(t) - Q_u(t), \quad Q_u^+(t) = \mathbb{E}[e^{-u\tau}\,Q_u(t-\tau)].$$

We thus know the Laplace transforms of all functions entering in (75) in terms of $L_\tau(\xi) := \mathbb{E}e^{-\xi\tau}$:

$$\widehat{U}(\xi) = \frac{1/\xi}{1-L_\tau(\xi)}, \quad \widehat{W}_u(\xi) = \frac{1}{\xi}\frac{L_\tau(u)-L_\tau(u+\xi)}{1-L_\tau(u+\xi)},$$
$$\widehat{H}_u(\xi) = \frac{1}{\xi^2}\frac{1-L_\tau(\xi)}{1-L_\tau(u+\xi)}, \quad \widehat{Q}_u^+(\xi) = \frac{L_\tau(u+\xi)}{\xi^2}\,\frac{\xi\mathbb{E}\tau - 1 + L_\tau(\xi)}{1-L_\tau(u+\xi)}.$$

(i) When τ is exponential, we already know that $U(t) = 1+\lambda t$ and that $W_u(t) = \lambda e^{-ut}/(\lambda+u)$ and, with $L_\tau(u) = \lambda/(\lambda+u)$, we obtain

$$\widehat{H}_u(\xi) = \frac{\lambda+u+\xi}{\xi(\lambda+\xi)(u+\xi)}, \quad \widehat{Q}_u^+(\xi) = \frac{1}{(u+\xi)(\lambda+\xi)},$$

that can easily be inverted to the nonnegative functions

$$H_u(t) = \frac{\lambda^2(1-e^{-ut}) - u^2(1-e^{-\lambda t})}{\lambda u(\lambda - u)}, \quad Q_u^+(t) = \frac{e^{-ut} - e^{-\lambda t}}{\lambda - u}.$$

The values of H_u and Q_u^+ at $u = \lambda$ should be interpreted as limits when $u \to \lambda$. Thus, $H_\lambda(t) = \lambda^{-1}[2 - (\lambda t + 2)]e^{-\lambda t}$, $Q_\lambda^+(t) = te^{-\lambda t}$. Substitute these functions in (75) to obtain the annouced formula.
(ii) When σ is exponential with rate μ, all functions in (75) are essentially Laplace tranforms of σ, for example, $\mathbb{E}W_u(\sigma) = \mu \widehat{W}_u(\mu)$. Hence

$$\widetilde{\mathbb{E}}[e^{-u\beta(0)}; \beta(0) > 0] = \frac{\mu \widehat{W}_u(\mu)\, \mu \widehat{H}_u(\mu) + \mu \widehat{Q}_u^+(\mu)}{\mathbb{E}\tau\, \mu \widehat{U}(\mu)},$$

and the formula is obtained because we know all Laplace transforms.
(iii) The formula readily follows from either (i) or (ii). □

Let us take a closer look at the law of the random variable ζ of Corollary 10(iii). Letting $\rho = \lambda\mu$ we have

$$\mathbb{E}e^{-u\mu\zeta} = \frac{1}{\rho+2} \frac{u^2 + (2\rho+1)u + \rho(\rho+2)}{(u+\rho)(u+1)^2}.$$

Inverting this Laplace transform, we find that $\mu\zeta$ has density

$$g_\rho(t) = \frac{1}{(\rho+2)(\rho-1)^2} \big[\rho e^{-\rho t} + (\rho^3 - 3\rho + 1 + \rho^2(\rho-1)t)e^{-t}\big],$$

for all values of $\rho \neq 1$ and, for $\rho = 1$, the density corresponds to the limit of this expression when $\rho \to 1$:

$$g_1(t) = \frac{1}{6}(t^2 + 2t + 2)e^{-t}.$$

We now pass on to computing first moments.

Lemma 9. *Consider the fully blocking system under stationarity assumptions. Then*

$$\widetilde{\mathbb{E}}\beta(0) = \frac{\mathbb{E}^* \left[\sum_{i=0}^{N-1} \tau_i T_i - \sigma_0 T_M \right]}{\mathbb{E}^* T_N} \tag{76}$$

Proof. Take $f(x) = x$ in (61) and regroup the terms there to obtain

$$\int_{T_0}^{T_N} \beta(t) dt = \sum_{i=0}^{N-1} \tau_i (T_i - T_M) + (T_N - \sigma_0)\, T_M = \sum_{i=0}^{N-1} \tau_i T_i - \sigma_0 T_M$$

and then use the Palm inversion formula. □

Next define

$$Z(t) = \mathbb{E} \sum_{i=0}^{N_t - 1} T_i, \quad t \geq 0. \tag{77}$$

Lemma 10. *Consider the fully blocking system and assume that (τ_n, σ_n), $n \in \mathbb{Z}$, is i.i.d. under $\mathbb{P}$ and such that $\mathbb{E}\tau_0 < \infty$. Assume further that τ_n is independent of σ_n for all n. Then*

$$\widetilde{\mathbb{E}}\beta(0) = \mathbb{E}\sigma + \frac{\mathbb{E}\left[\sum_{i=0}^{N-1} T_i\right]}{\mathbb{E}N} = \mathbb{E}\sigma + \frac{\mathbb{E}Z(\sigma)}{\mathbb{E}U(\sigma)},$$

where Z is the unique solution to the fixed-point equation

$$Z(t) = \mathbb{E}[Z(t-\tau)] + \mathbb{E}[\tau U(t-\tau)]$$

and has Laplace transform

$$\widehat{Z}(\xi) = \frac{\mathbb{E}\tau e^{-\xi\tau}}{\xi(1-\mathbb{E}e^{-\xi\tau})^2}.$$

Proof. The numerator of (76) is written as

$$\begin{aligned}\mathbb{E}^*\left[\sum_{i=0}^{N-1} \tau_i T_i - \sigma_0 T_M\right] &= \mathbb{E}^* \sum_{i=0}^{N-1} \tau_i T_i + \mathbb{E}^*\sigma_0(-T_M) \\ &= \mathbb{E}\tau\,\mathbb{E}\left[\sum_{i=0}^{N-1} T_i\right] + \mathbb{E}(-T_M|\psi_0 = 1)\,\mathbb{E}\sigma \\ &= \mathbb{E}\tau\,\mathbb{E}\left[\sum_{i=0}^{N-1} T_i\right] + \mathbb{E}T_N\,\mathbb{E}\sigma.\end{aligned}$$

Dividing this by $\mathbb{E}T_N = \mathbb{E}\tau\,\mathbb{E}N$ results in the first equality. Next use the function (77) to write $\mathbb{E}\left[\sum_{i=0}^{N-1} T_i\right] = \mathbb{E}Z(\sigma)$. The fixed point equation is obtained from first principles or by differentiating both sides of (67) with respect to u and letting $u \to 0$. The Laplace transform is obtained by taking the Laplace transform of both sides of the fixed-point equation. □

Corollary 11. *Let the assumptions of Theorem 4 hold true.*
(i) If the variables τ_n are exponential with rate λ, then

$$\widetilde{\mathbb{E}}\beta(0) = \mathbb{E}\sigma + \frac{\lambda}{2}\frac{\mathbb{E}\sigma^2}{1+\lambda\mathbb{E}\sigma}.$$

(ii) If the variables σ_n are exponential with rate μ, then, with $L_u = \mathbb{E}e^{-u\tau}$,

$$\widetilde{\mathbb{E}}\beta(0) = \frac{1}{\mu} + \frac{\mathbb{E}\tau e^{-\mu\tau}}{1-\mathbb{E}e^{-\mu\tau}}$$

(iii) If the τ_n are exponential with rate λ, and the σ_n are exponential with rate μ then, under $\widetilde{\mathbb{P}}$,

$$\widetilde{\mathbb{E}}\beta(0) = \frac{1}{\mu} + \frac{\lambda}{\mu(\lambda+\mu)}.$$

6 Further discussion

We studied the performance of two measures of freshness of information (AoI and NAoI) for two policies: fully push-out $\mathcal{P}$ and fully blocking $\mathcal{B}$ and computed their Laplace transforms in steady-state. We briefly hinted that the choice of policy is important and it is not clear that $\mathcal{P}$ always outperforms $\mathcal{B}$. The choice

model	push out ($\mathcal{P}$)		blocking ($\mathcal{B}$)	
	AoI ($\widetilde{\mathbb{E}}\alpha_{\mathcal{P}}(0)$)	NAoI ($\widetilde{\mathbb{E}}_{\mathcal{P}}\beta(0)$)	AoI ($\widetilde{\mathbb{E}}\alpha_{\mathcal{B}}(0)$)	NAoI ($\widetilde{\mathbb{E}}\beta_{\mathcal{B}}(0)$)
G/G	$\frac{\mathbb{E}\tau^2}{2\mathbb{E}\tau} + \frac{\mathbb{E}\tau\wedge\sigma}{\mathbb{P}(\tau\geq\sigma)}$	$\frac{\mathbb{E}\tau\wedge\sigma}{\mathbb{P}(\tau\geq\sigma)}$	$\frac{1}{\mu} + \frac{\mathbb{E}\tau^2}{2\mathbb{E}\tau} + \frac{\mathbb{E}\tau(U*U)(\sigma-\tau)}{\mathbb{E}U(\tau)}$	$\frac{1}{\mu} + \frac{\mathbb{E}Z(\sigma)}{EU(\sigma)}$
M/G	$\frac{1}{\lambda\mathbb{E}e^{-\lambda\sigma}}$	$\frac{1}{\lambda\mathbb{E}e^{-\lambda\sigma}} - \frac{1}{\lambda}$	$\frac{1}{\mu} + \frac{1}{\lambda} + \frac{\lambda}{2}\frac{\mathbb{E}\sigma^2}{1+\lambda\mathbb{E}\sigma}$	$\frac{1}{\mu} + \frac{\lambda}{2}\frac{\mathbb{E}\sigma^2}{1+\lambda/\mu}$
G/M	$\frac{\mathbb{E}\tau^2}{2\mathbb{E}\tau} + \frac{1}{\mu}$	$\frac{1}{\mu}$	$\frac{1}{\mu} + \frac{\mathbb{E}\tau^2}{2\mathbb{E}\tau} + \frac{\mathbb{E}\tau e^{-\mu\tau}}{1-\mathbb{E}e^{-\mu\tau}}$	$\frac{1}{\mu} + \frac{\mathbb{E}\tau e^{-\mu\tau}}{1-\mathbb{E}e^{-\mu\tau}}$
M/M	$\frac{1}{\lambda} + \frac{1}{\mu}$	$\frac{1}{\mu}$	$\frac{1}{\mu} + \frac{1}{\lambda} + \frac{\lambda}{\mu(\lambda+\mu)}$	$\frac{1}{\mu} + \frac{\lambda}{\mu(\lambda+\mu)}$

Table 1: Mean AoI and NAoI for different models of interarrival times (with $\mathbb{E}\tau = 1/\lambda$) and service times (with $\mathbb{E}\sigma = 1/\mu$) in the renewal case. We use queueing theoretic notation, where "G" means general distribution and "M" means memoryless (exponential).

depends on the distributions. To explain this, let us look at the formulas for the expectations of AoI and NAoI in various cases. They are summarized in Table 1 below.

Suppose first that we are in the "M/M" case, that is, both processes are Poissonian. Look at the last line of the table. Then the $\mathcal{P}$ policy outperforms the $\mathcal{B}$ policy because it gives smaller mean for AoI and NAoI.

Suppose next that we are in the "M/D" case, meaning that the reading times are all deterministic and all equal to a constant that, without loss of generality, can be taken to be equal to 1. Look at the second line of the table, remove the expectation and set $\sigma = 1/\mu = 1$. Using the inequality $e^x > 1 + x + x^2/2$ we can easily see that, for all $\lambda > 0$,

$$\frac{1}{\lambda e^{-\lambda}} > 1 + \frac{1}{\lambda} + \frac{\lambda}{2}\frac{1}{1+\lambda},$$

and this means that $\mathcal{B}$ outperforms $\mathcal{P}$ in this case. The intuition here is this: if σ is concentrated around its mean then it is better to completely read a message rather than push it out by an incoming one; the latter is more beneficial if, having partially read a message, we still know little about its actual duration.

To say that $\mathcal{B}$ outperforms $\mathcal{P}$ if σ is approximately deterministic is true only because we considered expectations performance measures If, for example, we consider tail probabilities as performance measures then more care is needed in order to justify this.

Even though we worked with Laplace transforms and derived, in certain cases, the density of AoI and the density of the NAoI conditional that it be positive, by Laplace transform inversion. We may alternately obtain expressions for the probability densities by using level-crossing arguments as in, e.g., [3]. We should also point out the genearility of the formulas obtained in Theorems 1, 3, 5 and 7: they hold only under stationarity assumptions. Therefore, we can, for example, incroporate situations where messages arrive according to more general than renewal processes.

Recall Remark 2 in Section 4.2. Since $\widetilde{\mathbb{E}}\alpha_{\mathcal{P}}(0) = \frac{\mathbb{E}\tau^2}{2\mathbb{E}\tau} + \widetilde{\mathbb{E}}\beta_{\mathcal{P}}(0)$, it follows that $\widetilde{\mathbb{E}}\alpha_{\mathcal{P}}(0)$ can be arbitrarily large if the variance of τ is large. This can be a drawback in using the AoI rather than the NAoI as a performance criterion. Indeed, as explained in the paper, if the arrivals are not controllable (and, typically, they are not) then it makes more sense to use NAoI as a measure of freshness of information.

Alternative definitions of age of information are possible and may be desirable. For example, a measure of freshness of information may involve message streams where the most recent message does *not* obsolete all previous ones. More specifically, assume that, upon arrival of a new message (with normalized "importance" 1), the importance of all prior messages can be diminished by a positive factor $\xi < 1$, and the objective could be to minimize the sum of the importance of all transmitted messages. This case may require a large (perhaps lossless) message buffer.

Consider now the claim made at the end of the Introduction: that bufferless systems perform best. We ran several simulations that support this. We merely present a typical one. Suppose all arrivals are

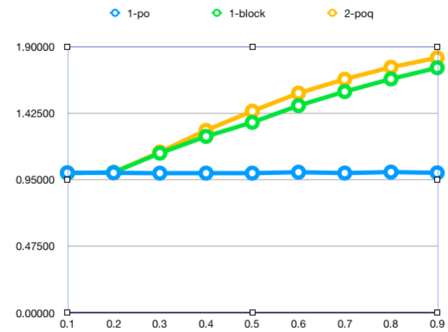

Figure 3: *Messages of fixed length equal to 1 arrive according to a Poisson(λ) process. The figure plots the mean NAoI as a function of λ in three cases: the bufferless fully push-out system (1-po), the bufferless fully blocking system (1-block) and a variant of the fully push-out sustem where a message can be temporarily stored in a buffer (2-poq). The curve in the last case is obtained via stochastic simulation.*

deterministic: messages arrive every $1/\lambda$ time units. Suppose message lengths are exponential with rate $\mu = 1$. Compare three systems. First the bufferless fully push-out system. Then Table 1, third row, tells us that the mean NAoI is $1/\mu = 1$. Second, the bufferless fully blocking system. Table 1 again tells us that the mean NAoI is

$$1 + (\lambda(e^{1/\lambda} - 1))^{-1}.$$

This, as a function of λ, is plotted in Figure 3; it is the curve labeled "1-block". Third, consider the following system that can store at most one message. Suppose a message, say message 1, is being read and that another message. say 2, arrives during the reading period of 1. Then it is stored. If no other message arrives while 1 is being read then message 2 starts being read and an empty position is created for the storage of a new message, say message 3, while 2 is being read. Suppose, however, that a message arrives while 1 is being read and 2 is in storage. Then the new message pushes message 2 out and occupies its position. This system is thus an extension of the bufferless fully push-out system. Simulations show that the mean NAoI is worse: it is plotted as the curve labeled "2-poq" in Figure 3.

The situation described in the last paragraph is typical. It is for this reason that we study bufferless systems. In the literature, systems with infinite storage have been studied. See, for instance, [2] where a LIFO (last-in first-out) infinite storage system with no service preemption is considered and where it is observed that serving an older message has no effect on the age of infrmation after a newer message has been served. Note that a LIFO system with nonzero buffer space and service preemption has the same NAoI as the bufferless push-out system.

Hence, the search for the optimal policy can be confined within the class of bufferless systems only. We proposed the new age of information as an alternative performance measure. Thus, if we need the age to be less than a given critical level, we need to find the best policy that minimizes the probability that the new age of information exceeds this critical level. Such problems are to be considered in future work.

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