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NIKOS E. FRANGOS
PETER IMKELLER

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Some inequalities for strong martingales (*)

by

Nikos E. FRANGOS

Hofstra University,
Department of Mathematics,
Hempstead, NY 11550

and

Peter IMKELLER

Mathematisches Institut der Ludwig-Maximilians-Universitat München,
Theresienstrasse 39,
8000 München 2, Federal Republic of Germany

ABSTRACT. — We extend Walsh's maximal inequality for strong martingales to transforms of strong martingales by bounded previsible processes and show that they converge a.s. Using Fefferman's inequality and bounded mean oscillation of dual optional projections of processes whose variation is bounded by a constant we derive Davis' inequality for strong martingales.

Key words : Two-parameter strong martingales, martingale transforms, maximal inequality, Davis' inequality.

RÉSUMÉ. — On étend l'inégalité maximale de Walsh pour des martingales fortes à des transformées de martingales fortes avec des processus prévisibles bornés et on montre qu'elles convergent p. s. En utilisant l'inégalité de Fefferman et BMO des projections duales optionnelles de processus dont la variation est bornée par une constante, on dérive l'inégalité de Davis pour les martingales fortes.

Classification A.M.S. : Primary 60 G 48, secondary 60 G 42, 60 E 15.

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INTRODUCTION

Among multi-parameter martingales, strong martingales have the closest relationships with one-parameter theory. This fact is underlined for example by Walsh's [11] maximal inequality which implies that, as in the theory of one-parameter martingales, L^1-boundedness ensures the existence of regular versions. This is in contrast to the behaviour of more general two-parameter martingales for which one needs $L\log^+ L$-boundedness (*see* Cairoli, Walsh [5]). Also, a version of the optional stopping theorem with respect to stopping domains holds for strong martingales. It implies that a discrete two-parameter space can be transformed in various ways by an increasing family of stopping domains into a one-parameter setting.

The latter fact is exploited in the first part of this paper where a version of Walsh's maximal inequality is shown to hold for transforms of strong martingales by bounded predictable processes (theorem 1). It is similar to the weak inequality for transforms of martingales by Burkholder ([2], [4]) which by now is classical. Also classical is Burkholder's ([2], [3]) result that transforms of L^1-bounded martingales by bounded predictable processes converge a. s. With the help of the inequality of theorem 1 this result is shown to hold for strong martingales (theorem 2).

In the second part, a proof of Davis' inequality for strong martingales is presented, the "easy half" of which has been known for some time (see Brossard [1], Theorem 3). For the "harder half", the domination of the maximum function by the square function, the methods of proof of the classical Davis' inequality via Fefferman's inequality (*see* Dellacherie, Meyer [7]) are used. Here it is important to know that dual optional projections of processes of variation bounded by constants have bounded mean oscillation. This proof, unfortunately, does not carry over to non-strong martingales.

0. NOTATIONS AND DEFINITIONS

The processes we consider are parametrized by $I=\mathbf{Z}_+^2$ or $I=[0, n]$ for some $n\in\mathbf{Z}_+^2$. Here intervals in $\mathbf{Z}_+^2$ are defined with respect to the usual partial ordering, coordinatewise linear order. Coordinates of "time points" $i\in\mathbf{Z}_+^2$ are usually denoted by lower indices, i. e. $i=(i_1, i_2)$. For intervals I in $\mathbf{Z}_+^2$ we write $I=I_1\times I_2$ with an obvious meaning. If there is no ambiguity, numbers $k\in\mathbf{Z}_+$ also denote the vector $(k, k)\in\mathbf{Z}_+^2$, for example $i-1=(i_1-1, i_2-1)$ for $i\in\mathbf{N}^2$. For functions $f: \mathbf{Z}_+^2\to\mathbf{R}$, $g: \mathbf{Z}_+\to\mathbf{R}$, intervals $J=[i, j]\subset\mathbf{Z}_+^2$, $K=[k, l]\subset\mathbf{Z}_+$, we define

$$\Box_J f=f(j)-f(i_1, j_2)-f(j_1, i_2)+f(i),$$
$$\Delta_K g=g(l)-g(k).$$

The filtration on our basic probability space $(\Omega, \mathscr{F}, P)$ is denoted by $(\mathscr{F}_i)_{i \in \mathbf{Z}_+^2}$. Moreover, for $i \in \mathbf{Z}_+^2$, $\mathscr{F}_{i_1}^1 = \bigvee_{j_2 \in \mathbf{Z}_+} \mathscr{F}_{(i_1, j_2)}$, $\mathscr{F}_{i_2}^2 = \bigvee_{j_1 \in \mathbf{Z}_+} \mathscr{F}_{(j_1, i_2)}$. All processes we consider are assumed to be zero on $\partial \mathbf{R}_+^2 \cap \mathbf{Z}_+^2$ (and real-valued). A process V is called "previsible", if V_i is $\mathscr{F}_{(i_1 - 1, i_2)} \vee \mathscr{F}_{(i_1, i_2 - 1)}$-measurable for any $i \in \mathbf{N}^2$. If A, B are processes, the symbol $A * B$ stands for the integral process of A w. r. to B, i. e. $A * B_i = \sum_{i \geqq j \in \mathbf{N}^2} A_j \, \square_{[j-1, j]} B$ for $i \in \mathbf{Z}_+^2$. *Mutatis mutandis* we define the integral process for one-parameter processes A, B and denote it by A.B. The "variation" of A is the random variable $\sum_{i \in I} |\square_i A|$. A random set $D \subset \Omega \times \mathbf{Z}_+^2$ is called "stopping domain", if $i \in D$ implies $[0, i] \subset D$ and $\{i \in D\} \in \mathscr{F}_{(i-1) \vee 0}$ for all $i \in \mathbf{Z}_+^2$ (*see* Walsh [11], p. 179). A process $(M_i)_{i \in I}$ is called "strong martingale", if M_i is $\mathscr{F}_i$-measurable and integrable and $E(\square_{[i-1, i]} M \mid \mathscr{F}_{i_1 - 1}^1 \vee \mathscr{F}_{i_2 - 1}^2) = 0$ for all $i \in I \cap \mathbf{N}^2$. It is obvious that if M is a strong martingale, V a bounded previsible process, $V * M$ is again a strong martingale. The "square function" of a strong martingale M is denoted by [M] and $[M] = \sum_{i \in \mathbf{N}^2 \cap I} (\square_{[i-1, i]} M)^2$. For the definition of F(D) (past of a stopping domain D) and M(D) (strong martingale M, stopped on D), *see* Walsh [11].

1. WEAK INEQUALITY AND CONVERGENCE OF TRANSFORMS

We use the fact that strong martingales are "one-parameter martingales in many disguises" to prove a weak inequality for their transforms by bounded previsible processes. This inequality implies, as in Walsh [11], that transforms of L^1-bounded strong martingales converge a. s. Remember that, like all processes we consider here, our strong martingales are supposed to vanish on the axes.

THEOREM 1. — *Let* $\alpha > 0$. *For any strong martingale* M, *any predictable process* V *which is bounded by* α *any* $\lambda > 0$

$$\lambda P(\sup_{i \in \mathbf{N}^2} |V * M|_i \geqq \lambda) \leqq 12 \alpha \sup_{i \in \mathbf{N}^2} \|M_i\|_1.$$

Proof. — An easy extension argument shows that we may confine our attention to strong martingales indexed by $I = [0, n]$ for some $n \in \mathbf{N}^2$. Fix a strong martingale M and for $\lambda > 0$ let

$$D^\lambda = \bigcup_{i \in I} \{[0, i+1] : |V * M|_j < \lambda \text{ for all } j \in [0, i]\} \cap I,$$

L^λ its "upper boundary" (*see* Walsh [11], pp. 180-183 for this and the following decomposition of martingales along stopping lines). For $i \in L^\lambda$ we have

$$\begin{aligned} V * M_i &= V * M(D^\lambda \cap [0, (i_1, n_2)]) \\ &\quad + V * M(D^\lambda \cap [0, (n_1, i_2)]) - V * M(D^\lambda) \\ &= V^\lambda * M_{(i_1, n_2)} + V^\lambda * M_{(n_1, i_2)} - V^\lambda * M_n, \quad (1.1) \end{aligned}$$

where $V^\lambda = 1_{\{. \in D^\lambda\}} V$. Note that, since $\{i \in D^\lambda\} \in \mathscr{F}_{i-1}$, $1 \leqq i \leqq n$, V^λ is again previsible and bounded by α. By definition of D^λ, (1.1) implies

$$\begin{aligned} P(\sup_{i \in I} |V * M|_i \geqq \lambda) &= P(\sup_{i \in L^\lambda} |V * M|_i \geqq \lambda) \\ &\leqq P(\sup_{i_1 \in I_1} |V^\lambda * M|_{(i_1, n_2)} \geqq \lambda/3) \\ &\quad + P(\sup_{i_2 \in I_2} |V^\lambda * M|_{(n_1 . i_2)} \geqq \lambda/3) \\ &\quad (\text{since } |V^\lambda * M|_n \leqq \sup_{i_1 \in I_1} |V^\lambda * M|_{(i_1, n_2)}). \quad (1.2) \end{aligned}$$

Consider the first term on the right hand side of (1.2). For $1 \leqq k \leqq n_1$, $0 \leqq l \leqq n_2$, let

$$D_{(k-1).n_2+l} := [(0, 0), (k-1, n_2)] \cup [(k-1, 0), (k, l)],$$
$$W^\lambda_{(k-1).n_2+l} := V^\lambda_{(k, l)}.$$

Then according to Walsh [11], p. 181,

$$N_k := M(D_k) \quad (1.3)$$

is a martingale with respect to the filtration $\mathscr{G}_k = \mathscr{F}(D_k)$, $0 \leqq k \leqq n_1 n_2$. Moreover, since by definition, $W^\lambda . N_{(i_1-1) n_2 + n_2} = V^\lambda * M_{(i_1, n_2)}$ for $i_i \in I_1$, we have

$$\sup_{i_1 \in I_1} |V^\lambda * M|_{(i_1, n_2)} \leqq \sup_{1 \leqq k \leqq n_1 n_2} |W^\lambda . N|_k,$$

where $W^\lambda . N$ is the transform of the one-parameter martingale N by the $(\mathscr{G}_k)_{0 \leqq k \leqq n_1 n_2}$-previsible process W^λ which, in addition, is bounded by α. Therefore, by theorem 2.1 of Burkholder [4]

$$\begin{aligned} \lambda P(\sup_{i_1 \in I_1} |V^\lambda * M|_{(i_1, n_2)} \geqq \lambda/3) &\leqq \lambda P(\sup_{1 \leqq k \leqq n_1 n_2} |W^\lambda . N|_k \geqq \lambda/3) \\ &\leqq 6 \alpha \| N_{n_1 n_2} \|_1 = 6 \alpha \| M_n \|_1. \quad (1.4) \end{aligned}$$

An analogous inequality for the second term on the right hand side of (1.2) is readily obtained with the help of a corresponding one-parameter arrangement of M with respect to the other coordinate axis. Substitution of (1.4) and its analogue in (1.2) gives the desired inequality. □

THEOREM 2. — *For any strong martingale* M *which is bounded in* L^1, *any bounded previsible process* V, *the transform* $V * M$ *converges a. s.*

Proof. — Fix an L^1-bounded strong martingale M and a bounded previsible process V. Observe that $V * M$ is a strong martingale. Use lemma 3.6 of Walsh [11], p. 184, to choose an increasing sequence $(n_k)_{k \in \mathbf{N}}$ in $\mathbf{N}^2$ and a sequence $(A_k)_{k \in \mathbf{N}}$ of sets such that

$$A_k \in \mathscr{F}_{n_k}, \qquad P(A_k) \geqq 1-2^{-k}, \qquad k \in \mathbf{N}, \tag{1.5}$$

$$E(1_{A_k} \left| V * M_n - V * M_{n_k} \right|) < 4^{-k} \quad \text{for all } n \geqq n_k, \qquad k \in \mathbf{N}. \tag{1.6}$$

Note that, according to Burkholder [2], p. 1496, the one-parameter martingale $(V * M_{n_k})_{k \in \mathbf{N}}$ has an a. s. limit $V * M_\infty$. Walsh's ([11], p. 185) proof now deduces from (1.4), (1.5) and theorem 1 that $V * M$ converges a. s. to $V * M_\infty$. □

2. DAVIS' INEQUALITY

In the proof of Theorem 1 we already made use of the remarkable fact that strong martingales can be arranged as one-parameter martingales in many ways. Due to this observation, one part of the inequality presented here has been known (see Brossard [1], p. 118). For the sake of completeness, however, we prove it here together with the "hard part" whose proof rests upon the observation that dual optional projections in the context of strong martingale theory can be defined in an essentially one-parameter way. In consequence of that, a version of the proof of Davis' inequality via Fefferman's inequality works.

PROPOSITION 1. — *There is a constant c such that for any strong martingale* M

$$E([M]^{1/2}) \leqq c\, E(\sup_{i \in \mathbf{N}^2} |M_i|).$$

Proof. — By monotone convergence, it is enough to consider martingales indexed by $I = [0, n]$ for some $n \in \mathbf{N}^2$. For $0 \leqq k \leqq n_1 n_2$, a strong martingale M, let D_k, $\mathscr{G}_k$, N_k, be defined as in the proof of theorem 1, (1.3). Then

$$[M] = \sum_{1 \leqq k \leqq n_1 n_2} (N_k - N_{k-1})^2$$

and

$$2 . \sup_{i \in I} |M_i| \geqq \sup_{0 \leqq k < n_1 n_2} |N_k|.$$

Hence, Davis' classical inequality, applied to the martingale $(N_k, \mathscr{G}_k)_{0 \leqq k \leqq n_1 n_2}$ gives the desired inequality. □

PROPOSITION 2. — *There is a constant c such that for any strong martingale* M

$$E(\sup_{i \in \mathbf{N}^2} | M_i) \leqq c\, E([M]^{1/2}).$$

Proof. — Monotone convergence again allows one to consider strong martingales indexed by $I=[0, n]$ for some $n=(n_1, n_2) \in \mathbf{N}^2$. We follow the ideas of the proof of Davis' inequality in Dellacherie, Meyer [7], p. 302. The following inequality will be established: there exists a constant c_1 such that for any $\mathscr{F}$-measurable function $S: \Omega \to I$, any strong martingale M

$$E(|M_S|) \leqq c_1\, E([M]^{1/2}). \tag{2.1}$$

(2.1) will imply the proposition, since a simple version of the theorem of measurable sections (*see* Dellacherie, Meyer [6], p. 105) allows one to choose an $\mathscr{F}$-measurable section $S: \Omega \to I$ of the set

$$\{(\omega, i): |M_i(\omega)| = \max_{j \in I} |M_j(\omega)|\}.$$

Now fix S and let $B = \operatorname{sgn}(M_S)\, 1_{[S, n]}$. Define the two-parameter process A by

$$\square_{[i-1, i]} A = E(\square_{[i-1, i]} B \mid \mathscr{F}^1_{i_1} \vee \mathscr{F}^2_{i_2}), \qquad 1 \leqq i \leqq n,$$

and the two one-parameter processes A^i by

$$\Delta_{[k-1, k]} A^i = E(\Delta_{[k-1, k]} B_{(., n_{3-i})} \mid \mathscr{F}^i_k), \qquad 1 \leqq k \leqq n_i, \quad i=1, 2.$$

Each one of the processes B, A, A^1, A^2 is of bounded variation, the variation of B being 1. For any strong martingale M we have

$$\begin{aligned}
E(|M_S|) = E(M * B_n) = E(M * A_n) \quad &(\text{definition of A})\\
= E((\square_{[., n]} M) * A_n) &+ E(M_{(n_1, .)} \cdot A_n)\\
+ E(M_{(., n_2)} \cdot A_n) &- E(M_n A_n)\\
= E(M_{(n_1, .)} \cdot A^2_{n_2}) &+ E(M_{(., n_2)} \cdot A^1_{n_1}) - E(M_n A_n)
\end{aligned}$$

$[E((\square_{[., n]} M) * A_n) = 0$ by strong martingale property; definition of $A^i]$

$$\begin{aligned}
= -E((\Delta_{[., n_2]} M_{(n_1, .)}) \cdot A^2_{n_2}) &- E((\Delta_{[., n_1]} M_{(., n_2)}) \cdot A^1_{n_1})\\
+ E(M_n A^2_{n_2}) &+ E(M_n A^1_{n_1}) - E(M_n A_n)\\
= E(M_n A^2_{n_2}) &+ E(M_n A^1_{n_1}) - E(M_n A_n)
\end{aligned} \tag{2.2}$$

First, consider the two one-parameter martingales

$$N^i := E(A^i_{n_i} \mid \mathscr{F}^i_{.}), \qquad i=1, 2.$$

By definition of A^i, Dellacherie, Meyer [7], p. 290 80 (*b*), yields the inequalities

$$E\Big(\sum_{j \leqq k \leqq n_i} (N^i_k - N^i_{k-1})^2 \mid \mathscr{F}^i_j\Big) \leqq 5, \qquad 1 \leqq j \leqq n_i, \quad i=1, 2.$$

Therefore, Fefferman's inequality with $H=K=1$ (*see* Dellacherie, Meyer [7], p. 295) implies that for any strong martingale M, $i=1, 2$

$$\left|E(M_n A^i_{n_i})\right| = \left|E\left(\sum_{1 \leqq k \leqq n_i} \Delta_k M_{(n_i,\,.)} \Delta_k N^i\right)\right|$$
$$\leqq \sqrt{10}\, E\left[\sum_{1 \leqq k \leqq n_i} (M_{(k,\, n_{3-i})} - M_{(k-1,\, n_{3-i})})^2\right]^{1/2}).$$

An iterated application of Chinchine's inequality as in Ledoux [9], p. 124, yields a constant c_2 such that for any strong martingale M

$$\left|E(M_n A^i_{n_i})\right| \leqq c_2\, E([M]^{1/2}), \qquad i=1, 2. \tag{2.3}$$

Next, consider the last term on the right hand side of (2.2). Observe that in consequence of the equations

$$E(A_n - A_{(k-1,\, n_2)} \mid \mathscr{F}^1_k)$$
$$= E\left(\sum_{(k,\, 1) \leqq i \leqq n} E(\square_{[i-1,\, i]} B \mid \mathscr{F}^1_{i_1} \vee \mathscr{F}^2_{i_2}) \mid \mathscr{F}^1_k\right)$$
$$= E(B_n - B_{(k-1,\, n_2)} \mid \mathscr{F}^1_k), \qquad 1 \leqq k \leqq n_1,$$

the processes $A_{(.,\, n_2)}$ and $B_{(.,\, n_2)}$ have the same "left potential" *see* (Dellacherie, Meyer [7], p. 166) which, by definition of B, is bounded by 1. Now let

$$N := E(A_n \mid \mathscr{F}^1_.).$$

Again by Dellacherie, Meyer [7], p. 290

$$E\left(\sum_{j \leqq k \leqq n_1} (N_k - N_{k-1})^2 \mid \mathscr{F}^1_j\right) \leqq 5, \qquad 1 \leqq j \leqq n_1.$$

As above, we can apply Fefferman's and Chinchine's inequalities to obtain for any strong martingale M

$$\left|E(M_n A_n)\right| \leqq c_2\, E([M]^{1/2}). \tag{2.4}$$

Now combine (2.2)-(2.4) to derive the desired inequality (2.1). This completes the proof. □

We finally add the results of the preceding propositions to state our main result.

THEOREM 3. — *There exist constants c_1, $c_2 > 0$ such that for any strong martingale* M

$$c_1\, E([M]^{1/2}) \leqq E\left(\sup_{i \in \mathbf{N}^2} |M_i|\right) \leqq c_2\, E([M]^{1/2}).$$

Remarks:

1. In [10], Mishura claims a result similar to Theorem 3. However the author only gives a proof of the (easy) left-hand side inequality. Further, Mishura's proof is long and we were unable to verify the validity of the arguments. For the (hard) right-hand side inequality, the author claims that similar arguments work. This seems not to be the case.

2. Theorem 3 can be extended to continuous parameter strong martingales. For the existence of quadratic variations of continuous parameter martingales see Imkeller [8].

3. The extension of the results of this paper to N-parameter strong martingales should present no difficulty.

REFERENCES

[1] J. Brossard, Regularité des Martingales à deux indices et inégalités de normes, *Lecture Notes in Mathematics*, No. **863**, 1981, pp. 91-121.

[2] D. L. Burkholder, Martingale Transforms, *Ann. Math. Stat.*, Vol. **37**, 1966, pp. 1494-1504.

[3] D. L. Burkholder, A Geometrical Characterization of Banach Spaces in which Martingale Difference Sequences are Unconditional, *Ann. Prob.*, Vol. **9**, 1981, pp. 997-1011.

[4] D. L. Burkholder, An Extension of a Classical Martingale Inequality, in *Probability Theory and Harmonic Analysis*, J. A. Chao and W. A. Woyczynski, Eds., Marcel Dekker, New York, 1985.

[5] R. Cairoli and J. B. Walsh, Stochastic Integrals in the Plane, *Acta Math.*, Vol. **134**, 1975, pp. 111-183.

[6] C. Dellacherie and P. A. Meyer, *Probabilités et Potentiel*, Ch. 1-IV, Hermann, Paris, 1980.

[7] C. Dellacherie and P. A. Meyer, *Probabilités et Potentiel*, Ch. V-VIII, Hermann, Paris, 1980.

[8] P. Imkeller, The Structure of Two-parameter Martingales and their Quadratic Variation, *Thesis*, Univ. München, 1986.

[9] M. Ledoux, Inégalités de Burkholder pour martingales indexées par $N \times N$, *Lecture Notes in Mathematics*, No. **863**, 1981, pp. 122-127.

[10] Ju S. Mishura, On Properties of the Quadratic Variation of Two-parameter Strong Martingales, *Theor. Probab. and Math. Statist.*, Vol. **25**, 1982, pp. 99-106.

[11] J. B. Walsh, Convergence and Regularity of Multiparameter Strong Martingales, *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, Vol. **46**, 1979, pp. 177-192.

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