The Annals of Probability
1987, Vol. 15, No. 3, 1097–1111

QUADRATIC VARIATION FOR A CLASS OF $L \log^+ L$-BOUNDED TWO-PARAMETER MARTINGALES

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Let $M = (M_t)_{t \in [0,1]^2}$ be a two-parameter $L \log^+ L$-bounded (not necessarily continuous) martingale. Assume that the marginal filtrations in the first and second directions are quasi-left continuous. We prove the existence of quadratic variation in the sense of convergence in probability. This is done first for bounded martingales. The extension to the general case is obtained by approximating a given martingale by its bounded truncations and using a two-parameter version of the square function inequality of Burkholder.

Introduction. One of the basic notions in the development of stochastic calculus is quadratic variation.

In the theory of one-parameter (semi-)martingales the existence of quadratic variation is immediate from the definition of the stochastic integral (see Métivier [17, pages 175, 176]). Moreover, once its existence is established for L^2-bounded martingales, the concept of localization based upon the notion of stopping time allows a simple extension to the larger class of local martingales.

In the theory of two-parameter processes, however, stopping and consequently localization are more difficult and by far less important (see, for example, [14]). The development of the notion of quadratic variation for two-parameter martingales reflects this fact. To mention only a few steps: In their pioneering paper, Cairoli and Walsh [5] initiated a stochastic calculus for two-parameter martingales. In [6], the same authors established the existence of quadratic variation in L^p-sense $p > \frac{1}{2}$ for bounded martingales with the Wiener filtration. Zakai [26] extends Cairoli's and Walsh's proof to continuous (locally) L^4-bounded martingales. Assuming that every L^2-bounded martingale admits a continuous version and using the inequalities of Burkholder, Davis and Gundy as an essential tool like his predecessors, Chevalier [7] constructs quadratic variation for a class of continuous martingales. Nualart [23] deals with L^2-bounded continuous martingales and, besides constructing the quadratic variation, proves sample continuity of it. Guyon and Prum [13], for their stochastic calculus of representable martingales (w.r.t. Wiener filtration), construct quadratic variation for these processes and also have a very detailed bibliography containing further references. Among the few papers containing results for noncontinuous martingales, we only mention Mishura [20], [21]. In the latter, which treats the existence of quadratic variation via decomposition of the (L^2-bounded) martingale into jump parts and a continuous component, some regularity assumptions are made concerning the martingale and the quadratic variation of its one-parameter sections.

Received June 1985; revised March 1986.

AMS 1980 *subject classifications*. Primary 60G44; secondary 60G07, 60G42.

Key words and phrases. Two-parameter martingales, quadratic variation, Burkholder inequality, dual previsible projections.

Here, we prove that $L\log^+L$-bounded martingales (not necessarily continuous) possess a quadratic variation provided the one-parameter "marginal" filtrations $\mathbb{F}^1$ and $\mathbb{F}^2$ are quasi-left continuous. To do this, we use a weak Burkholder-type inequality for square functions together with a truncation argument which allows one to reduce the $L\log^+L$-bounded case to the bounded case. Given the facts that stopping and localization are poor concepts in the two-parameter theory, $L\log^+L$ seems to be a "natural" boundary for the existence theorems of quadratic variation (see Cairoli and Walsh [5, page 137] and Bakry [1]). However, we could not make this statement precise.

In Section 1 we prove a weak Burkholder-type inequality for the square function of martingales indexed by $\mathbb{N}^2$. We first show that the one-parameter version can be extended from real-valued to Hilbert-valued martingales. Then, expressing the two-dimensional difference sequence of the martingale as a one-dimensional difference sequence of a Hilbert space-valued martingale, we obtain the desired generalization (Theorem 1.2).

In Section 2, we discuss the problem of the existence of quadratic variation for (continuous parameter) bounded martingales M first. Using some basic facts on the dual previsible projections (in the i direction of parameter space) A (A^i), $i = 1, 2$, of M^2, and Burkholder's L^p-inequalities for uniform integrability of the square sums, we prove that quadratic variation exists provided $M_{(1,\cdot)}$ and $M_{(\cdot,1)}$ are quasi-left continuous (Theorem 2.6). Finally, by means of Theorem 1.2, we are able to trace back the existence of quadratic variation for $L\log^+L$-bounded martingales to the case of the bounded truncations $E((-m \vee M_{\underline{1}}) \wedge m|\mathscr{F}_\cdot)$, $m \in \mathbb{N}$ (Theorem 2.9). Since quasi-left continuity of $M_{(1,\cdot)}$ and $M_{(\cdot,1)}$ may not be preserved by truncation, we have to make a slightly stronger assumption whereby $\mathbb{F}^1$ and $\mathbb{F}^2$ are quasi-left continuous. We could not decide whether Theorem 2.9 remains true under weaker assumptions than these.

Notation, preliminaries and definitions. We consider processes with parameter set T, where $T = \mathbb{N}$ or $\mathbb{N}^2$ in Section 1 and $T = \mathbb{I} = [0,1]^2$ in Section 2. T is endowed with the usual partial order, i.e., (coordinatewise) linear order. Time points in $\mathbb{I}$ are denoted by $t = (t_1, t_2)$; intervals in $\mathbb{I}$ by $J = J_1 \times J_2$. For $J =]s,t]$, we write $J^1 =]s_1, t_1] \times [0, s_2]$, $J^2 = [0, s_1] \times]s_2, t_2]$. If the corners are not specified in advance, we denote by s^J and t^J the lower and upper corner of an interval J. A partition of $\mathbb{I}$ by intervals is always understood to be generated by axial parallel lines. Its intervals are supposed to be left open and right closed in the relative topology of $\mathbb{I}$. The mesh of a partition $\mathbb{J}$ is defined to be

$$\sup_{J \in \mathbb{J}} \left(|s_1^J - t_1^J| \vee |s_2^J - t_2^J| \right).$$

An 0-sequence $(\mathbb{J}_n)_{n \in \mathbb{N}}$ of partitions is a sequence such that the mesh of $\mathbb{J}_n$ goes to zero as $n \to \infty$. For a partition $\mathbb{J}$, $\mathbb{J}_i$ is the set of all J_i, $i = 1, 2$, such that $J = J_1 \times J_2 \in \mathbb{J}$. For $r \in \mathbb{R}$, $\underline{r} = (r, r)$.

Given a function f: $T \to \mathbb{R}$, $s, t \in T$, $s \le t$ and setting $J =]s,t]$, denote

$$\Delta_J f = \begin{cases} f(t) - f(s), & \text{if } T \subset \mathbb{R}, \\ f(t) - f(t_1, s_2) - f(s_1, t_2) + f(s), & \text{if } T \subset \mathbb{R}^2. \end{cases}$$

If $J =]m, m+1]$ or $]m, m+1] \times]n, n+1]$ for $m, n \in \mathbb{N}$, we write $\Delta_m f$, respectively, $\Delta_{mn} f$. If A is an increasing function, i.e., $\Delta_J A \geq 0$ for any interval J of T, we write $A(J)$ instead of $\Delta_J A$.

Our basic probability space $(\Omega, \mathscr{F}, P)$ is assumed to be complete, the basic filtration $\mathbb{F} = (\mathscr{F}_t)_{t \in T}$ to be augmented by the zero sets of $\mathscr{F}$, and right continuous.

If $T \subset \mathbb{R}^2$, for $t = (t_1, t_2) \in T_1 \times T_2 = T$, define

$$\mathscr{F}^1_{t_1} = \bigvee_{t_2 \in T_2} \mathscr{F}_{(t_1, t_2)}, \qquad \mathscr{F}^2_{t_2} = \bigvee_{t_1 \in T_1} \mathscr{F}_{(t_1, t_2)}$$

and

$$\mathbb{F}^1 = \left(\mathscr{F}^1_{t_1}\right)_{t_1 \in T_1}, \qquad \mathbb{F}^2 = \left(\mathscr{F}^2_{t_2}\right)_{t_2 \in T_2}.$$

$\mathbb{F}^1$ and $\mathbb{F}^2$ are always assumed to fulfill the well known (F4) condition of Cairoli and Walsh [5]:

for all $t \in T$, X bounded, $\mathscr{F} - \mathscr{B}(\mathbb{R})$-measurable, we have

$$E(X|\mathscr{F}_t) = E\left(E\left(X|\mathscr{F}^1_{t_1}\right)|\mathscr{F}^2_{t_2}\right).$$

Let $(\mathscr{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$. Stochastic processes X with values in $\mathscr{H}$ are always assumed to be (strongly) $\mathscr{F} \otimes \mathscr{B}(T) - \mathscr{B}(\mathscr{H})$-measurable. In talking about $\mathscr{H}$-valued processes, "integrable" means Bochner integrable w.r.t. P and E, the Bochner integral.

For $p \in \{0\} \cup [1, \infty[$, $L^p(\Omega, \mathscr{F}, P)$ is the usual space of real-valued p-integrable random variables. It is a Banach space with norm $\|\cdot\|_p$ if $p \geq 1$ and a metrizable topological vector space with the topology of convergence in probability if $p = 0$. $L \log^+ L$ is the space of real-valued random variables X such that $E(|X| \log^+ |X|) < \infty$.

An $\mathscr{H}$-valued $\mathbb{F}$-adapted integrable process $M = (M_t)_{t \in T}$ is called a *martingale* (*strong martingale*, *submartingale*) if for all $s, t \in T$, $s \leq t$, we have

$$E(M_t|\mathscr{F}_s) = M_s \left(T \in \mathbb{R}^2_+ \text{ and } E\left(\Delta_{]s,t]} M | \mathscr{F}^1_{s_1} \vee \mathscr{F}^2_{s_2}\right) = 0,\right.$$

$$\left.\mathscr{H} = \mathbb{R} \text{ and } E(M_t|\mathscr{F}_s) \geq M_s\right).$$

M is *$L \log^+ L$-bounded* if $\sup_{t \in T} E(|M_t| \log^+ |M_t|) < \infty$. If $T = \mathbb{I}$, we assume $M_t = 0$ for $t \in \mathbb{I} \cap \partial \mathbb{R}^2_+$.

1. A two-parameter Burkholder inequality. Let M be a real-valued martingale indexed by $\mathbb{N}$, $M_1 = 0$. Let $[M] = (\sum_{k \in \mathbb{N}} (\Delta_k M)^2)^{1/2}$ be the square function of M. The following weak inequality for square functions, due to Burkholder [2], [3], is well known:

$$\lambda P([M] \geq \lambda) \leq 3 \sup_{n \in \mathbb{N}} E(|M_n|) \tag{1.1}$$

for every $\lambda > 0$.

Let $(\mathscr{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space, M an $\mathscr{H}$-valued martingale indexed by $\mathbb{N}$. Assume that each M is Bochner integrable. Define

$$[\![M]\!] = \left(\sum_{k \in \mathbb{N}} \langle \Delta_k M, \Delta_k M \rangle \right)^{1/2}.$$

Then for every $\lambda > 0$,

$$\lambda P([\![M]\!] \geq \lambda) \leq 3 \sup_{n \in \mathbb{N}} E(\|M_n\|). \tag{1.2}$$

The proof of (1.2) follows verbatim the lines of (1.1) given in [3]. Only the following well-known identity is needed:

$$\langle a, E(X) \rangle = E(\langle a, X \rangle),$$

where $a \in \mathscr{H}$ and X is any $\mathscr{H}$-valued Bochner integrable random variable.

Considering a martingale difference sequence, indexed by $\mathbb{N}^2$, as a difference sequence of a Hilbert-valued martingale indexed by $\mathbb{N}$ and using (1.2), one can deduce an analogue of (1.1) for two-parameter martingales (Theorem 1.2). A similar result using basically the same ideas was proved by Shieh [25], whose work was communicated to us by the referee. The second part of [25], dealing with continuous parameter martingales, however, is not correct. For the sake of completeness and because our inequality (1.4) is slightly different from that in [25], we present the main ideas of the proof. Let M be an $L \log^+ L$-bounded real-valued martingale indexed by $\mathbb{N}^2$, $M_1 = 0$. Then there exists an $L \log^+ L$-integrable random variable M_∞ such that

$$M_{mn} = E(M_\infty | \mathscr{F}_{mn}) = E\big(E\big(M_\infty | \mathscr{F}_m^1\big) | \mathscr{F}_n^2\big).$$

M_∞ is the a.s. limit of M. Let $[M] = (\sum_{k, l \in \mathbb{N}} (\Delta_{kl} M)^2)^{1/2}$ be the square function of M. In the proof of Theorem 1.2, we need a strengthening of Doob's inequality, due to Millet and Sucheston [19, page 23], which we recall for convenience.

LEMMA 1.1. *Let M be a real-valued $L \log^+ L$-bounded martingale indexed by $\mathbb{N}$. Then for every $\delta > 0$, $0 < \delta \leq 1$, there exists a constant $A(\delta)$ such that*

$$E\Big(\sup_{n \in \mathbb{N}} |M_n|\Big) \leq \delta + A(\delta) \max\big[E(|M_\infty|), E(|M_\infty| \log^+ |M_\infty|)\big]. \tag{1.3}$$

THEOREM 1.2. *Let M be a real-valued $L \log^+ L$-bounded martingale indexed by $\mathbb{N}^2$. Then for every $\delta > 0$, $0 < \delta \leq 1$, there exists a constant $A(\delta)$ such that*

$$\lambda P([M] \geq \lambda) \leq \delta + A(\delta) \max\big[E(|M_\infty|), E(|M_\infty| \log^+ |M_\infty|)\big] \tag{1.4}$$

for every $\lambda > 0$.

PROOF. (See also [25] and [12, page 183].) First define the l_2-valued random variable,

$$f = (\Delta_1 M, \Delta_2 M, \dots),$$

where

$$\Delta_k M = E\big(M_\infty | \mathscr{F}_k^1\big) - E\big(M_\infty | \mathscr{F}_{k-1}^1\big), \qquad k \in \mathbb{N}.$$

Then by Davis' inequality [8],

$$(1.5)\qquad E(\|f\|) = E\left(\left(\sum_{k\in\mathbb{N}} (\Delta_k M)^2\right)^{1/2}\right) \le cE\left(\sup_{n\in\mathbb{N}} |E(M_\infty|\mathscr{F}_n^1)|\right).$$

Let $F = (E(f\,|\mathscr{F}_n^2))_{n\in\mathbb{N}}$. Then F is an l_2-valued martingale. Observe that

$$\Delta_{kl}M = E(\Delta_k M|\mathscr{F}_l^2) - E(\Delta_k M|\mathscr{F}_{l-1}^2);$$

thus

$$[M] = [\![F]\!].$$

Hence for every $\lambda > 0$,

$$\begin{aligned}\lambda P([M] \ge \lambda) &= \lambda P([\![F]\!] \ge \lambda)\\ (1.6)\qquad &\le 3 \sup_{n\in\mathbb{N}} E(\|E(f\,|\mathscr{F}_n^2)\|) \quad \text{[by (1.2)]}\\ &\le 3E(\|f\|).\end{aligned}$$

Inequality (1.5) and Lemma 1.1 finish the proof. □

We note the following consequence of Theorem 1.2.

COROLLARY 1.3. *Let M be a real-valued $L\log^+ L$-bounded martingale indexed by $\mathbb{N}^2$. Then the square function $[M]$ of M is a.s. finite.*

2. Existence of the quadratic variation. We now consider continuous parameter martingales with parameter set $\mathbb{I}$. Before we start to discuss quadratic variation, let us have a brief comparative look at the one-parameter case. In this case, the technique of localization works. With its help, the existence theorem for quadratic variation can be easily transferred from the central L^2-theory to "L^0-theory," to yield quadratic variation for the large classes of local L^2-martingales (see for example Métivier [17, pages 175, 176], where the "transfer" works by having available a local stochastic integral of the martingale). For example, every continuous martingale, being locally bounded, possesses a continuous quadratic variation.

In the two-parameter case, however, things change drastically. Due to the nontotal order of the parameter space, the analogues of stopping times are much less important and, therefore, localization is a poor concept. Indeed, there are continuous L^2-martingales for which there exists no sequence of stopping domains $(D_n)_{n\in\mathbb{N}}$ converging to $\mathbb{R}_+^2$ such that the stopped martingales M^{D_n}, $n \in \mathbb{N}$, are bounded (see [14]). Therefore, unlike the one-parameter case, there is no easy way of extending results from the space of L^2-martingales to larger spaces.

Yet, L^2-theory is central and powerful for two-parameter martingales also, and we want to take advantage of that fact. Therefore, given an $L\log^+ L$-bounded martingale M, we need a sequence of "approximating martingales" $(M^n)_{n\in\mathbb{N}}$ which (a) is L^2-integrable and (b) approximates M well enough to guarantee convergence of the corresponding quadratic variations to be constructed with the help of L^2-theory. According to the preceding remarks we cannot expect to be able to work with a localizing sequence, so we resort to the truncated martingales

$$M_\cdot^n = E((-n \vee M_1) \wedge n | \mathscr{F}_\cdot), \qquad n \in \mathbb{N}.$$

$(M^n)_{n \in \mathbb{N}}$ certainly fulfills (a) and, as will be shown later, also (b), but the operation of truncating and taking conditional expectation is too "rough" to provide M^n with the same regularity properties as M. The trouble lies in the fact that continuity properties of M and the filtration do not coincide, as will be seen in the following example.

EXAMPLE 2.1. Let $B = (B_t)_{t \in \mathbf{R}_+}$ be a Brownian motion on $(\Lambda, \mathfrak{G}, Q)$, endowed with the completed, continuous filtration $\mathbb{G} = (\mathfrak{G}_t)_{t \in \mathbf{R}_+}$. Let

$$Y = 1_{[0,1/3]} + 1_{\{B_{2/3} - B_{1/3} > 0\}} 1_{]2/3,1]}.$$

Y is previsible and the stochastic integral process

$$N = \int Y \, dB$$

defines a continuous martingale w.r.t. $\mathbb{G}$. But N is even a martingale w.r.t. the filtration $\mathbb{H} = (\mathfrak{H}_t)_{t \in \mathbf{R}_+}$, where

$$\mathfrak{H}_t = \mathfrak{G}_t \quad \text{for } t < \tfrac{1}{3},$$

$$\mathfrak{H}_t = \mathfrak{G}_t \vee \sigma(B_{2/3} - B_{1/3}) \quad \text{for } t \geq \tfrac{1}{3},$$

completed by the zero sets of $\mathfrak{G}$. Indeed, let $s \leq t$ be given. In order to prove $E(N_t | \mathfrak{H}_s) = N_s$, it is enough to consider the only nontrivial case $\frac{1}{3} \leq s < \frac{2}{3}$. If $t \leq \frac{2}{3}$, we have

$$N_t = B_{1/3} = N_s,$$

and if $t > \frac{2}{3}$,

$$E(N_t | \mathfrak{H}_s) = E(E(N_t | \mathfrak{H}_{2/3}) | \mathfrak{H}_s) = E(E(N_t | \mathfrak{G}_{2/3}) | \mathfrak{H}_s)$$
$$= E(B_{1/3} | \mathfrak{H}_s) = B_{1/3} = N_s.$$

Of course, the filtration $\mathbb{H}$ is not quasi-left continuous (see Dellacherie [9, page 57]). We now concentrate on the parameter set $[0,1]$. We will prove that the truncations

$$N^\alpha := E((-\alpha \vee N_1) \wedge \alpha | \mathfrak{H}_\cdot), \qquad \alpha > 0,$$

"feel" the jump of the filtration at $\frac{1}{3}$. For convenience, we use the notation $\langle -\alpha, x, \alpha \rangle = (-\alpha \vee x) \wedge \alpha$, $x \in \mathbb{R}$, $\alpha > 0$. Fix $\alpha > 0$. For $t < \frac{1}{3}$ we have

$$N_t^\alpha = E(\langle -\alpha, B_{1/3} + (B_1 - B_{2/3}) 1_{\{B_{2/3} - B_{1/3} > 0\}}, \alpha \rangle | \mathfrak{G}_t)$$
$$= \tfrac{1}{2}\left[E(\langle -\alpha, B_{1/3}, \alpha \rangle | \mathfrak{G}_t) + E(\langle -\alpha, B_{1/3} + (B_1 - B_{2/3}), \alpha \rangle | \mathfrak{G}_t)\right],$$

whereas by definition of $\mathfrak{H}_{1/3}$,

$$N^{\alpha}_{1/3} = E\big(\langle -\alpha, B_{1/3} + (B_1 - B_{2/3})1_{\{B_{2/3}-B_{1/3}>0\}}, \alpha\rangle | \mathfrak{H}_{1/3}\big)$$
$$= 1_{\{B_{2/3}-B_{1/3}\le 0\}}\langle -\alpha, B_{1/3}, \alpha\rangle$$
$$+1_{\{B_{2/3}-B_{1/3}>0\}}E\big(\langle -\alpha, B_{1/3} + (B_1 - B_{2/3}), \alpha\rangle | \mathfrak{G}_{1/3}\big).$$

Therefore, using the abbreviations

$$X_1 := \langle -\alpha, B_{1/3}, \alpha\rangle, \qquad X_2 := E\big(\langle -\alpha, B_{1/3} + (B_1 - B_{2/3}), \alpha\rangle | \mathfrak{G}_{1/3}\big),$$

the left continuity of $\mathbb{G}$ gives

$$\begin{aligned} N^{\alpha}_{1/3-} &= \tfrac{1}{2}(X_1 + X_2), \\ N^{\alpha}_{1/3} &= 1_{\{B_{2/3}-B_{1/3}\le 0\}}X_1 + 1_{\{B_{2/3}-B_{1/3}>0\}}X_2. \end{aligned} \tag{2.1}$$

It is clear from (2.1) that N^{α} is not continuous at $\frac{1}{3}$ for all $\alpha > 0$. N^{α} being a bounded martingale, (2.1) even implies that $(N^{\alpha})^2$ is not quasi-left continuous in the usual sense (see Dellacherie [9, page 119]), i.e.,

$$E\Big(\big(N^{\alpha}_{1/3-}\big)^2\Big) \ne E\Big(\big(N^{\alpha}_{1/3}\big)^2\Big) \quad \text{for all } \alpha > 0. \tag{2.2}$$

Finally, to have a two-parameter model of a martingale whose truncations do not inherit its regularity properties, simply take $(\Lambda, \mathfrak{G}, Q)$ as before and

$$\mathscr{F}_t := \mathfrak{H}_{t_1\wedge t_2}, \qquad M_t := E(N_1|\mathscr{F}_t), \qquad t \in \mathbb{I}.$$

Now, as it happens and will be seen, even for bounded two-parameter martingales N we cannot establish the existence of quadratic variation unless $N_{(1,\cdot)}$ and $N_{(\cdot,1)}$ are quasi-left continuous. Therefore, our plan to obtain the quadratic variation of an $L\log^+L$-bounded martingale M from those of its truncations will not work out if we only assume that $M_{(1,\cdot)}$ and $M_{(\cdot,1)}$ are quasi-left continuous, since the preceding example shows that truncations do not necessarily inherit this property. But we can do with a slightly stronger assumption, namely, that filtrations $\mathbb{F}^1$ and $\mathbb{F}^2$ are quasi-left continuous. Then $M_{(1,\cdot)}$ and $M_{(\cdot,1)}$ together with their truncations are quasi-left continuous (see Dellacherie [9, page 112]). Taking this into account, we proceed along the following lines: (1) Prove the existence of quadratic variation for bounded martingales, thus for the truncations M^n of M under the assumption of quasi-left continuity; (2) using the inequality (1.4) to reduce the problem to the bounded case, obtain the quadratic variation of M under the assumption of (1).

Until further notice, assume now that M is a bounded martingale which is right continuous and possesses limits in all quadrants (see Bakry [1, page 46] and Méyer [18, page 35]). Let $(\mathbb{J}_n)_{n\in\mathbb{N}}$ be a 0-sequence of partitions of $\mathbb{I}$. We will show that for any $t \in \mathbb{I}$ the sequence $(\sum_{J\in\mathbb{J}_n}(\Delta_{J\cap[0,t]}M)^2)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega, \mathscr{F}, P)$. By standard arguments, the existence of quadratic variation follows from this and we may confine our attention to $t = \mathbf{1}$. We may also assume that $(\mathbb{J}_n)_{n\in\mathbb{N}}$ is "increasing," i.e., $\mathbb{J}_{n+1}$ is a refinement of $\mathbb{J}_n$ for all

$n \in \mathbb{N}$ (see Neveu [29, page 96]). The point of departure for the proof is the following decomposition of $(\Delta_J M)^2$, the J interval in $\mathbb{I}$, into one-parameter "boundary" terms, an M^2-term and two martingale terms (see Nualart [23, page 448]):

$$\begin{aligned}\Delta_J M^2 &= 2M_{s^J}\,\Delta_J M + 2\,\Delta_{J^1} M \Delta_{J^2} M \\ &\quad + 2\,\Delta_J M \Delta_{J^1} M + 2\,\Delta_J M \Delta_{J^2} M + (\Delta_J M)^2.\end{aligned} \tag{2.3}$$

Summation of (2.3) over $J \in \mathbb{J}_n$ and some algebra give

$$\begin{aligned}\sum_{J\in\mathbb{J}_n} (\Delta_J M)^2 &= \sum_{J_1 \in (\mathbb{J}_n)_1} \left(\Delta_{J_1 \times [0,1]} M\right)^2 + \sum_{J_2 \in (\mathbb{J}_n)_2} \left(\Delta_{[0,1]\times J_2} M\right)^2 \\ &\quad - M_1^2 + 2 \sum_{J\in\mathbb{J}_n} M_{s^J}\,\Delta_J M + 2 \sum_{J\in\mathbb{J}_n} \Delta_{J^1} M \Delta_{J^2} M,\end{aligned} \tag{2.4}$$

$n \in \mathbb{N}$. We will now establish the convergence in $L^2(\Omega, \mathscr{F}, P)$ of the terms of (2.4) separately, beginning with the first two. Obviously, they are square sums approximating the quadratic variations of the one-parameter martingales $M_{(1,\cdot)}$ and $M_{(\cdot,1)}$, respectively. It is enough to recall well-known results from one-parameter theory (see, for example, Dellacherie and Méyer [10, pages 250–255]). We can therefore concentrate on the last two terms, the martingale components of the square sum: The well-behaved first one converges in general, as the following proposition shows.

PROPOSITION 2.2. *Let M be a bounded martingale. Then for any 0-sequence $(\mathbb{J}_n)_{n\in\mathbb{N}}$ of partitions of $\mathbb{I}$, the sequence $(\sum_{J\in\mathbb{J}_n} M_{s^J}\,\Delta_J M)_{n\in\mathbb{N}}$ converges in $L^2(\Omega, \mathscr{F}, P)$.*

PROOF. Let A be the unique previsible increasing process of M^2 (see Merzbach [15, page 51] and Méyer [18, page 21]). For $m, n \in \mathbb{N}$, $n \le m$, we get, using the martingale property,

$$\begin{aligned}&\left\| \sum_{J\in\mathbb{J}_n} M_{s^J}\,\Delta_J M - \sum_{K\in\mathbb{J}_m} M_{s^K}\,\Delta_K M \right\|_2^2 \\ &\quad = \left\| \sum_{J\in\mathbb{J}_n} \sum_{J\supset H\in\mathbb{J}_m} (M_{s^J} - M_{s^H})\,\Delta_H M \right\|_2^2 \\ &\quad = \sum_{J\in\mathbb{J}_n} \sum_{J\supset H\in\mathbb{J}_m} \left\|(M_{s^J} - M_{s^H})\,\Delta_H M\right\|_2^2 \\ &\quad = \sum_{J\in\mathbb{J}_n} \sum_{J\supset H\in\mathbb{J}_m} E\Big((M_{s^J} - M_{s^H})^2 E\big((\Delta_H M)^2 | \mathscr{F}_{s^H}\big)\Big) \\ &\quad = \sum_{J\in\mathbb{J}_n} \sum_{J\supset H\in\mathbb{J}_m} E\Big((M_{s^J} - M_{s^H})^2 E\big(A(H) | \mathscr{F}_{s^H}\big)\Big) \\ &\quad = E\left(\int_{\mathbb{I}} Y_{n,m}\, dA\right),\end{aligned}$$

where $Y_{n,m} = \sum_{J \in \mathbb{J}_n} \sum_{J \supset H \in \mathbb{J}_m} (M_{s^J} - M_{s^H})^2 1_H$. Since M is bounded, $(Y_{n,m}: \mathbb{N} \ni n \le m \in \mathbb{N})$ is uniformly bounded. Moreover, since M possesses limits in the third quadrant, $Y_{n,m} \to 0$ $(n, m \to \infty,\ n \le m)$ a.s. Therefore, Lebesgue's dominated convergence theorem yields the desired result. □

The convergence of the last term in (2.4) is more delicate. Due to its structure, we can again make use of some knowledge on dual previsible projections of M^2, but, unlike Proposition 2.2, only in the two coordinate directions of parameter space separately. In the general case, the difficult part is then the control of the behavior of the term in the respective other direction. In case the i-projections A^i, $i = 1, 2$, of M^2, however, are increasing processes, things simplify considerably (for definition and basic properties of A^i, $i = 1, 2$, see Cairoli and Walsh [5, pages 117–121] and Merzbach [15, pages 50, 51]).

PROPOSITION 2.3. *Let M be a bounded martingale. Assume the dual previsible projections of M^2 in i-direction A^i, $i = 1, 2$, are increasing processes. Then for every 0-sequence of partitions $(\mathbb{J}_n)_{n \in \mathbb{N}}$ of $\mathbb{I}$, the sequence $(\sum_{J \in \mathbb{J}_n} \Delta_{J^1} M \Delta_{J^2} M)_{n \in \mathbb{N}}$ converges in $L^2(\Omega, \mathscr{F}, P)$.*

PROOF. Let $m, n \in \mathbb{N}$, $n \le m$, be given. Use the martingale property of M and the inequality $(x + y)^2 \le 2(x^2 + y^2)$, $x, y \in \mathbb{R}$, to get

$$\begin{aligned}
&\Big\| \sum_{J \in \mathbb{J}_n} \Delta_{J^1} M \Delta_{J^2} M - \sum_{K \in \mathbb{J}_m} \Delta_{K^1} M \Delta_{K^2} M \Big\|_2^2 \\
&= \Big\| \sum_{J \in \mathbb{J}_n} \sum_{J \supset H \in \mathbb{J}_m} \big[\Delta_{H_1 \times [0, s_2^J]} M \Delta_{[0, s_1^H] \times H_2} M - \Delta_{H^1} M \Delta_{H^2} M \big] \Big\|_2^2 \\
&= \sum_{J \in \mathbb{J}_n} \sum_{J \supset H \in \mathbb{J}_m} \big\| \Delta_{H_1 \times [0, s_2^J]} M \Delta_{[0, s_1^H] \times H_2} M - \Delta_{H^1} M \Delta_{H^2} M \big\|_2^2 \\
&= \sum_{J \in \mathbb{J}_n} \sum_{J \supset H \in \mathbb{J}_m} \big\| \Delta_{H_1 \times]s_2^J, s_2^H]} M \Delta_{[0, s_1^H] \times H_2} M \\
&\qquad + \Delta_{H_1 \times [0, s_2^J]} M \Delta_{]s_1^J, s_1^H] \times H_2} M \big\|_2^2 \qquad (2.5)\\
&\le 2 \Big[\sum_{J \in \mathbb{J}_n} \sum_{J \supset H \in \mathbb{J}_m} \big\| \Delta_{H_1 \times]s_2^J, s_2^H]} M \Delta_{[0, s_1^H] \times H_2} M \big\|_2^2 \\
&\qquad + \sum_{J \in \mathbb{J}_n} \sum_{J \supset H \in \mathbb{J}_m} \big\| \Delta_{H_1 \times [0, s_2^J]} M \Delta_{]s_1^J, s_1^H] \times H_2} M \big\|_2^2 \Big].
\end{aligned}$$

The arguments for the second term being symmetric, it is enough to estimate only the first one on the right-hand side of (2.5). By (F4) in Cairoli and Walsh [5] and elementary properties of increasing processes,

$$\sum_{J \in \mathbb{J}_n} \sum_{J \supset H \in \mathbb{J}_m} \left\| \Delta_{H_1 \times]s_2^J, s_2^H]} M \Delta_{[0, s_1^H] \times H_2} M \right\|_2^2$$

$$= \sum_{J \in \mathbb{J}_n} \sum_{J \supset H \in \mathbb{J}_m} E\big(A^1(H_1 \times]s_2^J, s_2^H])A^2([0, s_1^H] \times H_2)\big)$$

$$\leq \sum_{J \in \mathbb{J}_n} \sum_{J \supset H \in \mathbb{J}_m} E\big(A^1(H_1 \times]s_2^J, s_2^H])A^2([0,1] \times H_2)\big)$$

$$= \sum_{J_2 \in (\mathbb{J}_n)_2} \sum_{J_2 \supset H_2 \in (\mathbb{J}_m)_2} E\big(A^1([0,1] \times]s_2^J, s_2^H])A^2([0,1] \times H_2)\big)$$

$$= E\left(\int_{[0,1]} Z_{n,m} \, dA^2_{(1,\cdot)} \right),$$

where $Z_{n,m} = \sum_{J_2 \in (\mathbb{J}_n)_2} \sum_{J_2 \supset H_2 \in (\mathbb{J}_m)_2} A^1([0,1] \times]s_2^J, s_2^H])1_{H_2}$, $n, m \in \mathbb{N}$, $n \leq m$. Since M is bounded, A_1^1 is p-integrable for any $p > 0$. Since $A^2_{(1,\cdot)}$ is increasing, it possesses left limits. Therefore, $(Z_{n,m}: \mathbb{N} \ni n \leq m \in \mathbb{N})$ is uniformly bounded by an integrable process and $Z_{n,m} \to 0$ $(n, m \to \infty, n \leq m)$ a.s. Again, an appeal to Lebesgue's dominated convergence theorem finishes the proof. □

COROLLARY 2.4. *Let M be a bounded martingale. If M has orthogonal increments or path independent variation* (*see Merzbach and Nualart* [16]), *it possesses quadratic variation. In particular, this holds if M is a strong martingale.*

PROOF. If M has orthogonal increments or path independent variation, the hypotheses of Proposition 2.3 hold true (see [16], Proposition 2.5 and Theorem 2.7). Strong martingales have orthogonal increments. □

REMARK 1. For square integrable martingales with orthogonal increments and under an additional regularity assumption, Mishura's [20] paper has results on the existence of quadratic variation.

REMARK 2. Truncation martingales do not necessarily inherit the property of having orthogonal increments or path independent variation. Therefore, Proposition 2.3 and its corollary do not yield quadratic variations of $L \log^+ L$-bounded martingales with these properties via the methods of this paper.

Now assume that A^1 for instance, is not increasing. In this case, it is rather difficult to control its dependence on the second time parameter. We were not able to prove the existence of the limit of the last term in (2.4) without further assumptions. More precisely, we could not modify the proof of Proposition 2.3 to make it work if the dual previsible projections A^i, $i = 1, 2$, have jumps.

PROPOSITION 2.5. *Let M be a bounded martingale. Assume the processes $M_{(1,\cdot)}$ and $M_{(\cdot,1)}$ are quasi-left continuous, i.e., for any previsible $\mathbb{F}^1$-stopping time S ($\mathbb{F}^2$-stopping time T), we have $M_{(S,1)} = M_{(S-,1)}$ ($M_{(1,T)} = M_{(1,T-)}$). Then for every 0-sequence $(\mathbb{J}_n)_{n\in\mathbb{N}}$ of partitions of $\mathbb{I}$ the sequence $(\sum_{J\in\mathbb{J}_n}\Delta_{J^1}M\,\Delta_{J^2}M)_{n\in\mathbb{N}}$ converges in $L^2(\Omega, \mathscr{F}, P)$.*

PROOF. We resume the proof of Proposition 2.3 after (2.5). To estimate the first term on the right-hand side of (2.5) again, we use a submartingale argument and (F4) (in [5]) repeatedly to obtain for $n, m \in \mathbb{N}$, $n \le m$,

$$\sum_{J\in\mathbb{J}_n}\ \sum_{J\supset H\in\mathbb{J}_m} \left\| \Delta_{H_1\times]s_2^J, s_2^H]}M\,\Delta_{[0,s_1^H]\times H_2}M\right\|_2^2$$

$$\le \sum_{J\in\mathbb{J}_n}\ \sum_{J\supset H\in\mathbb{J}_m} E\Big(E\Big(\big(\Delta_{H_1\times]s_2^J,s_2^H]}M\big)^2\Big|\mathscr{F}_{s^H}\Big)E\Big(\big(\Delta_{[0,1]\times H_2}M\big)^2\Big|\mathscr{F}_{s^H}\Big)\Big)$$

$$= \sum_{J\in\mathbb{J}_n}\ \sum_{J\supset H\in\mathbb{J}_m} E\Big(E\Big(\big(\Delta_{H_1\times]s_2^J,s_2^H]}M\big)^2\Big|\mathscr{F}^1_{s_1^H}\Big)A^2_{(1,\cdot)}(H_2)\Big)$$

$$(2.6)\qquad \le \Bigg[E\Bigg(\sum_{J_2\in(\mathbb{J}_n)_2}\ \sup_{J_2\supset H_2\in(\mathbb{J}_m)_2} \Bigg[\sum_{J_1\in(\mathbb{J}_n)_1}\ \sum_{J_1\supset H_1\in(\mathbb{J}_m)_1} E\Big(\big(\Delta_{H_1\times]s_2^J,s_2^H]}M\big)^2\Big|\mathscr{F}^1_{s_1^H}\Big)\Bigg]^2\Bigg)\Bigg]^{1/2}$$

$$\times \Bigg[E\Bigg(\sum_{J_2\in(\mathbb{J}_n)_2}\big[A^2_{(1,\cdot)}(J_2)\big]^2\Bigg)\Bigg]^{1/2} \quad \text{(Cauchy–Schwarz)}.$$

Now observe that for previsible $\mathbb{F}^2$-stopping times T we have

$$E\big(M^2_{(1,T)}\big) = E\big(M^2_{(1,T-)}\big),$$

i.e., the process $M^2_{(1,\cdot)}$ is regular in the sense of Dellacherie [9, page 119]. Consequently, $A^2_{(1,\cdot)}$ is continuous (Dellacherie [9]). Furthermore, M being bounded, A^2_1 is p-integrable for all $p > 0$. Therefore, the second factor on the right-hand side of (2.6) converges to zero as $n \to \infty$. To finish the proof, we have to show that the first factor is uniformly bounded in $n, m \in \mathbb{N}$, $n \le m$. Apply Doob's inequality, Burkholder's L^p-inequalities twice for $p = 2$ and an inequality for previsible projections (see Burkholder, Davis and Gundy [4, page 232]) to find constants $c_1, \ldots, c_4$ independent of M such that for all $n, m \in \mathbb{N}$,

$n \le m$,

$$E\left(\sum_{J_2 \in (\mathbb{J}_n)_2} \sup_{J_2 \supset H_2 \in (\mathbb{J}_m)_2} \left[\sum_{J_1 \in (\mathbb{J}_n)_1} \sum_{J_1 \supset H_1 \in (\mathbb{J}_m)_1} E\left(\left(\Delta_{H_1 \times]s_2^J, s_2^H]} M\right)^2 | \mathscr{F}^1_{s_1^H}\right)\right]^2\right)$$

$$\le c_1 E\left(\sum_{J_2 \in (\mathbb{J}_n)_2} \left[\sum_{J_1 \in (\mathbb{J}_n)_1} \sum_{J_1 \supset H_1 \in (\mathbb{J}_m)_1} E\left(\left(\Delta_{H_1 \times J_2} M\right)^2 | \mathscr{F}^1_{s_1^H}\right)\right]^2\right) \quad \text{(Doob)}$$

$$\le c_2 E\left(\sum_{J_2 \in (\mathbb{J}_n)_2} \left[\sum_{J_1 \in (\mathbb{J}_n)_1} \sum_{J_1 \supset H_1 \in (\mathbb{J}_m)_1} \left(\Delta_{H_1 \times J_2} M\right)^2\right]^2\right) \quad \text{(projection inequality)}$$

$$\le c_3 E\left(\sum_{J_2 \in (\mathbb{J}_n)_2} \left(\Delta_{[0,1] \times J_2} M\right)^4\right) \quad \text{(Burkholder)}$$

$$\le c_3 \left[E\left(\sup_{J_2 \in (\mathbb{J}_n)_2} \left(\Delta_{[0,1] \times J_2} M\right)^4\right)\right]^{1/2} \left[E\left(\left[\sum_{J_2 \in (\mathbb{J}_n)_2} \left(\Delta_{[0,1] \times J_2} M\right)^2\right]^2\right)\right]^{1/2}$$

(Cauchy–Schwarz)

$$\le c_4 E\left(M_1^4\right) < \infty \quad \text{(Doob, Burkholder).} \qquad \square$$

Together with the preceding remarks, Propositions 2.3 and 2.5 yield the following result.

THEOREM 2.6. *Let M be a bounded martingale. Assume that $M_{(1,\cdot)}$ and $M_{(\cdot,1)}$ are quasi-left continuous. Then M possesses quadratic variation in $L^2(\Omega, \mathscr{F}, P)$.*

Finally, since quasi-left continuity of the filtration implies quasi-left continuity of the martingale, we get the following corollary.

COROLLARY 2.7. *Assume $\mathbb{F}^1$ and $\mathbb{F}^2$ are quasi-left continuous, i.e., for any previsible $\mathbb{F}^1$-stopping time S ($\mathbb{F}^2$-stopping time T) we have $\mathscr{F}^1_S = \mathscr{F}^1_{S-}$ ($\mathscr{F}^2_T = \mathscr{F}^2_{T-}$). Then every bounded martingale possesses quadratic variation in $L^2(\Omega, \mathscr{F}, P)$.*

PROOF. See Dellacherie [9, page 112]. □

REMARKS. Mishura [21] proves the existence of quadratic variation for square integrable martingales by a different method. He decomposes M into its various jump components and a continuous part and treats them separately. His results are valid under regularity assumptions which are not completely comparable with our assumptions. The fact that in the proof of Proposition 2.3 missing regularity of A^2 in the first time variable was, in a seemingly unnatural way, replaced by additional regularity in the second, may indicate that quasi-left continuity is not the crucial thing for the existence of quadratic variation. We were unable to see in which way a jump of $A^2_{(1,\cdot)}$ might influence the convergence of the left-hand side of (2.6).

To turn to the second step of our program now, let M be an $L\log^+ L$-bounded martingale. Again, we may assume that M is right continuous and possesses limits in all quadrants. Given a 0-sequence of partitions $(\mathbb{J}_n)_{n\in\mathbb{N}}$ of $\mathbb{I}$ as in Proposition 2.3, we will show that the sequence $\left(\sum_{J\in\mathbb{J}_n}\left(\Delta_{J\cap[0,t]}M\right)^2\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^0(\Omega,\mathscr{F},P)$, i.e., in probability, taking $t=\mathbf{1}$ for convenience again. Of course, we will hereby apply the knowledge we gained in the bounded case on the truncations M^m, $m\in\mathbb{N}$, of M. This in particular means that we need quasi-left continuity of $\mathbb{F}^1$ and $\mathbb{F}^2$. The following lemma relates the square sums of M and its truncations and will be essential in reducing the desired L^0-convergence to L^2-convergence in the bounded case.

LEMMA 2.8. *Let M be an $L\log^+ L$-bounded martingale,*

$$M^m = E((-m\vee M)\wedge m|\mathscr{F}_\cdot),$$

$m\in\mathbb{N}$, and $(\mathbb{J}_n)_{n\in\mathbb{N}}$ a sequence of partitions of $\mathbb{I}$. Then we have

$$\sum_{J\in\mathbb{J}_n}(\Delta_J M)^2 - \sum_{J\in\mathbb{J}_n}(\Delta_J M^m)^2 \to 0 \quad (m\to\infty) \text{ in } L^0(\Omega,\mathscr{F},P)$$

uniformly in $n\in\mathbb{N}$.

PROOF. First observe that for $n, m\in\mathbb{N}$ (see also Doléans [11, page 287])

$$\begin{aligned}&\left|\sum_{J\in\mathbb{J}_n}(\Delta_J M)^2 - \sum_{J\in\mathbb{J}_n}(\Delta_J M^m)^2\right|\\ &\quad\le \sum_{J\in\mathbb{J}_n}(\Delta_J(M-M^m))^2 + 2\sum_{J\in\mathbb{J}_n}|\Delta_J M|\,|\Delta_J(M-M^m)| \qquad (2.7)\\ &\quad\le \sum_{J\in\mathbb{J}_n}(\Delta_J(M-M^m))^2 + 2\left[\sum_{J\in\mathbb{J}_n}(\Delta_J M)^2\sum_{J\in\mathbb{J}_n}(\Delta_J(M-M^m))^2\right]^{1/2}.\end{aligned}$$

Now denote

$$X_{n,m} := \sum_{J\in\mathbb{J}_n}(\Delta_J(M-M^m))^2, \qquad Y_n := \sum_{J\in\mathbb{J}_n}(\Delta_J M)^2, \qquad n,m\in\mathbb{N}.$$

By Theorem 1.2, applied to the discrete two-parameter martingales $(M_{t^J}\colon J\in\mathbb{J}_n)$, $((M-M^m)_{t^J}\colon J\in\mathbb{J}_n)$, $n,m\in\mathbb{N}$, we have on the one hand

$$\lim_{\lambda\to\infty}\sup_{n\in\mathbb{N}} P(Y_n\ge\lambda) = 0, \qquad (2.8)$$

and on the other hand, for any $\lambda,\delta>0$ with $A(\delta)$ according to Theorem 1.2,

$$\begin{aligned}&\limsup_{m\to\infty}\sup_{n\in\mathbb{N}} P(X_{n,m}\ge\lambda)\\ &\quad\le \frac{1}{\lambda}\Big(\delta + A(\delta)\limsup_{m\to\infty}\max\left[E(|M_1-M_1^m|), E(|M_1-M_1^m|\log^+|M_1-M_1^m|)\right]\Big)\\ &\quad= \frac{\delta}{\lambda},\end{aligned}$$

which implies

$$\lim_{m \to \infty} \sup_{n \in \mathbb{N}} P(X_{n,m} \geq \lambda) = 0 \quad \text{for any } \lambda > 0. \tag{2.9}$$

Now combine (2.8) and (2.9) to obtain for $\lambda, \mu > 0$,

$$\begin{aligned} \limsup_{m \to \infty} \sup_{n \in \mathbb{N}} P(X_{n,m} Y_n \geq \lambda) &\leq \sup_{n \in \mathbb{N}} P(Y_n \geq \mu) + \lim_{m \to \infty} \sup_{n \in \mathbb{N}} P(X_{n,m} \geq \lambda/\mu) \\ &= \sup_{n \in \mathbb{N}} P(Y_n \geq \mu) \end{aligned}$$

and therefore

$$\lim_{m \to \infty} \sup_{n \in \mathbb{N}} P(X_{n,m} Y_n \geq \lambda) = 0 \quad \text{for any } \lambda > 0. \tag{2.10}$$

Finally use (2.9) and (2.10) to show that the left-hand side of (2.7) converges to zero as $m \to \infty$, uniformly in $n \in \mathbb{N}$. This completes the proof. □

The main result of this paper is now an easy consequence of Lemma 2.8 and Corollary 2.7.

THEOREM 2.9. *Assume $\mathbb{F}^1$ and $\mathbb{F}^2$ are quasi-left continuous. Then every $L\log^+ L$-bounded martingale possesses quadratic variation in $L^0(\Omega, \mathscr{F}, P)$.*

PROOF. For $k, n, m \in \mathbb{N}$ we have

$$\begin{aligned} \Bigl| \sum_{J \in \mathbb{J}_k} (\Delta_J M)^2 - \sum_{J \in \mathbb{J}_n} (\Delta_J M)^2 \Bigr| &\leq \Bigl| \sum_{J \in \mathbb{J}_k} (\Delta_J M)^2 - \sum_{J \in \mathbb{J}_k} (\Delta_J M^m)^2 \Bigr| \\ &\quad + \Bigl| \sum_{J \in \mathbb{J}_k} (\Delta_J M^m)^2 - \sum_{J \in \mathbb{J}_n} (\Delta_J M^m)^2 \Bigr| \\ &\quad + \Bigl| \sum_{J \in \mathbb{J}_n} (\Delta_J M^m)^2 - \sum_{J \in \mathbb{J}_n} (\Delta_J M)^2 \Bigr|. \end{aligned} \tag{2.11}$$

Apply Lemma 2.8 to the first and last terms on the right-hand side of (2.11) and Corollary 2.7 to the second term to show that the left-hand side goes to zero in $L^0(\Omega, \mathscr{F}, P)$ as $k, n \to \infty$. □

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