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Identifiability of Compound Poisson Distributions

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Abstract

Compound Poisson distributions (CPD's) are frequently used as alternatives in studying situations where a simple Poisson model is found inadequate to describe. In this paper an attempt is made to identify compound Poisson distributions when it is known that the conditional distribution of two random variables (r.v.'s) is compound binomial. Some interesting special cases and their application to accident theory are discussed.

1. Introduction

Let X, Y be two non-negative and integer-valued r.v.'s such that $X \geq Y$. It is known (see, e.g. Wang, 1975) that if the conditional distribution of Y given X is binomial with parameters n, p (i.e., $Y|(X=n)$ is binomial (n, p), $n=0, 1, 2, \ldots, 0<p<1$) then Y is a Poisson r.v. with parameter λp (Poisson (λp)), $\lambda>0$ if and only if (iff) X is a Poisson (λ) r.v. There has been an extensive use of this model in various practical situations.

Accident theory is a field where this model can have potential applications. The reason lies in the form invariance of the distributions of X and Y under the binomial assumption for the distribution of $Y|(X=n)$. Leiter and Hamdan's (1973) work provides an example in this direction. They have considered the random vector (X, Y) defined as above as a model for the interpretation of highway accidents. In this context, X represented the number of highway accidents in a given locality and for a given period of time and Y represented the number of fatal accidents among those X accidents over the same period. However, applying the model to bivariate accident data of this type, did not result in a satisfactory fit.

It seems reasonable, therefore, to question the model with respect to the assumptions on the distribution of $Y|(X=n)$ or X or both. One may consider, for instance, that either p or λ or both are not constants. Instead, one may regard them as being r.v.'s themselves with some distribution functions, say $F(p)$, $0<p<1$ and $G(\lambda)$, $0<\lambda<+\infty$ respectively. In this case, the distributions of $Y|(X=n)$ and X will be of a compound form. In particular, the distribution of $Y|(X=n)$ will be binomial compounded by $F(p)$ (binomial $(n, p) \wedge F(p)$) while that of X will by Poisson $(\lambda) \wedge G(\lambda)$. The question arises then, whether the distribution of Y can be identified from knowledge concerning the form of the distribution of X and vice versa. Any results in

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this direction will have important implications in the area of applications as one will not have to restrict oneself to the case of a constant p or λ. The assumption of p and λ being r.v.'s with distributions of general type offers a much more relaxed condition. Moreover, different forms of $F(p)$ and $G(\lambda)$ will lead to several alternative models.

The following sections provide some answers to the problem. Specifically, section 2 develops some theoretical results along the lines described above while section 3 examines the consequences of these results in the context of some applied problems.

2. The main results

Before stating the main results we list a few basic facts that will be utilized in the course of the proofs.

Let $Z_{(k)}$ denote the descending factorial i.e.

$$Z_{(k)} = Z(Z-1)\ldots(Z-k+1), \quad k=0,1,2,\ldots;(Z_{(0)}=1).$$

Then,

Z is binomial (n,p) implies that the *kth* factorial moment (f.m.) of Z is $EZ_{(k)} = n_{(k)}p^k$.

Also

$$Z \text{ is Poisson } (\lambda) \text{ implies } EZ_{(k)} = \lambda^k \tag{1}$$

Then, it follows immediately that

$Y|(X=n)$ is binomial $(n,p) \wedge F(p)$ implies

$$E(Y_{(k)}|X=n) = n_{(k)} \int_0^1 p^k \, dF(p)$$

whence

$$EY_{(k)} = EX_{(k)} \int_0^1 p^k \, dF(p) \tag{2}$$

Theorem 2.1. *Suppose that for the non-negative, integer-valued r.v.'s X, Y we have that*

$$P(Y=r|X=n) = \int_0^1 \binom{n}{r} p^r q^{n-r} \, dF(p), \quad r=0,1,\ldots;q=1-p \tag{3}$$

(i.e., $Y|(X=n)$ is binomial $(n,p)\wedge F(p)$). Then, Y is Poisson $(\lambda p)\wedge F(p)$ iff X is Poisson (λ).

Proof. The "if" part is straightforward. For the "only if" part suppose that Y is Poisson $(\lambda p)\wedge F(p)$, i.e., that

$$P(Y=r)=\int_0^1 \exp(-\lambda p)\frac{(\lambda p)^r}{r!}\,dF(p), \quad r=0,1,\dots$$

This implies that

$$EY_{(k)}=\int_0^1 \sum_{r=k}^{\infty} \exp(-\lambda p)\frac{(\lambda p)^r}{r!}\,r_{(k)}\,dF(p)$$

$$=\int_0^1 (\lambda p)^k\,dF(p), \quad k=1,2,\dots \tag{4}$$

On the other hand, because of (3) we have that relation (2) is valid. Combining (2) and (4) we find that

$$EX_{(k)}=\lambda^k, \quad k=1,2,\dots \tag{5}$$

Consequently, the f.m.'s of the distribution of the r.v. X are the same as those corresponding to the Poisson distribution. This implies that the r.v. X has the same moments as a Poisson variable. Since the Poisson distribution is uniquely determined by its moments the result follows.

Theorem 2.2. *Let X, Y be as in Theorem 2.1. In addition, assume that the distribution of X is determined uniquely by its f.m.'s and that*

$$\int_0^{\infty} e^{h\lambda}\,dG(\lambda)<\infty \quad \text{for some } h>0. \tag{6}$$

Then Y is Poisson $(\lambda p)\wedge G(\lambda)\wedge F(p)$ iff X is Poisson $(\lambda)\wedge G(\lambda)$.

Proof. The "if" part of the proof is straightforward. As far as the "only if" part is concerned, by following the argument of the proof of Theorem 2.1 we have that

$$EY_{(k)}=\int_0^{\infty} \lambda^k\,dG(\lambda)\int_0^1 p^k\,dF(p), \quad k=1,2,\dots$$

Therefore,

$$EX_{(k)}=\int_0^{\infty} \lambda^k\,dG(\lambda), \quad k=1,2,\dots \tag{7}$$

It is now known that the kth f.m. of the CPD is of the form (7). Since we have assumed that the distribution of X is uniquely determined by its f.m.'s and because of (6), it follows that

$$ES^X = E\sum_k X_{(k)}\frac{(s-1)^k}{k!} = \int_0^\infty \sum_k \frac{\lambda^k(s-1)^k}{k!}\,dG(\lambda)$$
$$= \int_0^\infty e^{\lambda(s-1)}\,dG(\lambda), \quad |s-1| \leq h.$$

Thus, by the uniqueness of probability generating functions, X is Poisson $(\lambda)\wedge G(\lambda)$.

Remarks

1. It can be observed that if $G(\lambda)$ is degenerate, Theorem 2.2 reduces to Theorem 2.1.

2. An interesting problem concerning Theorem 2.2 is that of relaxing the condition that the distribution of X is uniquely determined by its f.m.'s.

3. It is clear that for different forms of $F(p)$ and $G(\lambda)$ Theorems 2.1 and 2.2 provide characterizations for different forms of CPD's. Suppose, for instance that $p\sim$beta I (a, b) and $\lambda\sim$gamma $(a+b, m)$ i.e.,

$$dF(p) = \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)}p^{a-1}(1-p)^{b-1}\,dp, \quad 0<p<1$$

and

$$dG(\lambda) = \frac{m^{-(a+b)}}{\Gamma(a+b)}e^{-\frac{\lambda}{m}}\lambda^{a+b-1}\,d\lambda, \quad \lambda>0$$

where $a>0$, $b>0$, $m>0$. Then, by Theorem 2.2 $Y\sim$negative binomial $(a, m/(m+1))$ i.e.,

$$P(Y=y) = \binom{a+y-1}{y}\left(\frac{m}{m+1}\right)^y\left(\frac{1}{m+1}\right)^a, \quad y=0,1,2,\ldots$$

iff $X\sim$negative binomial $(a+b, m/(m+1))$.

Also, if $F(p)$ is degenerate and $\lambda\sim$gamma (a, m) then, from Theorem 2.1, $Y\sim$negative binomial $(a, mp/(mp+1))$ iff X-negative binomial $(a, m/(m+1))$.

4. Identifiability problems connecting the distributions of Y and $Y|X$ when X is known to follow a Poisson or a compound Poisson distribution are studied in Panaretos (1981).

5. Godambe (1977) has studied necessary and sufficient conditions for CPD's, with the compounding distribution of an exponential type, to be equivalently represented as Poisson sums. An interesting open problem here is to see how Godambe's results can be utilized so that Theorems 2.1 and 2.2 be extended to Poisson sums.

6. The model with $Y|(X=n)\sim$binomial $(n, p)\wedge F(p)$ can also be viewed as an extension of the damage model considered by Rao (1963). In this model X represents an original observation produced by a natural process (e.g.,

number of eggs laid by an insect). $Y|(X=n)$ is the damage process and Y is the resulting observation. Some of the aspects regarding an extension of the damage model have been examined in Panaretos (1979).

7. In the damage model set-up Rao (1963) observed that, with binomial damage, Y is negative binomial iff X is negative binomial. Remark 3 indicates that this property of the damage model is preserved if one allows the parameter p of the binomial damage process to vary according to a beta I law.

8. Going back to the accident situation examined by Leiter & Hamdan (1973) one may notice, from their numerical results, that the Poisson distribution fits the observed distribution of Y partricularly well. However, this is not the case as far as the distribution of X is concerned; the agreement between observed and Poisson expected frequencies is rather poor. This signals the need for examining, among other possibilities, the question as to whether the situation can be better explained by any of the models studied in this paper.

3. *Discussion*

The CPD which results from compounding the Poisson by a gamma distribution seems to have been first adopted in connection with applications to accident theory by Greenwood & Woods (1919). The underlying assumption was that the accident experience of each individual was Poisson with mean value λ varying from individual to individual in the gamma law. This led to the introduction of the concept of accident proneness. The results of Theorem 2.1 and 2.2 may be of some interest in this direction of accident theory, especially in connection with actuarial studies. In this context, X may denote the number of incurred accidents and Y can be the number of reported accidents. Here, one is justified to assume that each accident is reported with probability p which varies from accident to accident. (Social, legal or financial pressures may encourage one to underreport one's accidents). Consequently, the model with $Y|(X=n)$ following a compound binomial distribution might be appropriate. In this case, let the distribution of p be known and assume that there is evidence to suggest that Y is compound Poisson distributed. Then, by Theorems 2.1 and 2.2, the distribution of X will also be a compound Poisson and hence, one would be justified in suspecting the presence of accident proneness. Moreover, determining the exact form of the unknown distribution of X will be equivalent to identifying the form of the compounding process in the observable distribution of Y.

Let us now consider the results of Theorems 2.1 and 2.2 in relation to another distribution applicable to accident theory, namely the univariate generalized Waring distribution with parameters $\alpha, \beta, \varrho > 0$ (UGWD $(\alpha, \beta; \varrho)$) and probability function given by

$$p_r = \frac{\varrho_{(\beta)}}{(\alpha+\varrho)_{(\beta)}} \frac{\alpha_{(r)}\beta_{(r)}}{(\alpha+\beta+\varrho)_{(r)}} \frac{1}{r!}, \quad r=0,1,2,\ldots, \quad \alpha_{(\beta)} = \Gamma(\alpha+\beta)/\Gamma(\alpha).$$

(For more details concerning the structure and applications of this distribution see Irwin (1975), Xekalaki (1981).)

One may observe that if $Y|(X=n) \sim$ negative hypergeometric $(n; m, N)$ i.e., if

$$P(Y=r|X=n) = \frac{\binom{-m}{r}\binom{-N+m}{n-r}}{\binom{-N}{n}}, \quad r=0,1,2,\ldots,n, \quad N>m>0$$

and $X \sim$UGWD $(\alpha, N; \varrho)$ then $Y\sim$UGWD $(\alpha, m; \varrho)$.

In the case $N=m+1$ the converse is also true i.e., on the assumption that $Y|(X=n) \sim$ negative hypergeometric $(n; m, m+1)$ and $Y \sim$ UGWD $(\alpha, m; \varrho)$ then $X \sim$ UGWD $(\alpha, m+1; \varrho)$.

To prove this note that

$$q_r = \sum_{n=r}^{\infty} p_n \binom{m+r-1}{r} \Big/ \binom{m+n}{n}, \quad r=0,1,2,\ldots,$$

where $q_r=P(Y=r)$ and $p_n=P(X=n)$. This is a functional equation in p_n. Since $Y \sim$ UGWD $(\alpha, m; \varrho)$, a solution for p_n is the UGWD $(\alpha, m+1; \varrho)$. The uniqueness follows if one observes that

$$\sum_{n=r}^{\infty} (p_n - p_n^*) \Big/ \binom{m+n}{n} = 0, \quad r=0,1,2,\ldots,$$

where p_n^* is another solution. If H_r denotes the left hand side of this equation, it follows that $H_r - H_{r+1} = 0$, $r=0,1,2,\ldots$ i.e., $p_r = p_r^*$, $r=0,1,2,\ldots$

Taking now into account the fact that the negative hypergeometric $(n; m, N)$ can be viewed as binomial $(n, p) \underset{p}{\wedge}$ beta I $(m, N\text{-}m)$ and the UGWD $(\alpha, \beta; \varrho)$ as negative binomial $(\beta, c/(1+c)) \underset{c}{\wedge}$ beta II (α, ϱ) (see e.g. Xekalaki (1981)), where beta II (α, ϱ) is the distribution with density function

$$f(x) = \frac{\Gamma(\alpha+\varrho)}{\Gamma(\alpha)\Gamma(\varrho)} x^{\alpha-1}(1+x)^{-(\alpha+\varrho)}, \quad x>0$$

the above result can equivalently be stated as follows:

If $Y|(X=n) \sim$ binomial $(n, p) \wedge F(p)$ then $Y \sim$ Poisson $(\lambda p) \wedge G(\lambda) \wedge F(p)$ iff $X \sim$ Poisson $(\lambda) \wedge G(\lambda)$ where $F(p)$ and $G(\lambda)$ are the distribution functions of the standard power function (beta I $(m, 1)$) and gamma $(m+1, c) \underset{c}{\wedge}$ beta II (α, ϱ) laws respectively.

However, this result cannot be thought of as a special case of Theorem 2.2 as the UGWD is not uniquely determined by its f.m.'s. By contrast, the assumption that the distribution of X is uniquely determined by its f.m.'s is redundant in the case of the negative binomial distribution of Remark 3. Combining these two facts together with the fact that both the UGWD and the negative binomial distribution arise very often in the context of accident theory makes this assumption rather strong and hence, any results along the lines suggested by Remark 2 will be very interesting.

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