

A PROPERTY OF THE YULE DISTRIBUTION AND ITS APPLICATIONS

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ABSTRACT

The relationship  $Y = RX$  between two random variables  $X$  and  $Y$ , where  $R$  is distributed independently of  $X$  in  $(0,1)$ , is known to have important consequences in different fields such as income distribution analysis, inventory decision models, etc.

In this paper it is shown that when  $X$  and  $Y$  are discrete random variables, relationships of similar nature lead to Yule-type distributions. The implications of the results are studied in connection with problems of income underreporting and inventory decision making.

1. INTRODUCTION

The relationship between two random variables (r.v.'s)  $X$  and  $Y$  where  $Y < X$  is of potential importance in the context of economic models in various areas such as in income distribution analysis ( $X$  = true income,  $Y$  = reported income), productivity measurement ( $X$  = true labor input,  $Y$  = observed labor input) or in inventory decision making ( $X$  = demand for an item within a unit time

interval,  $Y$  = item unit in stock within the same time interval).  
 Krishnaji (1970b) has examined a model of multiplicative under-reporting of income in which the reported income  $Y$  related to the true income  $X$  through the relationship

$$Y = RX \quad (1)$$

where  $R$  was a r.v. independent of  $X$  and with support on an interval  $(a,b) \subseteq [0,1]$ . In particular, he examined the effect of (1) on Pareto income distributions by showing that if  $R$  has the density function (d.f.)

$$h(r) = pr^{p-1}, \quad p > 0, \quad 0 \leq r \leq 1 \quad (2)$$

then,  $X \stackrel{d}{=} Y | (Y > x_0)$ ,  $x_0 \geq 0$  if and only if (iff)  $X$  has a Pareto distribution on  $(x_0, +\infty)$ .

There are theoretical reasons for regarding the Pareto distribution as an approximation to a more general distribution defined by Yule (1924) with probability function (p.f.) given by

$$p_x = \frac{\rho x!}{(\rho+1)_{(x+1)}}, \quad \rho > 0, \quad x=0,1,2,\dots \quad (3)$$

where

$$a_{(b)} = \Gamma(a+b)/\Gamma(a), \quad a > 0, \quad b > 0.$$

These reasons can be traced in Irwin's (1975) derivation of the Pearson type VI distribution as the continuous analogue of the generalized Waring distribution (whose special case is the Yule distribution in (3)). It is worth noticing that Simon (1955) showed that distributions of incomes are expressible in terms of the Yule distribution. It would therefore be interesting to examine the behavior of this income distribution under a model that will imply underreporting of income. The following section looks into this problem.

## 2. THE MAIN RESULTS

Let  $X$  and  $Y$  be defined as before and let  $p_r$  and  $q_r$ ,  $r=0,1,2,\dots$  denote their p.f.'s respectively. Then, if incomes are underreported ( $P(Y < X) = 1$ ), it follows that the observed

income distribution is

$$q_r = P(Y=r) = P(Y=r | Y < X) = \frac{\sum_{x=r+1}^{\infty} P(Y=r | X=x)p_x}{\sum_{r=0}^{\infty} \sum_{x=r+1}^{\infty} P(Y=r | X=x)p_x},$$

i.e.,

$$q_r = \frac{1}{1-p_0} \sum_{x=r+1}^{\infty} P(Y=r | X=x)p_x, \quad r=0,1,2,\dots \quad (4)$$

An underreporting model of the multiplicative type as in (1), appropriately modified to account for discrete income distributions, might possibly be

$$Y = [RX], \quad \text{whenever } X > 0$$

with  $R$  defined as in (1) and  $[a]$  denoting the integral part of  $a$ . Krishnaji (1970a), though not specifically referring to economic models, showed that when  $R$  is uniformly distributed on  $(0,1)$  with d.f. given by (2) for  $p=1$ , then  $p_x$  has a zero-truncated Yule form iff  $p_r = q_r/(1-q_0)$ ,  $r \geq 1$ , i.e., iff  $X \stackrel{d}{=} Y | (Y > 0)$ .

This result, brought within the framework of income underreporting, implies that truncating the observed distribution of incomes at the point zero, we can recover information about the true distribution, provided of course that it is 0-truncated. This generates the question of what happens in cases where the observed distribution is truncated at a point  $k-1$ ,  $k > 1$ , or, more generally, in cases where we can only observe the tail frequencies of the distribution of  $Y$  beyond a point  $k-1$  which, of course, are more realistic assumptions. What can we then say about the true income distribution? Any result in this direction will be in analogy to Krishnaji's (1970b) result in the continuous case for the Pareto distribution. The answer to this question is given by the theorem given below. In the sequel, we adopt Krishnaji's (1970a) assumption for  $R$ , i.e., we consider the model

$$Y = [RX], \quad X > 0, \quad R \sim \text{uniform in } (0,1), \quad R \text{ independent of } X. \quad (5)$$

Note that from (5) and relationship (4) we have that

$$\begin{aligned} q_r &= \frac{1}{1-p_0} \sum_{x=r+1}^{\infty} P([RX]=r | X=x) p_x \\ &= \frac{1}{1-p_0} \sum_{x=r+1}^{\infty} P(r \leq RX < r+1 | X=x) p_x \\ &= \frac{1}{1-p_0} \sum_{x=r+1}^{\infty} P\left(\frac{r}{x} \leq R < \frac{r+1}{x}\right) p_x, \end{aligned}$$

i.e.,

$$q_r = \frac{1}{1-p_0} \sum_{x=r+1}^{\infty} \frac{p_x}{x}, \quad r=0,1,2,\dots \quad (6)$$

(An alternative consideration may be to assume that the underreporting chance mechanism is represented not by (5) but by the conditional distribution of  $Y$  given  $X$  with p.f.  $P(Y=y | X=x) = \frac{1}{x}$ ,  $0 \leq y < x$ . Then, under this model which again implies that  $Y < X$ , we can obtain relationship (6) using (4). In this way underreporting can be thought of as "additive damage" in the sense of Patil and Ratnaparkhi (1975). (See also Krishnaji (1970a) and Kekalaki (1980).)

**Theorem 1:** Let  $X, Y$  be nonnegative, integer-valued r.v.'s satisfying (5) (or (6)). Then,

$$q_r = c_0 p_r, \quad r = k, k+1, \dots, k \geq 0, c_0 > 0 \quad (7)$$

iff  $X$  has a "modified Yule" distribution, i.e., iff

$$p_x = p_k \frac{\binom{k+1}{x-k}}{\binom{k+c+1}{x-k}}, \quad x \geq k, \quad c^{-1} = c_0(1-p_0). \quad (8)$$

**Proof:** Necessity: Let (5) (or (6)) and (8) be true. Then

$$q_r = \frac{1}{1-p_0} \frac{\binom{k+1}{-k}}{\binom{k+c+1}{-k}} \frac{r!}{(c+1)_{(r)}} \sum_{x=0}^{\infty} \frac{\binom{r+1}{x}}{(c+r+2)_{(x)}}.$$

But,  $\sum_{x=0}^{\infty} \frac{\binom{r+1}{x}}{(c+r+2)_{(x)}} = \frac{\Gamma(c+r+2)\Gamma(c)}{\Gamma(c+1)\Gamma(c+r+1)}$  (see Erdélyi (1953)). Hence

$$\begin{aligned} q_r &= \frac{p_k}{1-p_0} \frac{\binom{k+1}{-k}}{\binom{k+c+1}{-k}} \frac{1}{(c+1)_{(r)}} \frac{c+r+1}{c} \\ &= c_0 p_k \frac{\binom{k+1}{r-k}}{\binom{k+c+1}{r-k}} = c_0 p_r, \quad r \geq k. \end{aligned}$$

Therefore necessity has been established.

Sufficiency: Assume that (5) (or (6)) and (7) are true.

Then, for  $r \geq k$ ,  $p_r = \frac{1}{c_0(1-p_0)} \sum_{x=r+1}^{\infty} \frac{p_x}{x}$ , which implies that

$p_r - p_{r+1} = c \frac{p_{r+1}}{r+1}$ , i.e.,  $p_{r+1} - \frac{r+1}{r+c+1} p_r = 0$ . Therefore

$$p_r = p_k \frac{(k+1)(r-k)}{(k+c+1)(r-k)}, \quad r \geq k.$$

What this theorem says in fact is that when  $p_x$  is untruncated then we can only know its tail frequencies explicitly. However, as the following corollary shows, if  $p_x$  is  $(k-1)$ -truncated, the constant  $p_k$  in (8) can be determined from the condition  $\sum_{x=k}^{\infty} p_x = 1$  and hence, full knowledge of the form of  $p_x$  is attainable.

**Corollary 1:** Let  $X$  be an integer-valued r.v. defined on the set  $\{k, k+1, \dots\}$ ,  $k \geq 0$ . Let  $Y$  be another integer-valued r.v. defined on  $\{0, 1, 2, \dots\}$  and such that  $q_r$  relates to  $p_r$  through relation (5) (or (6)). Then  $X \stackrel{d}{=} Y | (Y \geq k)$  iff  $X$  has a  $(k-1)$ -truncated Yule distribution with p.f. given by

$$p_r = \rho \frac{(k+1)(x-k)}{(\rho+k+1)(x-k+1)}, \quad r = k, k+1, \dots \quad \text{where } \rho = q_0/(1-q_0). \quad (9)$$

Clearly, Krishnaji's result follows from this corollary, for  $k = 1$ . Another interesting special case of Theorem 1 arises when  $Y$  is also truncated at  $k-1$ ,  $k \geq 0$ . This can be stated in the form of the following corollary.

**Corollary 2:** Let  $X, Y$  be r.v.'s on  $\{k, k+1, \dots\}$ ,  $k \geq 0$  and assume that (5) (or (6)) holds. Then,  $X \stackrel{d}{=} Y$  iff  $X \sim (k-1)$ -truncated Yule distribution with parameter  $\rho$  as in (9).

Hence, under (5) (or (6)), the true income distribution is identical to the observed income distribution only when the former has a Yule form.

It could be argued that the results of Corollaries 1 and 2 would be more useful in practice if they were obtained in terms of the distribution of  $Y$  which, after all, is what we observe.

Indeed, this is possible when  $k > 0$  by virtue of the following theorem.

**Theorem 2:** Let  $X, Y$  be r.v.'s defined on  $\{1, 2, \dots\}$  and  $\{0, 1, 2, \dots\}$  respectively and assume that (5) (or (6)) holds. Then  $X$  has a  $(k-1)$ -truncated Yule distribution ( $k > 0$ ) with parameter  $\rho > 0$  as in (9) iff  $Y$  has a distribution with p.f.

$$q_r = \begin{cases} \frac{\rho}{(\rho+1)k} & r = 0, 1, \dots, k-1 \\ \frac{1}{\rho+1} u_r & r = k, k+1, \dots \end{cases} \quad (10)$$

where  $u_r$  is given by the right-hand side of (9).

**Proof:** The "necessity" part is tedious but straightforward. For sufficiency assume that (10) holds and observe that from (5) (or (6)) we have

$$\frac{p_{r+1}}{r+1} = q_r - q_{r+1}, \quad r = 0, 1, 2, \dots \quad (11)$$

Then, for  $r = 1, 2, \dots, k-1$ , (11) implies that  $p_r = 0$ , i.e.,  $p_r$  is truncated at the point  $k-1$ . On the other hand, for  $r > k-1$  we obtain (from (10) and (11))  $\frac{p_{r+1}}{r+1} = \frac{\rho}{r+1} \frac{(k+1)(r-k+1)}{(k+\rho+1)(r-k+2)}$ , i.e.,

$$p_r = \rho \frac{(k+1)(r-k)}{(k+\rho+1)(r-k+2)}, \quad r = k+1, k+2, \dots \quad (12)$$

Notice now that  $q_0 = q_k + p_k/k$ , i.e.,  $\rho = \frac{p_k}{k} + \frac{\rho}{(\rho+1)(k+1)}$ . Hence,  $p_k = \rho/(\rho+k+1)$  which implies that  $p_r$  is given by (12) for  $r = k, k+1, \dots$ . This completes the proof of the theorem.

What has just been shown is that, under (5) (or (6)) the actual income distribution is a Yule truncated at  $k-1$  if and only if the observed income distribution has probabilities proportional to a  $(k-1)$ -truncated Yule, for  $r \geq k$  with the first  $k$  probabilities proportional to  $1/k$ :

Clearly, Theorems 1 and 2 imply the following corollaries.

**Corollary 3:** Let  $X, Y$  be defined as in Corollary 1 with  $k > 0$  and assume that (5) (or (6)) holds. Then  $X \stackrel{d}{=} Y | (Y \geq k)$  iff  $Y$  has a distribution given by (10).

**Corollary 4:** Let  $X, Y$  be defined as in Corollary 2 with  $k > 0$

and assume that (5) (or (6)) holds. Then  $X \stackrel{d}{=} Y$  iff  $Y$  is distributed as in (10).

Hence, according to the latter result, an observed income distribution of the form (10) identifies (under (5) or (6)) the true income distribution as being a Yule truncated at  $k-1$ .

### 3. SOME FURTHER APPLICATIONS

So far, we have restricted ourselves to income distribution problems. However, the method of analysis may be applied to a number of other models. The distribution in (6), for instance, often arises in connection with various inventory decision models (see, e.g., Prichard and Eagle (1965), Foster et al. (1971)). As an example, consider a reorder point system in which additional items are ordered whenever the stock  $Y$  falls to a particular level  $y_0$ . Prichard and Eagle (1965) imposed a decision function on the expected fraction of lead time out of stock which amounted to selecting the reorder point to be the value  $y_0$  of  $Y$  for which  $E(T | Y=y_0) - y_0 \sum_{x=y_0+1}^{\infty} p_x/x$  first becomes less than an administratively set constant  $\tau$ . Here  $T$  is the fraction of lead time for which the demand  $X$  exceeds the stock in hand. Unless  $p_x/x$  becomes small fairly rapidly, the algorithm for the determination of the value  $y_0$  can be tedious as it involves summation to infinity of such terms.

Theorem 1, however, can offer some help in cases of a Yule distributed demand. A model that can give rise to a Yule demand distribution may be the following.

Assume that in a warehouse, demands for an item occur at random and let their distribution be the Poisson with parameter  $\lambda > 0$ . Suppose that each demand is for a number of units which may be considered as a r.v. taking nonnegative and integer values according to a log series distribution with probability generating function (p.g.f.) given by  $g_\theta(s) = 1 - \log[e^\theta - (e^\theta - 1)s]/\lambda$ ,  $\theta > 0$ , where  $\theta$  refers to the buyer's behavior. Then, if the numbers of units ordered by different demands are independent, the overall distribution of demand will have p.g.f. given by  $G_\theta(s) = \exp[\lambda(g_\theta(s) - 1)]$

$= [e^\theta - (e^\theta - 1)s]^{-1}$  (i.e., geometric with parameter  $e^{-\theta}$ ). Assume further that  $\theta$  varies from buyer to buyer and let its distribution be the exponential with parameter  $\rho > 0$ . Then, the final resulting demand distribution is  $G(s) = \rho \sum_{r=0}^{\infty} s^r \int_0^1 \theta^\rho (1-\theta)^r d\theta$   
 $= \rho \sum_{r=0}^{\infty} r! s^r / (\rho+1)_{(r+1)}$ , i.e.,  $X \sim \text{Yule}(\rho)$ . Consider now an inventory situation where the Yule ( $\rho$ ) distribution may be appropriate for describing the demand fluctuations. Then Prichard and Eagle's decision rule becomes: choose as a reorder point the value  $y_0$  of the stock  $Y$  for which  $P(X > y_0) - y_0(1-p_0) P(X = y_0) \leq \tau$ , or, after some simplification, choose  $y_0$  so that  $\Gamma(y_0 + \rho + 1) / \Gamma(y_0 + 1) \geq \Gamma(\rho + 1) / \tau(\rho + 1)$ . This is clearly a much simpler expression in terms of  $y_0$ .

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