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On Recursive Evaluation of Mixed Poisson Probabilities and Related Quantities

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Recursive formulae are derived for the probabilities of a wide variety of mixed Poisson distributions. Known results are unified and extended. Related formulae are discussed for transformed mixing random variables, shifted and truncated mixing distributions, compound distributions, and tail probabilities. Applications of these models are briefly discussed. *Key words: Pearson system, Pareto, generalized gamma, transformed beta, Weibull, Delaporte, Rayleigh, compound distributions, tail probabilities, power transformation, Poisson arrival queue, hazard rate, force of mortality.*

1. INTRODUCTION AND BACKGROUND

Mixed Poisson distributions are widely used for modeling claim counts when the portfolio is thought to be heterogeneous, and the mixing distribution represents a measure of this heterogeneity. See, for example, Bühlmann (1970) or Beard et al. (1984). With the exception of negative binomial or Poisson-inverse Gaussian distributions (e.g. Willmot, 1987), most mixed Poisson distributions tend to be difficult to evaluate, a factor inhibiting their use. Other parametric mixing distributions are considered in Albrecht (1984) and Willmot (1986) and references therein.

In this paper it is demonstrated that simple recursive formulae hold for the mixed Poisson probabilities for a wide variety of continuous mixing distributions, even though explicit formulae are complicated. Thus, define

$$p_m = \int_{x_0}^{x_1} \frac{x^m e^{-x}}{m!} f(x)\, dx; \quad m = 0, 1, 2, \ldots \tag{1}$$

to be the mixed Poisson probabilities where $f(x)$ is a probability density function (pdf) and $0 \leqslant x_0 < x_1 \leqslant \infty$. While (1) may be difficult to evaluate for certain choices of $f(x)$, simple recursive relations may hold (i.e. when (7) below is satisfied). The mixed Poisson probability generating function (pgf) associated with (1) is

$$P(z) = \sum_{m=0}^{\infty} p_m z^m = \int_{x_0}^{x_1} e^{x(z-1)} f(x)\, dx, \quad |z| < 1. \tag{2}$$

In Section 2 a simple technique is presented for the derivation of recursive formulae for a wide variety of choices of $f(x)$. In addition to unifying known results for distributions such as the negative binomial, Sichel's distribution (gamma and generalized inverse Gaussian mixtures, respectively), and the Poisson-beta mixture, other mixing distributions such as the Pareto and generalized Pareto are considered. The situation where the hazard rate or force of mortality of the mixing distribution is a ratio of polynomials is discussed.

Power transformed mixing random variables such as the transformed or generalized gamma and the transformed beta are considered in Section 3. In addition, inverse random variables such as the inverse gamma are also discussed.

In Section 4 it is shown how the results may be easily extended to shifted mixed distributions, an example being the Delaporte distribution (e.g. Schröter, 1990) as well as truncated mixing distributions such as the truncated normal and the truncated gamma. Compound mixed Poisson recursions are discussed in Section 5.

Evaluation of the tail probabilities,

$$g_m = \sum_{k=m+1}^{\infty} p_k; \quad m = 0, 1, 2, \ldots \tag{3}$$

with generating function (e.g. Feller, 1968, p. 265)

$$G(z) = \sum_{m=0}^{\infty} g_m z^m = \frac{P(z) - 1}{z - 1} \tag{4}$$

is considered in Section 6 along with applications to queueing theory. Define

$$\bar{F}(x) = \int_x^{x_1} f(y)\, dy \tag{5}$$

(with the understanding that $\bar{F}(x) = 1$ if $x < x_0$ and $\bar{F}(x) = 0$ if $x > x_1$) to be the survivor function of the mixing distribution. It is well known and easily shown (using integration by parts on (2), for example) that

$$G(z) = \int_0^{\infty} e^{x(z-1)} \bar{F}(x)\, dx = \frac{e^{x_0(z-1)} - 1}{z - 1} + \int_{x_0}^{x_1} e^{x(z-1)} \bar{F}(x)\, dx. \tag{6}$$

Further remarks and extensions are considered in Section 7, including an exponential-inverse Gaussian mixture (e.g. Bhattacharya & Kumar, 1986).

2. A GENERAL RESULT

In this section a fairly general method is proposed for the derivation of recursive formulae for the evaluation of mixed Poisson probabilities. The basic methodology is presented, and various examples are given which both unify and extend known results.

Consider the mixed Poisson probabilities defined by (1) with pgf (2). Suppose that

$$\frac{d}{dx} \log f(x) = \frac{\eta(x)}{\psi(x)} = \frac{\sum_{n=0}^{i} \eta_n x^n}{\sum_{n=0}^{j} \psi_n x^n}. \tag{7}$$

Densities which satisfy (7) are a generalization of the Pearson system (e.g. Ord, 1972, pp. 8–9) since $d/dx \log f(x)$ may be expressed as a ratio of polynomials. For notational convenience, let $k = \max(i, j)$ and define $\eta_n = 0$ for $n \notin (0, 1, 2, \ldots, i)$ and $\psi_n = 0$ for $n \notin (0, 1, 2, \ldots, j)$. Then $\eta(x)$ and $\psi(x)$ may be assumed to be polynomials of degree k.

Note that

$$\frac{d}{dx}\{e^{x(z-1)}f(x)\psi(x)\} = e^{x(z-1)}f(x)\{z\psi(x) + \phi(x)\} \tag{8}$$

where

$$\phi(x) = \sum_{n=0}^{k} \phi_n x^n = \eta(x) + \psi'(x) - \psi(x). \tag{9}$$

Obviously, one has from (9) that

$$\phi_n = \eta_n + (n+1)\psi_{n+1} - \psi_n; \quad n = 0, 1, 2, \ldots, k. \tag{10}$$

One may then integrate (8) over (x_0, x_1) to get the differential equation

$$\sum_{n=0}^{k} \{z\psi_n + \phi_n\}P^{(n)}(z) = f(x_1)\psi(x_1)e^{x_1(z-1)} - f(x_0)\psi(x_0)e^{x_0(z-1)} \tag{11}$$

where $P^{(n)}(z) = \int_{x_0}^{x_1} x^n e^{x(z-1)} f(x)\,dx$ is the nth derivative of $P(z) = P^{(0)}(z)$. One may equate coefficients of z^m in (11) to get a recursive formula for the coefficients $\{p_m; m = 0, 1, 2, \ldots\}$.

After some rearrangement, this yields the recursion

$$\sum_{n=-1}^{k} \{\phi_n + m\psi_{n+1}\}(m+n)!p_{m+n} = f(x_1)\psi(x_1)x_1^m e^{-x_1} - f(x_0)\psi(x_0)x_0^m e^{-x_0}. \tag{12}$$

In (12), $p_{-1} = 0$, x_0^m is interpreted as 1 if $x_0 = m = 0$, and $x_1^m e^{-x_1}$ is interpreted as 0 if $x_1 = \infty$.

Note that (12) may be re-expressed as

$$\sum_{n=-1}^{k} \{\phi_n + m\psi_{n+1}\}(m+n)^{(n)}p_{m+n} = f(x_1)\psi(x_1)h_m(x_1) - f(x_0)\psi(x_0)h_m(x_0)$$

where $a^{(b)} = \Pi_{i=1}^{b}(a+1-i)$ and $h_m(\lambda) = \lambda^m e^{-\lambda}/m!$ is a Poisson probability. Analogously, (11) may be written as

$$\sum_{n=0}^{k} \{z\psi_n + \phi_n\}P^{(n)}(z) = f(x_1)\psi(x_1)P_{x_1}(z) - f(x_0)\psi(x_0)P_{x_0}(z)$$

where $P_\lambda(z) = e^{\lambda(z-1)}$ is a Poisson pgf.

Also, one can always arrange that (12) be a homogeneous difference equation (i.e. that the right hand side of (12) is identically 0). If the numerator and denominator of (7) are each multiplied by $x - x_0$ then $\psi(x_0) = 0$ and the terms involving x_0 on the right hand side of (11) and (12) both vanish. Similarly, multiplication by $x - x_1$ results in $\psi(x_1)$ equalling 0 and multiplication by $(x - x_0)(x - x_1)$ causes $\psi(x_0)$ and $\psi(x_1)$ to each be 0. This does, however, result in a differential equation of order $k+1$ or $k+2$ rather than k, however.

Alternatively, if one has already derived the nonhomogeneous equation (12), the homogeneous equation may be obtained from it. Equation (11) may be differentiated, and subtraction of (11) multiplied by x_0 from it eliminates the term involving x_0. The term involving x_1 may be eliminated in a similar manner.

Some examples of the use of (11) and (12) are now given.

2.1. *Negative binomial distribution*

Suppose that the mixing distribution is the gamma distribution with

$$f(x) = \frac{\mu(\mu x)^{\alpha-1}e^{-\mu x}}{\Gamma(\alpha)}, \quad x > 0$$

where $\mu > 0$ and $\alpha > 0$ so that $x_0 = 0$ and $x_1 = \infty$. One has

$$\frac{d}{dx}\log f(x) = \frac{\alpha - 1 - \mu x}{x}.$$

Thus $\psi(x) = x$ and (9) yields $\phi(x) = \alpha - (1+\mu)x$. Then (11) becomes

$$zP'(z) + \alpha P(z) - (1+\mu)P'(z) = 0,$$

and (12) is the usual negative binomial recursion (e.g. Panjer & Willmot, 1992)

$$(m+\alpha)p_m = (1+\mu)(m+1)p_{m+1}; \quad m = 0, 1, 2, \ldots. \qquad \square$$

2.2. *Sichel's distribution*

If the mixing distribution is the generalized inverse Gaussian distribution with pdf

$$f(x) = \frac{\mu^{-\lambda}x^{\lambda-1}e^{-\frac{x^2+\mu^2}{2\beta x}}}{2K_\lambda(\mu\beta^{-1})}, \quad x > 0$$

where $\mu > 0$, $\beta > 0$, and $-\infty < \lambda < \infty$ then the resulting mixture is Sichel's distribution (e.g. Panjer & Willmot, 1992). The Poisson-inverse Gaussian distribution is the special case $\lambda = -1/2$. Now

$$\frac{d}{dx}\log f(x) = \frac{\mu^2 + (\lambda-1)2\beta x - x^2}{2\beta x^2}.$$

Thus one may set $\psi(x) = 2\beta x^2$ and so

$$\phi(x) = \mu^2 + 2\beta(\lambda+1)x - (1+2\beta)x^2.$$

Hence from (11) one obtains the second order differential equation

$$2\beta zP''(z) + \mu^2 P(z) + 2\beta(\lambda+1)P'(z) - (1+2\beta)P''(z) = 0,$$

and (12) is the well-known recursion (e.g. Willmot & Panjer, 1987) for $m = 2, 3 \ldots, (1+2\beta)m(m-1)p_m = 2\beta(m-1)(m+\lambda-1)p_{m-1} + \mu^2 p_{m-2}$. $\square$

2.3. *Poisson beta*

If

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{x^{\alpha-1}(\mu-x)^{\beta-1}}{\mu^{\alpha+\beta-1}}, \quad 0 < x < \mu$$

where $\alpha > 0$, $\beta > 0$, and $\mu > 0$ then with $x_0 = 0$, $x_1 = \mu$, one has

$$\frac{d}{dx}\log f(x) = \frac{(\alpha - 1)\mu - (\alpha + \beta - 2)x}{x(\mu - x)}.$$

Thus one may choose $\psi(x) = \mu x - x^2$, yielding

$$\phi(x) = \mu\alpha - (\alpha + \beta + \mu)x + x^2$$

and so (11) becomes

$$z\{\mu P'(z) - P''(z)\} + \mu\alpha P(z) - (\alpha + \beta + \mu)P'(z) + P''(z) = 0.$$

Also, (12) becomes

$$(m + 2)(m + 1)p_{m+2} = (m + 1)(m + \alpha + \beta + \mu)p_{m+1} - \mu(\alpha + m)p_m$$

in agreement with Willmot & Panjer (1987). Note that boundary conditions are obtainable from the explicit expressions for the pgf and probabilities in terms of the confluent hypergeometric function M($\cdot$) (e.g. Abramowitz & Stegun, 1965, p. 504), namely

$$P(z) = M\{\alpha, \alpha + \beta, \mu(z - 1)\}$$

and for $m = 0, 1, 2, \ldots$

$$p_m = \frac{\mu^m \Gamma(\alpha + \beta)\Gamma(\alpha + m)}{\Gamma(m + 1)\Gamma(\alpha)\Gamma(\alpha + \beta + m)} M(\alpha + m, \alpha + \beta + m, -\mu)$$

$$= \frac{\mu^m e^{-\mu}\Gamma(\alpha + \beta)\Gamma(\alpha + m)}{\Gamma(m + 1)\Gamma(\alpha)\Gamma(\alpha + \beta + m)} M(\beta, \alpha + \beta + m, \mu).$$

If $\alpha = 1$, then the choice $\psi(x) = \mu - x$ implies that $\phi(x) = x - (\mu + \beta)$ and (11) becomes $\{\mu z - (\mu + \beta)\}P(z) + (1 - z)P'(z) = -\beta$. Differentiation of this equation gives the above equation with $\alpha = 1$ and hence the same recursion, but this latter form yields the additional information (with $z = 0$) that $p_1 = (\mu + \beta)p_0 - \beta$. In this case the factor $m + 1$ drops out and the distribution belongs to the class studied by Sundt (1992). On the other hand, if $\beta = 1$, then the choice $\psi(x) = x$ implies that $\phi(x) = \alpha - x$ and (11) becomes $\alpha P(z) + (z - 1)P'(z) = \alpha e^{\mu(z-1)}$. This yields the recursion $(m + 1)p_{m+1} = (m + \alpha)p_m - \alpha\mu^m e^{-\mu}/m!$ for $m = 0, 1, 2, \ldots$. As discussed following (12), the term $\alpha e^{\mu(z-1)}$ is eliminated by the original choice $\psi(x) = x(\mu - x)$ yielding the original recursion with $\beta = 1$. □

2.4. *Pareto and generalized Pareto mixtures*

Consider the generalized Pareto mixing density (e.g. Hogg & Klugman, 1984)

$$f(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\mu^\alpha x^{\beta - 1}}{(\mu + x)^{\alpha + \beta}}, \quad x > 0$$

where $\alpha > 0$, $\beta > 0$, and $\mu > 0$. Then

$$\frac{d}{dx}\log f(x) = \frac{(\beta - 1)\mu - (\alpha + 1)x}{x(\mu + x)}.$$

Hence, $\psi(x) = \mu x + x^2$ and (9) yields $\phi(x) = \beta\mu + (1 - \alpha - \mu)x - x^2$. From (11) one obtains

$$\beta\mu P(z) + \{\mu z + (1 - \alpha - \mu)\}P'(z) + (z - 1)P''(z) = 0$$

and (12) becomes for $m = 0, 1, 2, \ldots$

$$(m + 2)(m + 1)p_{m+2} = (m + 1)(m + 1 - \alpha - \mu)p_{m+1} + \mu(m + \beta)p_m.$$

To begin the recursion, one has from formula (13.2.5) on page 505 of Abramowitz & Stegun (1965) that explicit expressions for the probabilities are given for $m = 0, 1, 2, \ldots$ by

$$p_m = \frac{\mu^m \Gamma(\alpha + \beta)\Gamma(m + \beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(m + 1)} U(m + \beta, m + 1 - \alpha, \mu)$$

where $U(\cdot)$ is the confluent hypergeometric function of the second kind. Also, for large m one has the asymptotic formula (Willmot, 1990a)

$$p_m \sim \frac{\Gamma(\alpha + \beta)\mu^\alpha}{\Gamma(\alpha)\Gamma(\beta)} m^{-\alpha-1}, \quad m \to \infty.$$

Similarly, the pgf (2) is

$$P(z) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)} U\{\beta, 1 - \alpha, \mu(1 - z)\}.$$

The Pareto distribution is obtained with $\beta = 1$. In this case one has $\psi(x) = \mu + x$ and, from (9), $\phi(x) = -(\alpha + \mu) - x$. Thus, (11) becomes

$$\{\mu z - (\alpha + \mu)\}P(z) + (z - 1)P'(z) = -\alpha.$$

Differentiation yields the first differential equation with $\beta = 1$, but this latter form yields the additional information (with $z = 0$) that

$$p_1 = \alpha - (\alpha + \mu)p_0.$$

In this case (12) becomes for $m = 0, 1, 2, \ldots$

$$(m + 2)p_{m+2} = (m + 1 - \alpha - \mu)p_{m+1} + \mu p_m$$

and the distribution belongs to the class studied by Sundt (1992). □

2.5. Polynomial ratio hazard rate mixtures

In general, one may express the pdf $f(x)$ in terms of the force of mortality or hazard rate $\mu(x)$ as

$$f(x) = \mu(x)e^{-\int_0^x \mu(y)\,dy}$$

(e.g. Lawless, 1982). Thus one has

$$\frac{d}{dx}\log f(x) = \left\{\frac{d}{dx}\log \mu(x)\right\} - \mu(x).$$

If

$$\mu(x) = \frac{\theta(x)}{\gamma(x)} = \frac{\sum_{n=0}^{r} \theta_n x^n}{\sum_{n=0}^{k} \gamma_n x^n}$$

then

$$\frac{d}{dx} \log f(x) = \frac{\theta'(x)\gamma(x) - \theta(x)\gamma'(x) - \{\theta(x)\}^2}{\theta(x)\gamma(x)}$$

and so one may derive a recursive formula with $\psi(x) = \theta(x)\gamma(x)$. Polynomial hazard rate models (including linear hazard rate models such as the Rayleigh) are of this form (e.g. Gross & Clark, 1975).

Various transformations on the mixing random variable or the mixing density preserve the form (7). Examples include scale transformations, posterior densities in mixed Poisson processes (e.g. Willmot & Sundt, 1989b), as well as power transformed mixing random variables (Section 3 below) and shifted and truncated mixing distributions (Section 4 below).

3. POWER TRANSFORMED MIXING RANDOM VARIABLES

Property (7) is often preserved under various transformations on the mixing random variable. In other words, if X has pdf $f(x)$ which satisfies (7) and $Y \equiv d(X)$ with pdf $f_*(y)$, then $f_*(y)$ satisfies (7) for various choices of $d(\cdot)$.

An important class of transformations is the class of power transformed mixing random variables. In this case, the pdf is of the form $|c|x^{c-1}f(x^c)$ where $f(\cdot)$ is itself a pdf. If X has pdf $f(x)$, then $X^{1/c}$ has this pdf. If (7) holds, then one has

$$\frac{d}{dx} \log\{|c|x^{c-1}f(x^c)\} = \frac{c-1}{x} + cx^{c-1} \frac{\eta(x^c)}{\psi(x^c)}. \tag{13}$$

If c is an integer then (13) may also be expressed as a ratio of polynomials and the results of the previous section applied. Some examples are now given.

3.1. Transformed or generalized gamma mixtures

The transformed gamma distribution is obtained by a power transformation on a gamma random variable with density given in Example 2.1. Thus one has

$$f(x) = \frac{\mu^\alpha c x^{c\alpha - 1} e^{-\mu x^c}}{\Gamma(\alpha)}, \quad x > 0$$

where $\mu > 0$, $c > 0$, and $\alpha > 0$.

This distribution is considered by Hogg & Klugman (1984) and Lawless (1982). The Weibull is obtained with $\alpha = 1$ and the gamma with $c = 1$. The resulting mixed distribution is thus a generalization of the negative binomial distribution.

One has

$$\frac{d}{dx}\log f(x) = \frac{(c\alpha - 1) - \mu c x^c}{x}.$$

Thus if c is a positive integer one may set $\psi(x) = x$ and then (9) gives $\phi(x) = c\alpha - x - \mu c x^c$. From (11) one obtains the differential equation

$$zP'(z) + c\alpha P(z) - P'(z) - \mu c P^{(c)}(z) = 0$$

and from (12) the recursive formula

$$\mu c\left\{\prod_{i=1}^{c}(m+i)\right\}p_{m+c} + (m+1)p_{m+1} = (m+c\alpha)p_m.$$

The usual negative binomial recursion is recovered when $c = 1$. □

3.2. *Transformed beta*

Consider the transformed beta pdf

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{c\mu^{\alpha}x^{c\beta-1}}{(\mu+x^c)^{\alpha+\beta}}, \quad x > 0$$

where $\alpha > 0$, $\beta > 0$, $\mu > 0$, and $c > 0$ (e.g. Hogg & Klugman (1984, p. 185) or Panjer & Willmot (1992, pp. 122–123)). The generalized Pareto ($c = 1$), the Pareto ($c = 1, \beta = 1$), the Burr ($\beta = 1$), and the log-logistic ($\beta = 1, \alpha = 1$) are special cases. Since

$$\frac{d}{dx}\log f(x) = \frac{\mu(c\beta - 1) - (1 + c\alpha)x^c}{\mu x + x^{c+1}},$$

it follows that for c a positive integer, one may choose $\psi(x) = \mu x + x^{c+1}$. Then

$$\phi(x) = c\beta\mu - \mu x + c(1-\alpha)x^c - x^{c+1}$$

and so one obtains

$$(z-1)\{\mu P'(z) + P^{(c+1)}(z)\} + c\beta\mu P(z) + c(1-\alpha)P^{(c)}(z) = 0.$$

Hence one obtains for $m = 0, 1, 2, \ldots$

$$\left\{\prod_{i=1}^{c+1}(m+i)\right\}p_{m+c+1} = \left\{\{m + c(1-\alpha)\}\prod_{i=1}^{c}(m+i)\right\}p_{m+c} - \mu(m+1)p_{m+1} + \mu(m+c\beta)p_m.$$

In addition, one has the simple asymptotic formula (e.g. Willmot, 1990a)

$$p_m \sim \frac{c\mu^{\alpha}\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}m^{-c\alpha-1}, m \to \infty.$$

It is worth noting that if $c\beta = 1$, one may choose $\psi(x) = \mu + x^c$ and

$\phi(x) = -\mu - \{1 + c(\alpha - 1)\}x^{c-1} - x^c.$

This yields the differential equation

$$(z-1)\{\mu P(z) + P^{(c)}(z)\} - \{1 + c(\alpha - 1)\}P^{(c-1)}(z) = -\frac{\Gamma\left(\alpha + \frac{1}{c}\right)}{\Gamma(\alpha)\Gamma\left(1 + \frac{1}{c}\right)\mu^{\frac{1}{c}-1}}.$$

Differentiation yields the first differential equation, but this latter form yields the additional information (by setting $z = 0$) that

$$p_c = \frac{\Gamma\left(\alpha + \frac{1}{c}\right)\mu^{1-\frac{1}{c}}}{c!\,\Gamma(\alpha)\Gamma\left(1 + \frac{1}{c}\right)} - \left\{\alpha - 1 + \frac{1}{c}\right\}p_{c-1} - \frac{\mu}{c!}p_0. \qquad \square$$

An important subset of this class is the set of inverse transformations obtained with $c = -1$. In this case (13) becomes

$$-\frac{2}{x} - \frac{\eta(x^{-1})}{x^2\psi(x^{-1})}.$$

This expression is easily expressed as a ratio of polynomials by multiplying by a sufficiently high power of x. Many distributions correspond to this case such as the inverse Weibull, inverse transformed gamma, and the following distribution.

3.3. Inverse gamma

Consider the reciprocal of a gamma variable with pdf given in Example 2.1. Then

$$f(x) = \frac{\mu^\alpha x^{-(\alpha+1)}e^{-\mu/x}}{\Gamma(\alpha)}, \quad x > 0,$$

where $\mu > 0$, $\alpha > 0$ and

$$\frac{d}{dx}\log f(x) = \frac{\mu - (\alpha + 1)x}{x^2}.$$

Thus $\psi(x) = x^2$, implying that $\phi(x) = \mu + (1 - \alpha)x - x^2$ and (11) becomes

$$\mu P(z) + (1 - \alpha)P'(z) + (z - 1)P''(z) = 0.$$

In this case one has for $m = 0, 1, 2, \ldots$

$$(m + 2)(m + 1)p_{m+2} = (m + 1)(m + 1 - \alpha)p_{m+1} + \mu p_m.$$

Explicit expressions for $P(z)$ and $\{p_n, n = 0, 1, 2, \ldots\}$ are obtainable in terms of the modified Bessel function of the third kind (e.g. Jørgensen, 1982). Thus one obtains

$$P(z) = \frac{2\{\mu(1-z)\}^{\frac{\alpha}{2}} K_\alpha\{2\sqrt{\mu(1-z)}\}}{\Gamma(\alpha)}$$

and for $m = 0, 1, 2, \ldots$

$$p_m = \frac{2\mu^{\frac{m+\alpha}{2}} K_{m-\alpha}(2\sqrt{\mu})}{\Gamma(\alpha)\Gamma(m+1)}.$$

Also, it follows from Willmot (1990a) that

$$p_m \sim \frac{\mu^\alpha}{\Gamma(\alpha)} m^{-\alpha-1}, \quad m \to \infty.$$

As mentioned by Jørgensen (1982), the reciprocal gamma distribution is a limiting case of the generalized inverse Gaussian distribution, implying that the present mixed distribution is a limiting case of Sichel's distribution (2.2). □

4. SHIFTED AND TRUNCATED MIXING DISTRIBUTIONS

Other transformations also preserve the form (7). One such operation involves shifting the density $f(x)$, leading to a mixing density of the form $f(x-\lambda_0)$, $x_0 + \lambda_0 < x < x_1 + \lambda_0$. In this case (2) is replaced by

$$\int_{x_0+\lambda_0}^{x_1+\lambda_0} e^{x(z-1)} f(x-\lambda_0)\,dx = e^{\lambda_0(z-1)} \int_{x_0}^{x_1} e^{x(z-1)} f(x)\,dx$$

which implies that the mixed distribution is the convolution of a Poisson distribution with mean λ_0 and the mixed distribution mixed by the unshifted pdf $f(x)$.

If $f(x)$ satisfies (7), then

$$\frac{d}{dx}\log f(x-\lambda_0) = \frac{\eta(x-\lambda_0)}{\psi(x-\lambda_0)} = \frac{\sum_{n=0}^{i} \eta_n^* x^n}{\sum_{n=0}^{j} \psi_n^* x^n} \tag{14}$$

where $\{\eta_0^*, \eta_1^*, \ldots, \eta_i^*\}$ and $\{\psi_0^*, \psi_1^*, \ldots, \psi_i^*\}$ are constants. Thus, (14) is of the same form as (7), implying that a differential equation of the form (11) holds for the pgf. As in the discussion following (12), one may wish to eliminate any non-zero Poisson terms by multiplying the numerator and denominator of (14) by one or more of the factors $(x - x_0 - \lambda_0)$ and $(x - x_1 - \lambda_0)$. In addition, it is worth noting that if the mixed Poisson distribution with unshifted mixing distribution belongs to the class studied by Sundt (1992), then a recursion for the mixed Poisson distribution with shifted mixing distribution may also be found using the techniques described in Section 4 of that paper. This is the case in the next two examples.

4.1. Delaporte's distribution

Ruohonen (1988), Willmot & Sundt (1989a), and Schröter (1990) considered the mixture by a shifted gamma distribution with pdf

$$f(x) = \frac{\mu^\alpha (x - \lambda_0)^{\alpha - 1} e^{-\mu(x - \lambda_0)}}{\Gamma(\alpha)}, \quad x > \lambda_0$$

where $\mu > 0$, $\alpha > 0$, and $\lambda_0 > 0$. Thus,

$$\frac{d}{dx} \log f(x) = \frac{\alpha - 1 + \mu\lambda_0 - \mu x}{x - \lambda_0},$$

implying that one may choose $\psi(x) = x - \lambda_0$, and so $\phi(x) = \alpha + \lambda_0(1 + \mu) - (1 + \mu)x$. Thus the pgf $P(z) = e^{\lambda_0(z-1)}\{1 - \mu^{-1}(z - 1)\}^{-\alpha}$ satisfies

$$\{-\lambda_0 z + \alpha + \lambda_0(1 + \mu)\}P(z) + \{z - (1 + \mu)\}P'(z) = 0,$$

and equating coefficients of z^m yields (with $p_{-1} = 0$) for $m = 0, 1, 2, \ldots$ the recursion of Schröter (1990), namely

$$(1 + \mu)(m + 1)p_{m+1} = \{\alpha + \lambda_0(1 + \mu) + m\}p_m - \lambda_0 p_{m-1}. \quad \square$$

4.2. Shifted Pareto

Albrecht (1984) considers the Poisson mixed over the density

$$f(x) = \frac{\alpha}{\mu}\left(\frac{x}{\mu}\right)^{-\alpha-1}, \quad x > \mu$$

where $\alpha > 0$ and $\mu > 0$. Evidently $f(x + \mu)$ is the Pareto density in 2.4. Thus the pgf is given by

$$P(z) = \alpha e^{\mu(z-1)} U\{1, 1 - \alpha, \mu(1 - z)\}$$

and from Albrecht (1984) explicit expressions for the probabilities are given for $m > \alpha$ by

$$p_m = \frac{\alpha\mu\Gamma(m - \alpha)}{\Gamma(m + 1)}\{1 - \Gamma(m - \alpha, \mu)\}$$

where $\Gamma(n, x)$ is the incomplete gamma function (Hogg & Klugman, 1984, p. 219). Also, from Willmot (1990a),

$$p_m \sim \alpha\mu^\alpha m^{-\alpha-1}, \quad m \to \infty.$$

Since

$$\frac{d}{dx} \log f(x) = -\frac{\alpha + 1}{x},$$

the choice $\psi(x) = x$ implies that $\phi(x) = -\alpha - x$ and from (11),

$$\alpha P(z) + (1 - z)P'(z) = \alpha e^{\mu(z-1)},$$

leading to the recursion for $m = 0, 1, 2, \ldots$

$$(m+1)p_{m+1} = (m-\alpha)p_m + \alpha\mu^m e^{-\mu}/m!.$$

The Poisson term may be eliminated by the choice $\psi(x) = x(x-\mu) = x^2 - \mu x$, implying that $\phi(x) = \mu\alpha + (\mu + 1 - \alpha)x - x^2$. Thus one obtains from (11) the relation $\mu\alpha P(z) + \{(\mu + 1 - \alpha) - \mu z\}P'(z) + (z-1)P''(z) = 0$. This yields the alternate formula for $m = 0, 1, 2, \ldots$

$$(m+2)(m+1)p_{m+2} = (m+1)(m+1+\mu-\alpha)p_{m+1} - \mu(m-\alpha)p_m. \qquad \square$$

Suppose that the mixing distribution is obtained by truncating a known density, i.e. the mixing distribution is of the form $Kf(x)$, $x_0^* < x < x_1^*$ where $K^{-1} = \int_{x_0^*}^{x_1^*} f(x)\,dx$. Clearly, (7) is unaffected by this transformation since

$$\frac{d}{dx}\log\{Kf(x)\} = \frac{d}{dx}\log f(x) \tag{15}$$

and the only change in (11) is the replacement of x_0 and x_1 by x_0^* and x_1^* respectively. One may wish to eliminate any non-zero Poisson terms in the same manner as in the discussion following (12).

4.3. Truncated gamma

As an alternative to Ruohonen's (1988) proposal of shifting the gamma density, one could consider instead the Poisson mixed by the truncated density

$$f(x) = \frac{\mu(\mu x)^{\alpha-1}e^{-\mu x}}{\Gamma(\alpha)\{1-\Gamma(\alpha, \mu x_0)\}}, \quad x > x_0$$

where $\Gamma(\alpha, x)$ is the incomplete gamma function (Hogg & Klugman, 1984, p. 219). As in 2.1, the pgf

$$P(z) = \int_{x_0}^{\infty} e^{x(z-1)}f(x)\,dx = \frac{\mu^\alpha\{1-\Gamma(\alpha, (\mu+1-z)x_0)\}}{(\mu+1-z)^\alpha\{1-\Gamma(\alpha, \mu x_0)\}}$$

satisfies the differential equation

$$\alpha P(z) + \{z-(1+\mu)\}P'(z) = -x_0 f(x_0)e^{x_0(z-1)}$$

from which one obtains the recursion for $m = 0, 1, 2, \ldots$

$$(1+\mu)(m+1)p_{m+1} = (m+\alpha)p_m + f(x_0)x_0^{m+1}e^{-x_0}/m!.$$

Alternatively, elimination of the Poisson term with $\psi(x) = x(x_0 - x) = x_0 x - x^2$ implies that $\phi(x) = \alpha x_0 - \{(\alpha+1) + x_0(1+\mu)\}x + (1+\mu)x^2$, and, from (11)

$$\alpha x_0 P(z) + \{x_0 z - (1 + \alpha + (1+\mu)x_0)\}P'(z) + \{(1+\mu) - z\}P''(z) = 0$$

from which one obtains the recursion for $m = 0, 1, 2, \ldots$

$$(1+\mu)(m+2)(m+1)p_{m+2} = (m+1)\{m+1+\alpha+x_0(1+\mu)\}p_{m+1} - x_0(m+\alpha)p_m.$$

4.4. Truncated normal

Suppose that the Poisson mixing density is given by

$$f(x)=\frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}\left\{1-\Phi\left(\frac{x_0-\mu}{\sigma}\right)\right\}},\quad x>x_0$$

where $\Phi(\cdot)$ is the standard normal distribution function. Then

$$\frac{d}{dx}\log f(x)=\frac{\mu-x}{\sigma^2}.$$

If $\psi(x)=\sigma^2$ then $\phi(x)=(\mu-\sigma^2)-x$ and (11) yields

$$\{\sigma^2 z+(\mu-\sigma^2)\}P(z)-P'(z)=-\sigma^2 f(x_0)e^{x_0(z-1)}.$$

Equation of the coefficients of z^m yields, for $m=0,1,2,\ldots$

$$(m+1)p_{m+1}=(\mu-\sigma^2)p_m+\sigma^2 p_{m-1}+\sigma^2 f(x_0)x_0^m e^{-x_0}/m!,$$

with $p_{-1}=0$ and $x_0^m=1$ if $x_0=m=0$. If $x_0>0$ then the choice $\psi(x)=\sigma^2(x_0-x)$ implies that

$$\phi(x)=\{x_0(\mu-\sigma^2)-\sigma^2\}-(x_0+\mu-\sigma^2)x+x^2.$$

This results in

$$\{\sigma^2 x_0 z+x_0(\mu-\sigma^2)-\sigma^2\}P(z)-\{\sigma^2 z+x_0+\mu-\sigma^2\}P'(z)+P''(z)=0.$$

This yields the alternate recursion, for $m=0,1,2,\ldots$

$$(m+2)(m+1)p_{m+2}=(x_0+\mu-\sigma^2)(m+1)p_{m+1}+\{\sigma^2(m+1)-x_0(\mu-\sigma^2)\}p_m-\sigma^2 x_0 p_{m-1}$$

with $p_{-1}=0$. Explicit expressions for the probabilities to begin the recursion follow from the pgf, namely,

$$P(z)=\frac{e^{\mu(z-1)+\frac{1}{2}(\sigma(z-1))^2}\left\{1-\Phi\left(\sigma(1-z)+\frac{x_0-\mu}{\sigma}\right)\right\}}{1-\Phi\left(\frac{x_0-\mu}{\sigma}\right)}.\qquad\square$$

Similarly, recursions may be derived for other distributions whose support is restricted through truncation to non-negative values, including the Cauchy and Student's t-distribution.

5. COMPOUND DISTRIBUTIONS

Often in actuarial applications one is interested in the compound distribution $\{q_0, q_1, q_2, \ldots\}$ with pgf

$$Q(z)=\sum_{m=0}^{\infty} q_m z^m=P\{C(z)\}\tag{16}$$

where $C(z)=\Sigma_{m=0}^{\infty} c_m z^m$ is the pgf of the associated claim size distribution and $P(z)$ is given by (2). In general, recursive formulae for $\{q_0, q_1, q_2, \ldots\}$ may be derived from the difference equation (12) for $\{p_0, p_1, p_2, \ldots\}$ as in Willmot & Panjer (1987) or directly from the differential equation (11).

In what follows it is assumed that the support of the claim size distribution is discrete on the non-negative integers. For continuous claim size distributions, the treatment is similar and hence is omitted.

A straightforward treatment may be accorded the special case of (7) when the Poisson mixing density satisfies

$$\frac{d}{dx}\log f(x)=\frac{\eta_0+\eta_1 x}{\psi_0+\psi_1 x}. \tag{17}$$

Then the mixed Poisson pgf (2) satisfies (11), i.e.

$$\{\psi_0 z+(\eta_0+\psi_1-\psi_0)\}P(z)+\{\psi_1 z+(\eta_1-\psi_1)\}P'(z)$$
$$=(\psi_0+\psi_1 x_1)f(x_1)e^{x_1(z-1)}-(\psi_0+\psi_1 x_0)f(x_0)e^{x_0(z-1)}.$$

If z is replaced by $C(z)$ then a differential equation results for $Q(z)$, namely

$$\{\psi_0 C(z)+(\eta_0+\psi_1-\psi_0)\}Q(z)C'(z)+\{\psi_1 C(z)+(\eta_1-\psi_1)\}Q'(z)$$
$$=(\psi_0+\psi_1 x_1)f(x_1)C'(z)e^{x_1\{C(z)-1\}}-(\psi_0+\psi_1 x_0)f(x_0)C'(z)e^{x_0\{C(z)-1\}}.$$

For notational convenience, define

$$C_2(z)=\sum_{m=0}^{\infty} c_m^{*2} z^m=\{C(z)\}^2, \tag{18}$$

and

$$B_x(z)=\sum_{m=0}^{\infty} b_m(x) z^m=\begin{cases}\psi_0 f(0)C(z), & x=0\\ x^{-1}(\psi_0+\psi_1 x)f(x)e^{x\{C(z)-1\}}, & 0<x<\infty.\\ 0, & x=\infty\end{cases} \tag{19}$$

Then one has

$$(\eta_1-\psi_1)\{zQ'(z)\}+\psi_1 C(z)\{zQ'(z)\}+\frac{\psi_0}{2}\{zC_2'(z)\}Q(z)$$
$$+(\eta_0+\psi_1-\psi_0)\{zC'(z)\}Q(z)+zB'_{x_0}(z)=zB'_{x_1}(z). \tag{20}$$

A recursive formula for $\{q_0, q_1, q_2, \ldots\}$ is obtained by equating coefficients of z^m in (20), yielding for $m=1,2,3,\ldots,$

$$(\psi_1-\eta_1)q_m=\sum_{k=0}^{m}\left\{\left(\psi_1+\frac{k}{m}(\eta_0-\psi_0)\right)c_k+\frac{\psi_0}{2}\cdot\frac{k}{m}c_k^{*2}\right\}q_{m-k}+b_m(x_0)-b_m(x_1) \tag{21}$$

beginning with $q_0=P(c_0)$. If $c_0>0$ one would have to solve (21) for q_m.

From (19), $b_m(0)=\psi_0 f(0)c_m$ and $b_m(\infty)=0$. For $x\in(0,\infty)$, $b_m(x)=x^{-1}(\psi_0+\psi_1 x)f(x)\tau_m(x)$ where $\Sigma_{m=0}^{\infty}\tau_m(x)z^m=e^{x\{C(z)-1\}}$. Evidently, $\{\tau_0(x),\tau_1(x),\ldots\}$ are compound Poisson probabilities which may be computed from the usual recursion (e.g. Panjer & Willmot, 1992, p. 171), i.e. $\tau_m(x)=m^{-1}x\,\Sigma_{k=1}^{m}kc_k\tau_{m-k}(x)$ for $m=1,2,3,\ldots$ beginning with $\tau_0(x)=e^{-x(1-c_0)}$.

Some examples are now given, some of which may also be derived by applying theorem 9 in Sundt (1992).

5.1. Compound Poisson-truncated gamma

From 4.3,

$$\frac{d}{dx}\log f(x)=\frac{\alpha-1}{x}-\mu$$

and $\eta_0=\alpha-1$, $\eta_1=-\mu$, $\psi_0=0$, and $\psi_1=1$. Then (21) becomes

$$(1+\mu)q_m=\sum_{k=0}^{m}\left\{1+\frac{k}{m}(\alpha-1)\right\}c_kq_{m-k}+b_m(x_0),$$

generalizing the usual compound negative binomial recursion since $b_m(0)=0$. The recursion begins with $q_0=P(c_0)$ where $P(z)$ is given in 4.3. □

5.2. A compound Poisson-beta mixture

Willmot (1986) considered the Poisson mixed over the beta (2.3) with $\alpha=1$, i.e.

$$f(x)=\frac{\beta(\mu-x)^{\beta-1}}{\mu^{\beta}},\quad 0<x<\mu.$$

One has

$$\frac{d}{dx}\log f(x)=\frac{1-\beta}{\mu-x}$$

and so $\eta_0=1-\beta$, $\eta_1=0$, $\psi_0=\mu$, and $\psi_1=-1$. Then (21) becomes

$$q_m=\sum_{k=0}^{m}\left\{\left(1+\frac{k}{m}(\mu+\beta-1)\right)c_k-\frac{\mu}{2}\frac{k}{m}c_k^{*2}\right\}q_{m-k}-\beta c_m.$$ □

5.3. Compound Poisson-Pareto

As in 2.4, if

$$f(x)=\alpha\mu^{\alpha}(\mu+x)^{-\alpha-1},\quad x>0$$

then $\eta_0=-(\alpha+1)$, $\eta_1=0$, $\psi_0=\mu$, and $\psi_1=1$. Then (21) becomes

$$q_m=\sum_{k=0}^{m}\left\{\left(1-\frac{k}{m}(1+\alpha+\mu)\right)c_k+\frac{\mu}{2}\cdot\frac{k}{m}c_k^{*2}\right\}q_{m-k}+\alpha c_m.$$ □

5.4. Compound Poisson-truncated normal

From 4.4, $\eta_0=\mu$, $\eta_1=-1$, $\psi_0=\sigma^2$, and $\psi_1=0$. Thus (21) yields the recursion

$$q_m=\sum_{k=0}^{m}\frac{k}{m}\left\{(\mu-\sigma^2)c_k+\frac{\sigma^2}{2}c_k^{*2}\right\}q_{m-k}+b_m(x_0).$$ □

5.5. Compound Delaporte

From 4.1, $\eta_0 = \alpha - 1 + \mu\lambda_0$, $\eta_1 = -\mu$, $\psi_0 = -\lambda_0$, and $\psi_1 = 1$. Thus (21) becomes the recursion of Schröter (1990), namely

$$(1+\mu)q_m = \sum_{k=0}^{m} \left\{ \left(1 + \frac{k}{m}(\alpha - 1 + (1+\mu)\lambda_0)\right) c_k - \frac{\lambda_0}{2} c_k^{*2} \right\} q_{m-k}. \qquad \square$$

5.6. Compound Poisson-shifted Pareto

From 4.2, $\eta_0 = -(\alpha + 1)$, $\eta_1 = 0$, $\psi_0 = 0$, and $\psi_1 = 1$. Thus from (21),

$$q_m = \sum_{k=0}^{m} \left\{1 - \frac{k}{m}(\alpha + 1)\right\} c_k q_{m-k} + b_m(\mu). \qquad \square$$

6. TAIL PROBABILITIES AND APPLICATIONS

One can always derive a recursive formula for the tails (3) from the recursion for the original probabilities by virtue of the relation (4) which may be rewritten as

$$P(z) = 1 + (z-1)G(z). \tag{22}$$

Differentiation of (22) yields, with $G^{(0)}(z) = G(z)$,

$$P^{(n)}(z) = nG^{(n-1)}(z) + (z-1)G^{(n)}(z), \tag{23}$$

valid for $n = 1, 2, 3, \ldots$. A differential equation in $P(z)$ may be expressed as a differential equation in $G(z)$ by direct substitution from (22) and (23) into (11) and a recursion for $\{g_n;\ n = 0, 1, 2, \ldots\}$ then derived. One could also insert $p_n = g_{n-1} - g_n$ in (12).

A more direct approach may be used when the survivor function $\bar{F}(x)$ is of a relatively simple form. In these situations (6) may be employed.

Define

$$A(z) = \sum_{m=0}^{\infty} a_m z^m = \frac{e^{x_0(z-1)} - 1}{z-1}. \tag{24}$$

Then

$$a_m = \sum_{j=m+1}^{\infty} \frac{x_0^j e^{-x_0}}{j!}, \quad m = 0, 1, 2, \ldots \tag{25}$$

and (6) may be written as

$$\int_{x_0}^{x_1} e^{x(z-1)} \bar{F}(x)\, dx = G(z) - A(z). \tag{26}$$

In general, it is known that

$$\bar{F}(x) = e^{-\int_0^x \mu(y)\, dy} \tag{27}$$

where $\mu(x)$ is the force of mortality. Thus, if the force of mortality is a ratio of polynomials as in 2.5, then

$$\frac{d}{dx}\{e^{x(z-1)}\bar{F}(x)\gamma(x)\} = e^{x(z-1)}\bar{F}(x)\{z\gamma(x) + \gamma'(x) - \theta(x) - \gamma(x)\}.$$

Integration of this expression over the range of support, i.e. (x_0, x_1), yields the differential equation

$$\sum_{n=0}^{\max(k,r)} \{z\gamma_n + (n+1)\gamma_{n+1} - (\theta_n + \gamma_n)\}\{G^{(n)}(z) - A^{(n)}(z)\} + \gamma(x_0)\, e^{x_0(z-1)} = 0 \tag{28}$$

since $\bar{F}(x_1) = 0$ and $\bar{F}(x_0) = 1$. In (28), it is assumed that $G^{(0)}(z) = G(z)$, $A^{(0)}(z) = A(z)$, $\gamma_n = 0$ if $n \notin \{0, 1, 2, \ldots, k\}$ and $\theta_n = 0$ if $n \notin \{0, 1, 2, \ldots, r\}$. A recursive formula for $\{g_n; n = 0, 1, 2, \ldots\}$ is easily obtained from (28) by equating coefficients of z^n.

It is worth noting that in this case, (28) is normally a lower order differential equation than would be obtained by first using $\psi(x) = \theta(x)\gamma(x)$ to obtain a differential equation for $P(z)$, and then using (22) and (23) to obtain one for $G(z)$.

Mixed Poisson distributions of this type arise naturally in connection with Poisson arrival queue length distributions. Models of this sort have been used in connection with loss reserving (e.g. Willmot, 1990b). For example, for the $M/G/1$ queue length distribution, assume that arrivals occur according to a Poisson process with rate λ and service occurs according to the service distribution function $F(x)$ with mean μ. Let $\rho = \lambda\mu < 1$. Then the equilibrium probability generating function of the number in the system may be expressed as (e.g. Tijms, 1986, section 4.4.3)

$$Q(z) = 1 - \rho + \rho z H(z) \tag{29}$$

where

$$H(z) = \frac{(1-\rho)K(z)}{1-\rho K(z)} \tag{30}$$

and

$$K(z) = \int_0^\infty e^{\lambda x(z-1)}\left\{\frac{1-F(x)}{\mu}\right\} dx. \tag{31}$$

Evidently, (31) is essentially of the form discussed in this section, so if the service time hazard rate is a ratio of polynomials, the coefficients of $K(z)$ may be obtained. Also, (30) is a compound geometric pgf so the associated distribution may be obtained recursively (e.g. Sundt & Jewell, 1981). Finally (29) gives the required queue length probabilities. A wide variety of Poisson arrival queue length distributions involving mixed Poisson pgf's of the form (31) are discussed by Neuts (1986).

7. FURTHER REMARKS

In this paper it is demonstrated how to derive recursive formulae for a wide variety of mixed Poisson probability distributions and related quantities by first obtaining a differential equation for the associated generating function. The approach requires polynomial type representations for the density or survivor function. In other cases it may be necessary to replace $\psi(x)$ formally by $\psi(x, z)$ in (8) in order to derive a differential equation for $P(z)$. The idea is to select a function $\psi(x, z)$ such that $\partial/\partial x\{e^{x(z-1)}f(x)\psi(x, z)\}$ equals $e^{x(z-1)}f(x)$ multiplied by a (hopefully simple) power series in x and z. Integration then yields a differential equation for $P(z)$.

7.1. Exponential-inverse Gaussian mixture

If the mixing distribution is the exponential-inverse Gaussian distribution (e.g. Bhattacharya & Kumar, 1986) with pdf

$$f(x) = \mu(1 + 2\beta x)^{-1/2}e^{\frac{\mu}{\beta}\{1-(1+2\beta x)^{1/2}\}}, \quad x > 0,$$

then

$$\frac{d}{dx}\log f(x) = -\frac{\beta}{1+2\beta x} - \mu(1+2\beta x)^{-1/2}.$$

In this case, however, one may multiply $e^{x(z-1)}f(x)$ by

$$\psi(x, z) = (1+2\beta x)^{1/2}\{1 + \mu^{-1}(1+2\beta x)^{1/2}(z-1)\}.$$

One has

$$e^{x(z-1)}f(x)\psi(x, z) = \{\mu + (z-1)(1+2\beta x)^{1/2}\}e^{x(z-1)+\frac{\mu}{\beta}\{1-(1+2\beta x)^{1/2}\}}$$

and differentiation with respect to x yields, after some algebra,

$$\frac{\partial}{\partial x}\{e^{x(z-1)}f(x)\psi(x, z)\} = e^{x(z-1)}f(x)\left\{\frac{\beta}{\mu}(z-1) - \mu - \frac{(z-1)^2}{\mu}(1+2\beta x)\right\}.$$

Integration over $(0, \infty)$ yields

$$1 - \mu - z = \frac{\beta}{\mu}(z-1)P(z) - \mu P(z) - \frac{(z-1)^2}{\mu}\{P(z) + 2\beta P'(z)\}$$

which may be written as

$$2\beta(1 - 2z + z^2)P'(z) = (\mu - \mu^2) - \mu z + \{(\mu^2 + \beta - 1) + (2-\beta)z - z^2\}P(z),$$

yielding the recursive formula (by equating coefficient of z^m)

$$2\beta p_1 = (\mu - \mu^2) + (\mu^2 + \beta - 1)p_0,$$

$$4\beta p_2 = (\mu^2 + 5\beta - 1)p_1 + (2-\beta)p_0 - \mu,$$

and for $m = 2, 3, 4, \ldots$

$$2\beta(m+1)p_{m+1} = \{\mu^2 + \beta(1+4m) - 1\}p_m + \{2 - \beta(2m-1)\}p_{m-1} - p_{m-2}.$$

The pgf (2) can be put into a recognizable form by making the change of variable

$$y = (2\beta)^{-1}(1-z)\{\mu(1-z) + (1+2\beta x)^{1/2}\}^2.$$

This yields

$$P(z) = \frac{\mu\Gamma(\frac{1}{2})e^{h(z)}}{\{2\beta(1-z)\}^{1/2}}\left\{1 - \Gamma\left(\frac{1}{2}, h(z)\right)\right\}$$

where $h(z) = (2\beta)^{-1}(1-z)\{1+\mu(1-z)^{-1}\}^2$ and $\Gamma(n, x)$ is the incomplete gamma function (e.g. Hogg & Klugman, 1984, p. 219). Recursive evaluation of the probabilities may begin with $p_0 = P(0)$. □

Numerical aspects of the use of these recursions such as stability have not yet been examined. Similarly, initial or boundary conditions such as asymptotic or exact expressions for certain probabilities need to be obtained.

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